Statistics of shocks in a toy model with heavy tails

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We study the energy minimization for a particle in a quadratic well in the presence of short-ranged heavy-tailed disorder, as a toy model for an elastic manifold. The discrete model is shown to be described in the scaling limit by a continuum Poisson process model which captures the three universality classes. This model is solved in general, and we give, in the present case (Frechet class), detailed results for the distribution of the minimum energy and position, and the distribution of the sizes of the shocks (i.e., switches in the ground state) which arise as the position of the well is varied. All these distributions are found to exhibit heavy tails with modified exponents. These results lead to an "exotic regime" in Burgers turbulence decaying from a heavy-tailed initial condition.

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I. INTRODUCTION AND MODEL

Strongly pinned elastic objects, such as interfaces, occur in nature in the presence of substrate impurity disorder which exhibits large fluctuations. The ground-state configuration is determined by a competition between the energy cost of deforming the interface and the energy gain in exploring larger regions of disorder. In the well-studied case of Gaussian disorder, no impurity site particularly stands out and the optimum arises from a global optimization. The typical interfaces are rough, with nontrivial roughness exponents $u \sim L^{\zeta}$, where u is the deformation field and L an internal coordinate scale. The total optimal energy H fluctuates from sample to sample with another exponent $H \sim L^{\theta}$. For directed lines (i.e., internal dimension $d = 1$) wandering in one dimension, $\zeta = 2/3$ and $\theta = 1/3$, which in turn are related to the exponents of the standard universality class for the Kardar-Parisi-Zhang growth equation [1].

In some physical systems however, the picture is completely different: a small fraction of the impurity sites produce a finite contribution to the total pinning energy, and the interface is deformed over large macroscopic scales, pinned specifically on those particular regions. One can see realizations of that situation in various areas such as transition in chemical reaction of BZ type, or in granular flows [2]. One expects that the usual critical exponents are modified, but much less is known in this case, both about equilibrium (e.g., ground states) and about nonequilibrium dynamics (e.g., depinning).

The present paper focuses on heavy-tailed disorder, which is paradigmatic of that situation, and whose probability distribution function[1] (PDF), $P(V)$, shows an algebraic tail. In terms of the cumulative distribution function (CDF), denoted $P_<(V) = \int_{-\infty}^{V} P(V')dV'$, we have

$$P_<(V) \simeq \frac{A}{(-V)^{\mu}} \quad \text{for } V \to -\infty. \tag{1}$$

As was found in numerous works, such a scale-free distribution often leads to behaviors dominated by rare events. They have been much studied in the context of diffusion in random media, where they generate anomalous diffusion [3]. More recently, heavy-tailed randomness was studied in the context of spin glasses and random matrices [4,5]. For instance, in [6] it was found that the PDF of the maximal eigenvalue of a large random matrix with i.i.d. entries distributed as changes from the standard Tracy-Widom distribution (the Gaussian universality class) to a Frechet distribution as μ is decreased below $\mu = 4$.

Only a few works address the pinning problem in the presence of heavy tails. In [6] it was argued, based on a Flory argument, that for a directed polymer in the so-called $(1+1)$-dimensional geometry (meaning internal dimension $d = 1$ and displacement $u \in R^D$ with $D = 1$), for $\mu < 5$ the roughness and energy exponents at $T = 0$ change to $\zeta = (1+\mu)/(2\mu - 1)$ and $\theta = 3/(2\mu - 1)$. For $\mu \geqslant 5$ one recovers the above-mentioned values for Gaussian disorder, i.e., the tails have subdominant effect. While some mathematical results are available for $\mu < 2$ [7], little is presently known rigorously for general μ or on the effect of a nonzero temperature on the problem [8].

In this paper we solve the much simpler case of a particle, which can be seen as the limit $d = 0$ of the elastic interface problem. We consider the minimization problem:

$$H(r) = \min_u H(r,u) = H(r,u(r)), \tag{2}$$

$$H(r,u) = \frac{m^2}{2}(u - r)^2 + V(u), \tag{3}$$

where $V(u)$ is a random potential (a random function of u) and we define $u(r) = \mathrm{argmin} H(r,u)$ the position of the minimum. The quadratic term confines the position u of the particle and mimics the elastic term for interfaces (see Fig. 1). More precisely, this model can be extended to an interface in a quadratic well and there m sets an internal length $L_m = 1/m$ [9]. The PDF of $u(r) - r$ and $H(r) - \overline{H(r)}$ (where we denote by $\overline{\cdots}$ the average over the disorder) are independent of r if $V(u)$ is statistically invariant by translation. Hence one can again define the exponents, as $m \to 0$ (see Sec. II A for more details):

$$u(r) - r \sim m^{-\zeta}, \quad H(r) - \overline{H(r)} \sim m^{-\theta}. \tag{4}$$

This "toy model" has been much studied in the context of disordered systems for Gaussian disorder. It also appears in the context of the decaying Burgers equation with random initial

[1] Also called probability density function below.

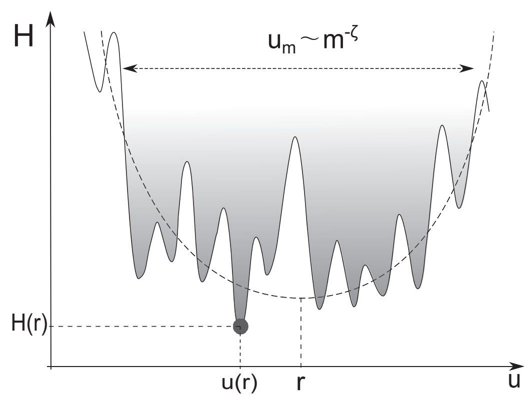

FIG. 1. Particle in a random potential landscape confined by an elastic force (i.e., a quadratic potential centered at r). $u(r)$ is the position with minimal total energy $H(r)$. Its fluctuations from sample to sample scale as $u_m \sim m^{-\zeta}$.

conditions, in the limit of vanishing viscosity (see Appendix A for details of the mapping). The case of short-range correlations corresponding to a short-range potential $V(u)$ was solved in the seminal paper of Kida [10]. An elegant derivation using replica was also given in [11]. Other derivations are given in [12] (Appendix J) and [13] (Appendix A). The case of Brownian correlations for $V(u)$ is related to the Sinai model studied in [12,14–18]. Other type of correlations have been studied in [19–23].

Here we consider the case where (i) correlations of $V(u)$ are short range and (ii) the PDF of $V(u)$ contains heavy tails. We then ask how the exponents and the PDF of $u(r)$ and $H(r)$ depend on the heavy-tail exponent μ. Another interesting observable are the jumps of the process $u(r)$. Indeed in the limit of small m the process $u(r)$ consists mostly of jumps called "static avalanches" or shocks (see below), and one defines the shock sizes $s = u(r^+) - u(r^-)$.

To be specific we solve here two variants of the model as follows.

(i) The discrete model: one starts with u on a discrete lattice and i.i.d. random variables $V(u)$. In the limit $m \to 0$ by rescaling the position u the process converges to a continuum limit.

(ii) The second is defined directly in the continuum for u: there $V(u)$ is defined as a Poisson point process.

Both models enjoy the same universal scaling limit.

In the absence of the quadratic well, $H = \min_u V(u)$ and the discrete problem reduces to the standard extreme value statistics problem. It must then be defined for a fixed system size $u = 1,\ldots,N$. For i.i.d. random variable (or weakly correlated ones) H then grows to infinity with the system size N and, after a proper rescaling, the PDF of $a_N H + b_N$ converges to one of the famous three universality classes [24]: (i) Gumbel when $P(V)$ decays faster than a power law; (ii) Frechet of index μ when $P(V)$ decays as a power law (1), and Weibull when $P(V)$ vanishes below some threshold (e.g., for $V < 0$). In the presence of the confining quadratic well, the same three classes survive: the Kida case belongs to the Gumbel class, while the heavy tail case belongs to the Frechet class. There are, however, some different universal features, such as the exponents and the distributions of shock sizes and minimum position.

In this paper we derive a general formula for the PDF of the position of the minimum $u(r)$, and for the distribution of the shock sizes s. Although our formula is valid for the three universality classes, we give a detailed calculation in the case of the Frechet class with power-law exponent μ. We find that both distributions exhibit algebraic tails with modified exponents. These results are extended to space dimension $D > 1$.

Note that some of our results were anticipated in the context of the decaying Burgers equation. In [25] Bernard and Gawedzki looked for universality classes distinct from Kida for statistically scale-invariant velocity fields: they focused on the Weibul class and called it an "exotic regime" for Burgers turbulence. In [26], a more general study was presented, encompassing the three regimes. However, in none of these works the distribution of the shock sizes was obtained. The present work thus gives results on another exotic regime for decaying Burgers turbulence.

Note that the nonequilibrium version of this toy model, where one studies the dynamics of a particle pulled quasistatically by the harmonic well in the random potential $V(u)$ was studied in [27]. The three universality classes were also found to appear, and the distribution of the avalanche sizes were obtained for the three classes.

In Sec. II, we solved the discrete toy model and obtain the joint PDF of the energy and the position in the small m limit. In Sec. III we consider the Poisson process model, and derive the shock size distribution. In Sec. IV, we consider the discrete toy model in higher dimension. Finally, in Sec. V, we discuss the case of a more general elastic manifold of internal dimension d using Flory arguments. The Appendixes contains the mapping to Burgers, and mode details.

II. FROM THE DISCRETE MODEL TO THE CONTINUUM: ONE-POINT DISTRIBUTIONS

A. Scaling exponents and dimensionless units

We now start from the discrete model where $u \in Z$ and $V(u)$ are i.i.d. random variables drawn from the distribution $P(V)$. We show that one obtains a nontrivial continuum limit in the limit $m \to 0$ upon rescaling of u (in what we call dimensionless units below). This procedure makes the universality appear clearly.

Let us study first the one-point distributions. For that purpose we can set $r = 0$ and consider $H = H(r = 0)$. The probability that the minimum total energy H is attained in position u with a value of the disorder V is equal to the product of (i) the probability $P(V)$ of having V in u and (ii) the probability to have higher total energies on all the other sites $u' \neq u$. It is thus given by the infinite product:

$$p(u,V) = P(V) \prod_{u' \neq u} P_> \left(H - \frac{m^2 u'^2}{2} \right). \tag{5}$$

To study the limit of small m, it is convenient in the following to absorb the dependence with m in the units (u_m, H_m, V_m) defined for the variables (u,H,V), respectively. One can then recover the dimensionful results by the substitution in all dimensionless results:

$$u \to u/u_m = m^{\zeta} u,$$
$$V \to V/V_m = m^{\theta} V,$$
$$H \to H/H_m = m^{\theta} H.$$

Except if stated, we work now in the dimensionless system of units defined above. Without loss of generality, A in Eq. (1) has been set to 1 by a rescaling of V.

At this stage the exponents θ and ζ are not specified. To obtain a nontrivial limit one needs to scale V as m^2u^2 which imposes the exponent relation:

$$\theta = 2\zeta - 2, \tag{6}$$

which is known in the directed polymer context as the STS relation [28].

The joint PDF Eq. (5) for the optimal position u and the value of the random potential V on the optimal site then becomes, in the small m limit,

$$\begin{aligned} p(u,V) &= m^{-\zeta-\theta} P(m^{-\theta} V) \prod_{u' \neq u} P_{>}\left[m^{-\theta}\left(H - \frac{u'^2}{2}\right)\right] \\ &\approx \frac{\mu}{|V|^{1+\mu}} \exp\left\{ - \int du' m^{-\zeta} P_{<}\left[m^{-\theta}\left(H - \frac{u'^2}{2}\right)\right]\right\} \\ &\quad \times \theta_{H<0} \\ &\approx \frac{\mu}{|V|^{1+\mu}} \exp\left(-F_\mu \left| V + \frac{u^2}{2}\right|^{\frac{1}{2}-\mu}\right) \theta_{V+\frac{u^2}{2}<0}, \end{aligned} \tag{7}$$

where $H = V + \frac{u^2}{2}$ and we denote everywhere $\theta_{x<0}$ the characteristic function of the interval (Heaviside function). Here and below we denote

$$F_\mu = \frac{\sqrt{2\pi}\,\Gamma[\mu - 1/2]}{\Gamma[\mu]}. \tag{8}$$

The joint PDF of u and H is simply $p(u, V = H - \frac{u^2}{2})$. Going from the infinite product to the exponential in the second line of Eq. (7) requires that $P_>(\cdot) \sim 1$ at all sites, or equivalently $H < 0$, which is verified for m small enough. The final expression for the joint PDF Eq. (7) is normalized to unity $\int dV\, du\, p(u,V) = 1$, which shows that we have correctly taken the small mass limit (no regions have been overlooked). More precisely, and as is further explained in Appendix B, as $m \to 0$ (the continuum limit), the rescaled cumulative (CDF) $m^{-\zeta} P_<(m^{-\theta} y)$ converges to $\frac{\theta_{-y}}{(-y)^{1+\mu}}$ [under the condition that the right tail is in $o(V^{-(1+\mu)})$; cf. Appendix B]. Hence only the contribution of the left tail of $P_<(\cdot)$ contributes to the integral in Eq. (7), a typical behavior in power-law statistics, and one can readily replace $P_<(\cdot)$ by its asymptotic expression (such estimates can be established rigorously by the use of Tauberian theorems [29]). This implies the second relation:

$$\zeta = \mu\theta, \tag{9}$$

which leads to

$$\zeta = \frac{2\mu}{2\mu - 1}, \tag{10}$$

$$\theta = \frac{2}{2\mu - 1}. \tag{11}$$

One could wonder about the existence of a threshold value μ_c above which the algebraic decay of the tails is fast enough to recover the behavior in the Gaussian disorder $\zeta = \zeta_{SR}$ and $\theta = \theta_{SR}$ (where SR stands for *short-range Gaussian disorder*). One notes that, unlike the directed polymer (see Sec. V), such a finite critical value μ_c for the disorder tail doesn't exist. In other words, any power-law tail matters. More precisely, one can say that $\mu_c = +\infty$. In that limit, indeed, $\zeta \to 1$ which is the value for the Gumbel class [12]. There is an interesting crossover in that limit where the leading contribution goes from the bulk of $P(V)$ (as is the case for the Gumbel class) to the tail (for the present power-law case).

B. Results for the one-point distributions

From Eq. (7), one can obtain the joint distribution of (H,V). Taking into account the Jacobian $\frac{\partial(u,V)}{\partial(H,V)} = [\sqrt{2}(H-V)]^{-1/2}$ and a factor of 2 from integration over positive and negative u yields

$$p(H,V) = \frac{\mu\sqrt{2}}{|V|^{1+\mu}\sqrt{H-V}} e^{-F_\mu |H|^{\frac{1}{2}-\mu}} \theta_{H<0,V<H}. \tag{12}$$

After integration, one obtains the various marginal distributions of H, V, and u. First we obtain

$$p(H) = \frac{(\mu - \frac{1}{2})F_\mu}{|H|^{\mu+\frac{1}{2}}} e^{-F_\mu |H|^{\frac{1}{2}-\mu}} \theta_{H<0}. \tag{13}$$

Hence the PDF of the total energy H is a Frechet distribution. On one hand, this appears as natural since we are dealing with extreme value statistics of heavy-tailed distributions. However, the index of the Frechet distribution is not μ (as would be naively expected) but $\mu - 1/2$, which is thus a correction coming from the competition with the elastic energy. As the particle chooses amongst the deepest sites, the distribution of its energy acquires a power-law tail which is even broader than the initial disorder. It is easy to extend the above calculation to a generalized elastic energy growing as u^α, the modified index being then $\mu - 1/\alpha$.

Next we also obtain the PDF of the potential V at the position of the minimum as

$$p(V) = \frac{\mu}{|V|^{\mu+1}} \phi_\mu(|V|)\theta_{V<0}, \tag{14}$$

where we have defined the auxiliary function:

$$\phi_\mu(x) = \sqrt{2}\int_0^x \frac{dy}{\sqrt{x-y}} e^{-F_\mu y^{\frac{1}{2}-\mu}}. \tag{15}$$

Note that the factor $\phi_\mu(|V|)$ gives the relative change of the tail of the PDF of the potential at the optimal site with respect to the tail of the original PDF of the disorder. For $|V|$ of order one it is of order one; hence the original tail exponent is not

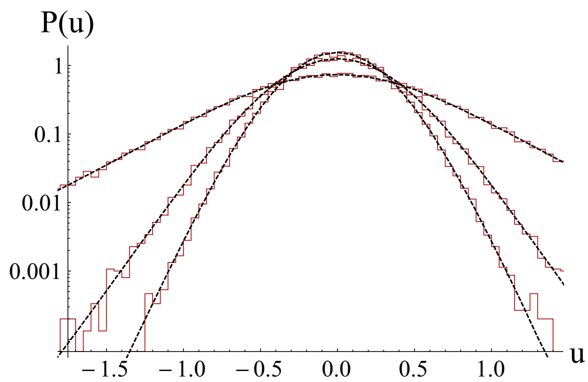

FIG. 2. (Color online) Comparison of the PDF for the position u as given in Eq. (17) (black dashed lines) with numerical simulations (red solid lines). The algebraic tails are clearly distinguishable as straight lines on the semilogarithmic plot. From the narrowest shape to the broadest (corresponding to the fatter tails), $\mu = 15$, 10, and 4. The sample size is $N = 5 \times 10^5$.

changed, but the amplitude is changed.[2] For large negative V, it diverges; hence we find

$$p(V) \simeq \frac{2\sqrt{2}\mu}{|V|^{\mu+\frac{1}{2}}}, \quad V \to -\infty, \tag{16}$$

which is again the original tail but with the same shift in the exponent $\mu \to \mu - \frac{1}{2}$ as noticed above, and a different amplitude. This surprising shift can be recovered by invoking results from record statistics theory. Consider a realization of the disorder with a particularly deep minimum, where the particle sits. From record statistics, it is known that the tail of the minimum of N heavy-tailed random variables decays as $\sim \frac{N}{V^{1+\mu}}$. Balancing elastic energy and potential leads to $E \sim u^\alpha \sim V$, and then to $N \sim u \sim V^{1/\alpha}$. Hence the dependence of N with V, inherent to the fact that large deviations in the disorder allows the particle to explore a larger space, leads to a modified exponent $\mu - 1/\alpha$ of the tail of H and V.[3]

Finally we obtain the PDF of the optimal position u of the particle as

$$p(u) = \mu\, \psi_\mu\left(\frac{u^2}{2}\right) \tag{17}$$

in terms of the auxiliary distribution:

$$\psi_\mu(x) = \int_0^\infty \frac{dy}{(x+y)^{\mu+1}} e^{-F_\mu y^{\frac{1}{2}-\mu}}. \tag{18}$$

The PDF of u decreases from a constant at $u = 0$ to a power law at large u. The position of the particle is thus heavy tailed as well as its PDF decays as

$$p(u) \simeq \frac{2^\mu}{u^{2\mu}}, \quad |u| \to +\infty. \tag{19}$$

[2] One should keep in mind that here V denotes the dimensionless potential; hence it is deep in the tail, since we use units of $V_m \sim m^{-\theta}$.

[3] We thank Jean-Philippe Bouchaud for helping to set up this argument.

The moments $\overline{u^{2n}}$ thus exist only for $2n < 2\mu - 1$ and are given in Appendix C. The comparison with numerics is made in Fig. 2. Finally, note that for $\mu < \frac{1}{2}$ the particle explores the whole space $u \sim W$, as the energy of the optimal site $\sim u^{1/\mu}$ grows faster than the elastic energy $\sim u^2$.

We note that the PDF of the "elastic energy" $E = u^2/2$ has also a tail:

$$p(E) \simeq \frac{1}{\sqrt{2}} \frac{1}{E^{\frac{1}{2}+\mu}}, \tag{20}$$

with exponent $\mu - \frac{1}{2}$ analogous to Eq. (16) for large values.

To conclude, the typical H, V of order one are already drawn in the original tail of $P(V)$ with exponent μ (since we work in the units $m^{-\theta}$) and the rare events acquire a tail with exponent $\mu - \frac{1}{2}$.

III. STATISTICS OF THE SHOCKS

As the center of the harmonic potential r is shifted, the optimal position $u(r)$ of the particle is changed as shown in Fig. 3. This corresponds to a jumpy motion of the particle; each jump is called a shock because corresponding to traveling shocks in the Burgers velocity field (see Appendix A). We now introduce the Poisson process model.

A. General case

1. Poisson process model and one-point distribution

The computation on the discrete model being rather cumbersome, we follow [25] and start directly in the continuum by distributing the random energies over the line as a Poisson process over the plane (V,u) of density $f(V)dV\,du$. Each cell of size $dV\,du$ is then either occupied or not, depending on the value of the random potential V_i at site u_i. This means that the potential is defined only at the u_i with values $V(u_i) = V_i$ and that

$$H(r) = \min_j H_j(r) = \min_j \left(V_j + \frac{(u_j - r)^2}{2}\right), \tag{21}$$

$$u(r) = \text{argmin} H_j(r). \tag{22}$$

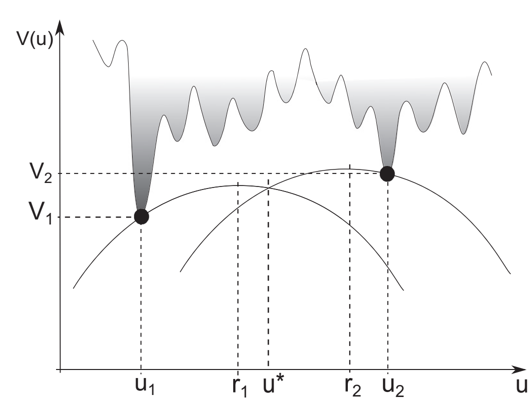

FIG. 3. Parabola construction for the minimization problem: when the center r of the parabola is shifted from r_1 to r_2, the position of the particle moves from u_1 to u_2. For given r_1 and r_2, the intersection of both the parabola is called u^*. More details are displayed in Appendix E.

We denote the primitive $F(x) = \int_{-\infty}^{x} f(t)dt$ and assume that $F(+\infty) = +\infty$. We now calculate, using methods similar to the one of [25], the one- and two-point characteristic function of the field $u(r)$.

For the one-point function we can choose $r = 0$, and define $u = u(0)$. Using formulas similar to Eq. (5) we find for the joint distribution of position and potential at the minimum

$$p(u,V)dV\,du = f(V)dV\,du \prod_{dV'du'} \left(1 - \theta_{V'+\frac{u'^2}{2}<V+\frac{u^2}{2}} f(V')dV'du'\right). \quad (23)$$

From the infinitesimal version of Eq. (5), and after the change of variables $z = u'$, $\phi = V + \frac{u^2}{2}$, the one-point distribution of the position of the minimum can be expressed as

$$p(u) = \int d\phi\, f\left(\phi - \frac{u^2}{2}\right) \exp\left[-\int dz\, F\left(\phi - \frac{z^2}{2}\right)\right]. \quad (24)$$

It is easy to check the normalization $\int du\, p(u) = 1$ by noting that the integral is a total derivative. This result is valid for arbitrary Poisson measure $f(V)$. As we discuss below one can recover the results of the previous section in a particular case.

2. *Shock and droplet size distributions*

To describe the statistical properties of the jumps of the optimal position $u(r)$ of the particle as r is varied one defines the shock density as

$$\rho(s) = \lim_{\delta r \to 0^+} \frac{1}{\delta r} \overline{\delta[u(r+\delta r) - u(r) - s]}. \quad (25)$$

Another definition, equivalent in the present case, uses the decomposition

$$u(r) = \sum_i s_i \theta_{r>r_i} + \tilde{u}(r), \quad (26)$$

where $\tilde{u}(r)$ is the smooth part of the field $u(r)$, which, for the Poisson process model can be set to zero. For other models in the same universality class this part is subdominant. The shock density is then defined as [9]

$$\rho(s) = \overline{\delta(r - r_i)\delta(s - s_i)}, \quad (27)$$

where the (r_i, s_i) are the positions and sizes of the shocks. Note that all the $s_i > 0$.

The shock density is intimately related to another quantity, the droplet density $D(s)$, namely the probability density for the total energy $H_j(r)$ in (21) for a given r, to exhibit two degenerate minima at positions u_1 and u_2, separated in space by $s = u_2 - u_1$ (see Fig. 4). By construction $D(s)$ is a symmetric function $D(s) = D(-s)$ and has dimension $1/(sE)$, where E is an energy. More precisely, it is defined as $D(s) = \int du_1 du_2 \delta(s - u_2 + u_1) p(u_1, u_2, 0)$, where $p(u_1, u_2, E)$ is the probability density for the absolute minimum in u_1 and the secondary minimum in u_2 separated by $E > 0$ in energy. Note that the knowledge of this function allows one to study more generally the statistics of several interesting observables (e.g., the position u) at low (but nonzero) temperature (for more details of the procedure, see, e.g. [12,30]).

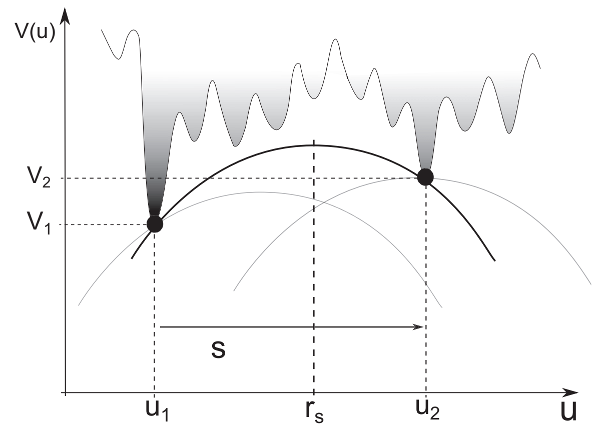

FIG. 4. Discontinuous motion of the particle can be decomposed in shocks. Those shocks occur (here in r_s) while the parabola is shifted and touches the potential at two positions u_1 and u_2, as depicted. The size of the shock is denoted $s = u_2 - u_1$.

As before, we denote the minimal total energy $\phi = H(u_1) = H(u_2)$. Requiring all the other sites to have higher total energy induces a factor $\exp[-\int F(\phi - z^2/2)dz]$ similar to Eq. (24). Then the integrated probability over the value ϕ of the minimum and the positions u_1 and u_2 at fixed $s = u_2 - u_1$ lead to

$$D(s) = \int d\phi\, du_1 du_2 f\left(\phi - \frac{u_1^2}{2}\right) f\left(\phi - \frac{u_2^2}{2}\right) \times \exp\left[-\int F\left(\phi - \frac{z^2}{2}\right)dz\right] \delta(s - u_2 + u_1). \quad (28)$$

The relation between the shock and the droplet density can be written (see Ref. [12], Secs. IV B 5 and E 4) for $s > 0$:

$$\rho(s) = s D(s) \theta_{s>0}. \quad (29)$$

The factor s originates from the change of variable from energy to position as $\frac{\partial H}{\partial r}$ noting that a small change in the position of the parabola around the point of degeneracy amounts to shift the relative energies of the two states by

$$\delta H = \delta r \times (u_1 - u_2). \quad (30)$$

Using this relation, from Eq. (29) we now obtain the shock density, which can be rewritten as, for $s > 0$,

$$\rho(s) = \frac{s}{2} \int d\phi\, dz\, f\left(\phi - \frac{(z-s)^2}{8}\right) \times f\left(\phi - \frac{(z+s)^2}{8}\right) e^{-\int dz' F(\phi - \frac{z'^2}{2})}, \quad (31)$$

where we denoted $z = u_1 + u_2$.

From the shock density one can define a normalized size probability distribution as

$$\rho(s) = \rho_0 p(s), \quad (32)$$

where $\int_0^\infty ds\, p(s) = 1$ and ρ_0 is the total shock density. The density $\rho(s)$ satisfies the following "normalization" identity:

$$\int_0^\infty ds\, s\, \rho(s) = 1, \quad (33)$$

which expresses that all the motion occurs in the shocks. Similarly $D(s)$ satisfies $\int_{-\infty}^{+\infty} ds\, s^2 D(s) = 2$. This identity, proved in Appendix D, is a signature of the STS relations which originate from the statistical translational invariance of the problem.

As a consistency check, $\rho(s)$ can also be extracted from the small separation behavior of the two-point characteristic function of the position field $u(r)$, for $r > 0$:

$$\overline{e^{\lambda[u(r)-u(0)]}} = 1 + r \int_0^\infty ds\, \rho(s)(e^{\lambda s} - 1) + O(r^2). \quad (34)$$

The calculation of this function is more cumbersome and displayed in Appendix E. As shown there, by identification in the above formula one recovers Eq. (31).

B. Scale invariance and universality classes

From Eq. (31), one can read the distribution of the shock sizes for any disorder in the continuum Poisson process model. For this model to be a "fixed point" (i.e., continuum limit) of a more general class of models (e.g., the discrete model studied in Sec. II as $m \to 0$) one should in addition require scale invariance. Then, similar to the usual problem of extremal statistics [31], and to the problem of the driven particle [27], three different classes of universality emerge. The nice feature of the Poisson process model is that it contains the three scale-invariant models.

1. *Three universality classes*

Let us consider again the minimization problem (22) in a dimensionful form:

$$H_m(r) = \min_j \left(V_j + m^2 \frac{(u_j - r)^2}{2} \right). \quad (35)$$

Let us require that $H_m(r)$ is scale invariant *in law*, i.e., that $H_m(m^{-\zeta} r)$ has the same distribution as $m^{-\theta} H_{m=1}(r)$, possibly up to an additive constant in H. One easily sees that it implies that $f(m^\theta V) = m^{-(\theta+\zeta)} f(V + C_m)$ and the STS exponent relation (6). There are three type of solutions.

(i) The "Gumbel" class, where the disorder left tail is exponentially fast decaying. This case corresponds to the well-known Kida statistics of the Burgers equation [10], and is obtained for a Poisson density $f(\phi) = e^\phi$ with the density of shocks:

$$\rho(s) = \frac{1}{2\sqrt{\pi}} s\, e^{-s^2/4}. \quad (36)$$

(ii) The "Weibull" class, where the disorder is bounded from below. It corresponds to the Poisson process model with $f(\phi) = \frac{1}{\phi^{1+\mu}} \theta_{\phi>0}$ with $-\infty < \mu < -1$. This model was studied in [25].

(iii) The "Frechet" class, the focus of the present paper, where the disorder presents an algebraic left tail, accounting for rare but large events. It corresponds to the choice $f(\phi) = \frac{1}{|\phi|^{1+\mu}} \theta_{\phi<0}$. As discussed above, this choice represents the continuous limit of the system defined in Sec. II.

Note that in all three classes the exponents are given by (10), the Gumbel class corresponding to $\mu = +\infty$ (with additional logarithmic corrections in that case).

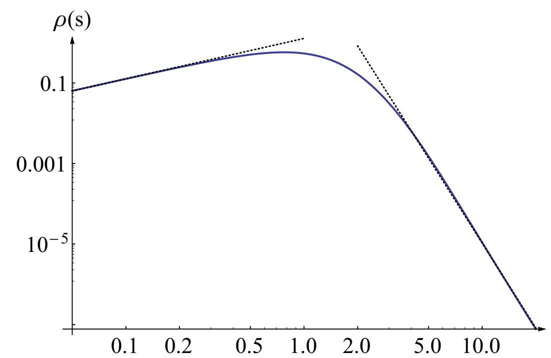

FIG. 5. (Color online) PDF $\rho(s)$ of the shock size, plotted from Eq. (37) for $\mu = 3/2$. In black dotted lines are the asymptotics for small and large s as given by Eqs. (38) and (39).

We now study in more detail the distribution of shock sizes in the Frechet class, and compare to the classical Kida statistics.

2. *Shock size distribution in the Frechet universality class*

Let us consider the Poisson process model with the choice

$$f(\phi) = \frac{\mu}{(-\phi)^{1+\mu}} \theta_{\phi<0},$$

$$F(\phi) = \frac{1}{(-\phi)^{\mu}} \theta_{\phi<0} + \infty \times \theta_{\phi>0}.$$

With this choice one sees that Eq. (24) for $p(u)$ for the Poisson model becomes identical (identifying $y = -\phi$) to Eq. (17) for the discrete model with the same constant F_μ given by (8). Note that the exponential factor in Eq. (24) vanishes if $\phi > 0$; hence the ϕ integration is in effect restricted to $\phi < 0$.

We now consider the shock size distribution from Eq. (31):

$$\rho(s) = \mu^2 s \int_0^\infty dz \int_{-\infty}^0 d\phi\, \exp(-F_\mu |\phi|^{\frac{1}{2}-\mu}) \times \left[\left(\frac{(z+s)^2}{8} - \phi \right) \left(\frac{(z-s)^2}{8} - \phi \right) \right]^{-(1+\mu)} \quad (37)$$

and we assume here $\mu > 1/2$. This distribution is plotted Fig. 5.

This function does not exhibit any divergence for small shock sizes, rather it behaves similarly to the Kida distribution at small s with

$$\rho(s) \simeq C_\mu s \quad (38)$$

and the constant C_μ is displayed in Appendix F. The main difference arises in the behavior of the large shocks. Instead of the exponential tail $e^{-s^2/2}$ in the Kida case, it shows algebraic tails of the form (see Fig. 5)

$$\rho(s) \simeq \frac{2^{2+\mu} \mu}{s^{\tau'}} \text{ for large } s, \quad (39)$$

with the decay exponent τ' for the right tail:[4]

$$\tau' = 1 + 2\mu. \tag{40}$$

To obtain this result from Eq. (37) one notes that it is the region for z near s which contributes most; hence one shifts $z \to z + s$ in Eq. (37) and replaces $\frac{1}{8}(z+2s)^2 - \phi \to s^2/2$ in the first factor. The remaining integral can be extended from $z \in [-\infty,\infty]$ and can then be performed exactly, being related to the normalization of the distribution $p(u)$ of a single minimum (17): one uses $\int dz\, \psi_\mu(z^2/8) = 2/\mu$. Note that since we assumed $\mu > 1/2$ it implies that $\tau' > 2$; hence the integral (33) exists, as required. However, the second moment of the shock size, $\int_0^{+\infty} ds\, s^2 \rho(s)$, is finite only for $\mu > 1$.[5]

Finally, it is useful to recall for comparison the avalanche size distribution for the nonequilibrium version of this model, i.e., the quasistatic depinning. There the jumps occur between the metastable states actually encountered in the driven dynamics as r increases, which are different from the absolute energy minima. The result of [27] for the Frechet class for the normalized distribution is

$$p(s) = \frac{(\alpha+1)(\alpha+2)}{\Gamma\left(2+\frac{1}{\alpha}\right)} \int_0^{+\infty} \frac{dy}{(y+s)^{3+\alpha}} e^{-y^{-\alpha}}, \tag{41}$$

where the local disorder *force* is short-range distributed with a heavy-tail index $\mu = 1 + \alpha$. The large s behavior is also a power law $p(s) \sim s^{-(2+\alpha)} \sim s^{-(1+\mu)}$.

IV. MODEL IN DIMENSION $D > 1$

The methods of solution presented in the previous sections can be extended to the toy model of the particle (i.e., $d = 0$) in general (external) space dimension $\mathbf{u} \in R^D$. The position of the minimum when the quadratic well is centered in $\mathbf{r} \in R^D$ is now denoted as $\mathbf{u}(\mathbf{r})$, a vector process which exhibits jumps; in fact it is constant on cells in R^D, separated by shock walls with discontinuities where it jumps by $\mathbf{s}$. To generalize most of the calculations one must simply replace the integrals over the spatial variable u by integrals over vectors $\mathbf{u}$. The new scaling exponents necessary to retain invariance of the tail of the potential are

$$\zeta = \frac{2\mu}{2\mu - D}, \tag{42}$$

$$\theta = \frac{2D}{2\mu - D}, \tag{43}$$

which reduce to Eq. (10) for $D = 1$ and still satisfy the relation (6). Let us first discuss one-point probabilities, hence setting $\mathbf{r} = 0$.

[4]We use the notation τ' to distinguish from the exponent for the divergence of *small* shocks usually called τ.

[5]In the functional RG this quantity equals $-\Delta'(0^+)/m^4$, while the second moment of $p(u)$ in Eq. (17) is $m^2\overline{u^2} = \Delta(0)$ (which exists only for $\mu > 3/2$), where $\Delta(u)$ is the correlator of the renormalized disorder (see [9,12] for definitions).

A. One-point distribution

Due to the rotational invariance of the elastic energy, one readily obtains the joint distribution:

$$p(\mathbf{u}, V) = \frac{\mu}{|V|^{1+\mu}} e^{-F_{\mu,D}|H|^{\frac{D}{2}-\mu}} \theta_{H<0}, \tag{44}$$

where $H = V + \frac{u^2}{2}$. It is normalized to unity $\int d^D\mathbf{u}\, dV\, p(\mathbf{u}, V) = 1$ and we have defined

$$F_{\mu,D} = S_D 2^{D/2-1} \frac{\Gamma[D/2]\Gamma[\mu - D/2]}{\Gamma[\mu]}, \tag{45}$$

where S_D is the surface of the unit sphere in dimension D ($S^1 = 2$). From this we extract the joint distribution of V and H as

$$p(V,H) = S_D 2^{\frac{D}{2}-1} (H - V)^{\frac{D}{2}-1} \frac{\mu}{|V|^{1+\mu}} \tag{46}$$

$$\times \exp\left(-F_{\mu,D}|H|^{\frac{D}{2}-\mu}\right)\theta_{H<0,V<H}, \tag{47}$$

which exhibit a "level repulsion" between H and V for $D > 2$.

The marginal distribution for H is again a Frechet with index now $\mu - \frac{D}{2}$:

$$p(H) = \frac{\left(\mu - \frac{D}{2}\right) F_{\mu,D}}{|H|^{\mu - \frac{D}{2}+1}} e^{-F_{\mu,D}|H|^{\frac{D}{2}-\mu}} \theta_{H<0}, \tag{48}$$

while the PDF of V takes the form

$$p(V) = \frac{\mu S_D}{2^{1-D/2}|V|^{\mu+1}} \phi_\mu^D(|V|)\theta_{V<0}, \tag{49}$$

where we have defined

$$\phi_\mu^D(x) = \int_0^x \frac{dy}{(x-y)^{1-D/2}} e^{-F_{\mu,D} y^{\frac{D}{2}-\mu}}. \tag{50}$$

Finally, the distribution of the optimal position is

$$p(\mathbf{u}) = \mu\, \psi_\mu^D\left(\frac{u^2}{2}\right), \tag{51}$$

where

$$\psi_\mu^D(x) = \int_0^\infty \frac{e^{-F_{\mu,D} y^{\frac{D}{2}-\mu}}}{(x+y)^{\mu+1}} \tag{52}$$

and, interestingly, the tail exponent of $P(\mathbf{u})$ is independent of D:

$$p(\mathbf{u}) \simeq \frac{2^\mu}{u^{2\mu}}, \quad |u| \to +\infty, \tag{53}$$

while the PDF for the radius $|u|$ decays as $\simeq 2^\mu S_D/|u|^{2\mu+1-D}$.

Note that the condition for the thermodynamic limit to be defined is now $\mu > \frac{D}{2}$, as the typical minimum site energy at a distance u of the center grows as $u^{D/\mu}$.

B. Droplet and shock densities

Note that the formula for the droplet density also generalizes easily in D dimension as

$$D(\mathbf{s}) = \int d\phi\, d^D\mathbf{u}_1 d^D\mathbf{u}_2\, f\left(\phi - \frac{u_1^2}{2}\right) f\left(\phi - \frac{u_2^2}{2}\right)$$

$$\times \exp\left[-\int F\left(\phi - \frac{z^2}{2}\right) d\mathbf{z}\right] \delta^D(\mathbf{s} - \mathbf{u}_2 + \mathbf{u}_1), \tag{54}$$

where $\vec{s}$ is the vector joining the two degenerate minima. It is now normalized as

$$\int d^D\mathbf{s}\, s^2 D(\mathbf{s}) = 2D \tag{55}$$

as shown in Appendix D. The shock density is now defined by reference to a direction of unit vector $\mathbf{e}_x$ as

$$\rho(\mathbf{s}) = \lim_{\delta r\to 0^+} \frac{1}{\delta r}\overline{\delta^D[\mathbf{u}(\mathbf{r}+\delta r\mathbf{e_x}) - \mathbf{u}(\mathbf{r}) - \mathbf{s}]}. \tag{56}$$

Since Eq. (30) generalizes to $\delta H = \delta r \mathbf{e_x}\cdot(\mathbf{u}_1 - \mathbf{u}_2)$, one sees that the relation between the shock and droplet densities is now

$$\rho(\mathbf{s}) = s_x D(\mathbf{s})\theta_{s_x>0}, \tag{57}$$

where $s_x = \mathbf{s}\cdot\mathbf{e}_x$ denotes the component of the jump along the direction x.

Using isotropy it now enjoys the normalization

$$\int_{s_x>0} d^D\mathbf{s}\, s_x \rho(\mathbf{s}) = 1, \tag{58}$$

which, again, expresses that all motion when $\mathbf{r}$ varies along a line occurs in shocks. Note that the relation Eq. (57), combined with the isotropy of $D(\mathbf{s})$ implies a number of relations[6] between moments, for instance,

$$\langle s_x^2\rangle = 2\langle s_y^2\rangle, \tag{59}$$

as well as $\langle s_x^4\rangle = \frac{8}{3}\langle s_y^4\rangle = 4\langle s_x^2 s_y^2\rangle$ and so on *provided these moments exist*, i.e., that the tail of $D(\mathbf{s})$ decays fast enough.[7]

It is interesting to note that Eqs. (54) and (57) factorize in the Kida (i.e., Gumbel) universality class [i.e., with the choice $f(\phi) = e^{\phi}$] leading to the simple result, after some Gaussian integrations:

$$\rho(\mathbf{s}) = \frac{s_x}{(4\pi)^{\frac{D}{2}}} e^{-s_x^2/4} e^{-\mathbf{s}_\perp^2/4}, \tag{60}$$

where we denote $\mathbf{s} = (s_x, \mathbf{s}_\perp)$ and $s_\perp$ represents the "wandering" part of the shock motion, transverse to the shift direction of the parabola. For instance, in two dimension $\mathbf{s} = (s_x, s_y)$, Eq. (60) reads $\rho(\mathbf{s}) = \rho_{D=1}(s_x) D_{D=1}(s_y)$. Hence, in the Kida case, higher dimension statistics of the shocks are completely solved from the $D = 1$ case.

The Frechet case, however, does not simplify as nicely. One now obtains

$$\rho(s) = \mu^2 \frac{s_x}{2^D}\int_0^\infty d^D\mathbf{z}\int_{-\infty}^0 d\phi\, \exp\left(-F_{\mu,D}|\phi|^{\frac{D}{2}-\mu}\right) \times\left[\left(\frac{(\mathbf{z}+\mathbf{s})^2}{8} - \phi\right)\left(\frac{(\mathbf{z}-\mathbf{s})^2}{8} - \phi\right)\right]^{-(1+\mu)} \tag{61}$$

and we assume here $\mu > D/2$. The tail for large $s = |\mathbf{s}|$ is obtained, by manipulations similar to the case $D = 1$ as

$$\rho(s) \simeq \frac{2^{2+\mu}\mu s_x}{s^{2+2\mu}} \text{ for large } s. \tag{62}$$

Interestingly, going to higher dimensions allows the fluctuations of the particle motion to spread even more. To illustrate that fact one can compute the marginal shock density along $\mathbf{e}_x$ defined as

$$\rho(s_x) = \int_{s_\perp} \rho(\mathbf{s}) = s_x\theta_{s_x>0}\int_{s_\perp} D(\mathbf{s}). \tag{63}$$

After some integrations from Eq. (61) one finds

$$\rho(s_x) = \mu^2 F_{\mu+1,D-1}^2 s_x \int_0^\infty dz \int_{-\infty}^0 d\phi\, e^{-F_{\mu,D}|\phi|^{\frac{D}{2}-\mu}} \times\left[\left(\frac{(z+s_x)^2}{8} - \phi\right)\left(\frac{(z-s_x)^2}{8} - \phi\right)\right]^{-(\frac{3-D}{2}+\mu)}. \tag{64}$$

Hence a formula very similar to Eq. (37), but with a modified exponent $\tilde{\mu} = \mu - (D-1)/2$, leading to an asymptotic algebraic decay of the shock size along x with exponent $\tau' = 2 - D + 2\mu$. The thermodynamic condition $\mu > D/2$ again ensures that the normalization integral (58) exists.

V. ELASTIC MANIFOLDS: RECALLING THE GENERAL FLORY ARGUMENT

We now check that the obtained values for the exponents agree with the general argument. For this we now recall the Flory argument given in [6] for the directed polymer, which we straightforwardly generalize to a manifold of internal dimension d (internal coordinate $x \in R^d$) with D displacement components $u \in R^D$. We consider that the random potential $V(x,u)$ lives in a total embedding space dimension $d + D$ and has short-range correlations with a heavy-tailed PDF (1) indexed by μ. Assume that a piece of size L (in x) explores typically $W \sim L^\zeta$ in dimension D. The volume explored by the manifold is $L^d W^D$; hence the minimal value of V on this volume behaves as $\sim(L^d W^D)^{1/\mu}$. This leads to $\mu\theta = d + D\zeta$. Imposing again that elasticity and disorder scale the same way (this is guaranteed by the general STS symmetry, i.e., statistical invariance under tilt) leads to $\theta = 2\zeta + d - 2$. Hence we obtain

$$\zeta = \frac{d + \mu(2-d)}{2\mu - D}, \tag{65}$$

$$\theta = \frac{2d + D(2-d)}{2\mu - D}, \tag{66}$$

with the (naive) threshold value beyond which one (presumably) recovers Gaussian disorder universality class:

$$\mu_c = \frac{d + D\zeta_{SR}}{d - 2 + 2\zeta_{SR}}, \tag{67}$$

where ζ_{SR} is the roughness exponent for short-range Gaussian disorder. For $d = 0$ one recovers the above values Eq. (42) and Eq. (43) for the toy model in general dimension D. For $d = 1$, $\zeta_{SR} = 2/3$ and $\theta_{SR} = 1/3$, which gives the value $\mu_c = 5$ given in [6] and recalled in the Introduction. It is interesting

---

[6] These are easily shown, e.g., by integrating with respect to $D(\mathbf{s}) \to e^{-\mu s^2}$, since any isotropic distribution can be represented as a superposition of such weights.

[7] The relation (59) is believed to be more general (i.e., to extend to interfaces) and was anticipated in [32], where it was related via the functional RG to the existence of a cusp in the effective action of the theory (see also [33]).

to note that at the upper-critical dimension $d_{uc} = 4$, $\zeta_{SR} = 0$; hence the critical value is $\mu_c = 2$.

VI. CONCLUSION

In the present paper we have studied the toy model for the interface, i.e., a point in a random potential, in presence of heavy-tailed disorder with exponent μ. In the scaling regime it leads to a universality class analogous to the Frechet class for extreme value statistics. It was found that all the relevant distributions (minimum energy, position, and sizes of shocks) exhibit also power-law tails with modified exponents continuously dependent on μ. Hence the presence of heavy tails in the underlying disorder pervades through all observables and modifies the behavior for every value of μ. That has to be compared with the directed polymer problem, where the effect of heavy tails disappears in favor of a "Gaussian" behavior for $\mu > 5$.

In addition, we have obtained here the shock size distribution for an "exotic" example of decaying Burgers turbulence, close from the Kida class because of the short-range correlations in the initial potential, but markedly different because of the heavy tails.

Finally, because of these heavy tails the functional RG method which, in its present form, is based [12,27] on the existence of the moments of the position of the minimum $u(r)$ cannot be applied in a standard way (at least in $d = 0$). We hope our study will inspire progress on the more general problem of the elastic manifold in the heavy-tailed disorder.

ACKNOWLEDGMENT

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APPENDIX A: EXOTIC REGIME IN DECAYING BURGERS TURBULENCE

The above particle model is directly related to the Burgers equation for a velocity field $\mathsf{v}(r,t)$, a simplified version of Navier-Stokes used to model compressible fluids:

$$\partial_t \mathsf{v} = \nu \partial_r^2 \mathsf{v} - \tfrac{1}{2}\partial_r \mathsf{v}^2. \tag{A1}$$

This equation can be integrated using the Cole-Hopf transformation. Here we study only the inviscid limit (of zero viscosity $\nu = 0^+$). In that case the solution is given by

$$\mathsf{v}(r,t) = \partial_r H(r) = \frac{r - u(r)}{t}. \tag{A2}$$

In terms of (3) one defines the "time" t as

$$t = m^{-2} \tag{A3}$$

and the initial condition

$$\mathsf{v}(r,t=0) = \partial_r H(r)|_{t=0} = \partial_r V(r), \tag{A4}$$

where $V(u)$ is the bare disorder of the toy model. In this paper we focused on the case when $V(u)$ is short-range correlated with a heavy tail. This corresponds to a well defined but peculiar type of distribution for the initial velocity field: it also has a tail exponent μ, but exhibits local anticorrelations so that $V(u)$ remains short-range correlated [if $\mathsf{v}(r,t=0)$ was short-range correlated with a heavy-tail distribution, that would correspond to $V(u)$ following a Levy process, either a Brownian motion for $\mu > 2$ or a Levy flight for $\mu < 2$].

As is well known evolution from a smooth initial condition presents shocks in finite time, i.e., the velocity field $\mathsf{v}(r,t)$ does not remain continuous but presents (negative) jumps in a discrete set of locations r_α, where $\mathsf{v}(r_\alpha^+,t) - \mathsf{v}(r_\alpha^-,t) = \Delta\mathsf{v} < 0$. These correspond to the (positive) jumps in $u(r)$, more precisely one has $\Delta\mathsf{v} = -S/t$, where S is the dimensionful shock size $S = u_m s = m^{-\zeta} s$ with the dimensionless size s studied in the present paper. To translate our results in terms of velocity jumps in Burgers, one thus just identifies $\Delta\mathsf{v} = -t^{\frac{\zeta}{2}-1} s$ (indeed the length scale is $m^{-\zeta} = t^{\zeta/2}$), where ζ is given by Eq. (42).

Finally, the time dependence of the mean energy density E is given by $E = \frac{1}{2}\mathsf{v}^2 \sim t^{-(2-\zeta)} = t^{-2(\mu-D)/(2\mu-D)}$, which recovers the result of [26]. Note that the regime $D/2 < \mu < D$ is very peculiar since it predicts an energy density growing instead of decaying, as discussed there.

APPENDIX B: FROM INFINITE PRODUCT TO INTEGRAL

To understand better the convergence to the continuum limit let us first choose a Pareto distribution, i.e., with a hard cutoff,

$$P_>(V) = \left(1 - \frac{1}{(-V)^\mu}\right)\theta_{V<V_0}, \tag{B1}$$

and consider again the infinite product Eq. (5). It can be rewritten, in the rescaled units, i.e., $u \to m^{-\zeta}u$, $V \to m^{-\theta}V$ as (taking into account the Jacobian involved in the rescaling)

$$p(u,V) = m^{-(\zeta+\theta)} \frac{\mu}{(m^{-\theta}|V|)^{1+\mu}} \theta_{V<V_0 m^\theta} \times \prod_{u' \neq u} \theta\left(H - \frac{u'^2}{2} < V_0 m^\theta\right) e^{\ln[1 - m^{\mu\theta}(-H + \frac{u'^2}{2})^{-\mu}]}. \tag{B2}$$

We see here that for $m \to 0$ it vanishes unless $H - \frac{u'^2}{2} < 0$ for all $u' \neq u$, but since in that limit the lattice grid tends to continuum, this condition becomes equivalent to $H < 0$. Since $V < H$ we do not need to retain the constraint $V < 0$. The infinite product becomes an integral, and the logarithm can be expanded, leading to

$$p(u,V) = \frac{\mu}{|V|^{1+\mu}} \theta_{H<0} e^{-\int du' (-H + \frac{u'^2}{2})^{-\mu}},$$

which leads to the result given in the text.

The mechanism holds for more general distributions with the same tail. As discussed in the text the rescaled $P_>(m^{-\theta}y)$ converges to unity for $y < 0$ and to zero for $y > 0$ so the precise shape of the distribution does not matter. More precisely, the weight of the events with $H > 0$ vanishes. To illustrate the point consider the worst case, i.e., when $P_>(V)$ is slowly decaying on the positive V side, e.g., as $V^{-\alpha}$. Then, for $H > 0$ (and $m \to 0$), there is an additional factor:

$$\approx \prod_{u' \neq u} \frac{\theta_{H - \frac{u'^2}{2} > 0}}{m^{-\alpha\theta}\left(H - \frac{u'^2}{2}\right)^\alpha} \simeq m^{\alpha\theta} e^{-\int_{-\sqrt{2H}}^{\sqrt{2H}} du' \ln(H - \frac{u'^2}{2})} = O(m^{\alpha\theta}),$$

since the integral is convergent, and this factor kills the contribution of the events with $H > 0$ (more precisely all the events with $H > -m^{-\gamma}$ with any $0 < \gamma < \theta$, in the original units).

APPENDIX C: MOMENTS OF u

From Eq. (17) and Eq. (18), we find the moments, for any real $n > 0$ such that $2n < 2\mu - 1$:

$$\overline{u^{2n}} = F_\mu^{\frac{2n}{2\mu-1}} \frac{2^n \Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{\mu-\frac{1}{2}-n}{\mu-\frac{1}{2}}\right)\Gamma\left(\mu+\frac{1}{2}-n\right)}{\sqrt{\pi}\,\Gamma\left(\mu+\frac{1}{2}\right)}.$$

The $2n$th moment thus diverges as $n \to \mu - \frac{1}{2}|^-$ as

$$\overline{u^{2n}} \simeq \frac{2^\mu}{\mu - \frac{1}{2} - n}. \tag{C1}$$

APPENDIX D: NORMALIZATION OF THE SHOCK DENSITY

A consistency check for the shock density is to check the normalization given in Eq. (33), i.e., $\int ds\, s\, \rho(s) = 1$. We recall that

$$I = \int_{s>0} ds\, s\, \rho(s) = \frac{1}{2}\int_s s^2 D(s) = \frac{1}{2}\int du_1 du_2 d\phi (u_1-u_2)^2 \times f\left(\phi - \frac{u_1^2}{2}\right) \times f\left(\phi - \frac{u_2^2}{2}\right) e^{-\int dz' F(\phi - \frac{(z')^2}{2})}. \tag{D1}$$

Due to the symmetry in the variables (u_1,u_2), one can only consider, for example,

$$\begin{aligned} I_{u_1} &= \int du_1 du_2 d\phi u_1^2 f\left(\phi - \frac{u_1^2}{2}\right) \times f\left(\phi - \frac{u_2^2}{2}\right) e^{-\int dz' F(\phi - \frac{(z')^2}{2})} \\ &= -\int du_1 d\phi\, u_1^2 f\left(\phi - \frac{u_1^2}{2}\right) \partial_\phi e^{-\int dz' F(\phi - \frac{(z')^2}{2})} \\ &= \int du_1 d\phi\, u_1^2 \partial_\phi f\left(\phi - \frac{u_1^2}{2}\right) e^{-\int dz' F(\phi - \frac{(z')^2}{2})}, \end{aligned} \tag{D2}$$

where we used the fact that, because of the limits $f(\phi) \to 0$ at $\phi \to -\infty$ and $F(\phi) \to \infty$ at $+\infty$, the boundary terms vanish. Considering the argument $\phi - u_1^2/2$ in $f(\cdot)$, one has the equivalence of the operators $\partial_\phi \leftrightarrow -u_1^{-1}\partial_{u_1}$ acting on $f(\cdot)$. Switching to ∂_{u_1} derivatives in Eq. (D2), and integrating by parts once again,

$$\begin{aligned} I_{u_1} &= -\int du_1 d\phi\, u_1 \partial_{u_1} f\left(\phi - \frac{u_1^2}{2}\right) e^{-\int dz' F(\phi - \frac{(z')^2}{2})} \\ &= \int du_1 d\phi\, f\left(\phi - \frac{u_1^2}{2}\right) e^{-\int dz' F(\phi - \frac{(z')^2}{2})} = 1, \end{aligned}$$

where again the boundary terms vanish due to $f(\phi - u^2/2) \to 0$ for $u \to \pm\infty$. Hence $I = \frac{1}{2}(I_{u_1} + I_{u_2}) = 1$ and the normalization is properly recovered. The deeper reason behind these identities arises from the STS symmetry, i.e., the fact that the disorder is statistically translationally invariant (see, e.g. [12,30]).

Note that all the steps of this calculation easily generalize to higher D, the only change being that now $u_1^2\partial_\phi \equiv -\mathbf{u}_1 \cdot \nabla_{\mathbf{u}_1}$ acting on $f(\phi - u_1^2/2)$. The final result is then $I = D$ as discussed in the text.

APPENDIX E: TWO-POINTS FUNCTION

Let us consider the joint probability that (V_1,u_1) and (V_2,u_2) realize the minimum total energy respectively when the quadratic well is centered in r_1 and when it is centered in r_2, in the same realization of the disorder. The minimal energies are denoted by

$$H_j = V_j + \frac{(u_j - r_j)^2}{2}, \quad j = 1,2. \tag{E1}$$

This probability reads

$$\begin{aligned} &p(V_1,u_1,V_2,u_2) dV_1 du_1 dV_2 du_2 \\ &\quad = f(V_1) f(V_2) dV_1 du_1 dV_2 du_2 \\ &\quad \times \prod_{\substack{dV_j',du_j' \\ u_1' < u^* \\ u_2' > u^*}} \left(1 - \theta_{V_1' + \frac{(u_1'-r_1)^2}{2} < V_1 + \frac{(u_1-r_1)^2}{2}} f(V_1') dV_1' du_1'\right) \\ &\quad \times \left(1 - \theta_{V_2' + \frac{(u_2'-r_2)^2}{2} < V_2 + \frac{(u_2-r_2)^2}{2}} f(V_2') dV_2' du_2'\right), \end{aligned} \tag{E2}$$

where u^* is the intersection abscissa of the two parabola, as represented in Fig. 3 given by

$$H_1 - \frac{(u^* - r_1)^2}{2} = H_2 - \frac{(u^* - r_2)^2}{2}, \tag{E3}$$

whose common value is denoted ϕ below. The additional Heaviside functions ensure that the random potential lies above these two parabola and touches those parabola on the two points u_1 and u_2.

The characteristic function can then be written

$$\begin{aligned} \langle e^{\lambda[u(r_2)-u(r_1)]}\rangle &= \int dV_1 dV_2 du_1 du_2 e^{\lambda(u_2-u_1)} \left[f(V_1)\delta_{V_2=V_1,u_2=u_1} \right. \\ &\quad \left. + f(V_1) f(V_2) \theta_{u_1<u^*<u_2}\right] \\ &\quad \times e^{-\int_{u<u^*} F(H_1 - \frac{(u-r_1)^2}{2}) - \int_{u>u^*} F(H_2 - \frac{(u-r_2)^2}{2})}, \end{aligned} \tag{E4}$$

where the first term accounts for the contribution when there is no shock between r_1 and r_2 and the second when there is at least one. Let us now perform the change of variables:

$$\begin{aligned} x &= \frac{r_2 - r_1}{2} \quad \text{and} \quad y = u^* - \frac{r_1 + r_2}{2}, \\ z &= u - r_1 \quad \text{and} \quad z' = r_2 - u, \\ z_1 &= u_1 - r_1 \quad \text{and} \quad z_2 = r_2 - u_2, \\ \phi &= H_1 - \frac{(x+y)^2}{2} = H_2 - \frac{(x-y)^2}{2}. \end{aligned} \tag{E5}$$

Hence $x + y = u^* - r_1$ and $x - y = r_2 - u^*$. In terms of the auxiliary functions,

$$\begin{aligned} J_+(\phi,y,x) &= \int_{z_1 \leqslant x+y} dz_1 f\left(\phi + \frac{(x+y)^2 - z_1^2}{2}\right) e^{-\lambda z_1}, \\ J_-(\phi,y,x) &= \int_{z_2 \leqslant x-y} dz_2 f\left(\phi + \frac{(x-y)^2 - z_2^2}{2}\right) e^{-\lambda z_2}, \\ I_+(\phi,y,x) &= \int_{z \leqslant x+y} dz\, F\left(\phi + \frac{(x+y)^2 - z^2}{2}\right), \\ I_-(\phi,y,x) &= \int_{z' \leqslant x-y} dz' F\left(\phi + \frac{(x-y)^2 - z'^2}{2}\right), \end{aligned}$$

the characteristic function of the difference $u(r_2) - u(r_1)$ takes the form

$$\langle e^{\lambda[u(x)-u(-x)]}\rangle = \int d\phi\, dy[f(\phi) + 2x\, e^{2\lambda x} J_+(\phi,y,x)J_-(\phi,y,x)] \times \exp[-I_+(\phi,y,x) - I_-(\phi,y,x)], \quad \text{(E6)}$$

where the $2x = r_2 - r_1$ factor comes from the Jacobian $dV_1 dV_2 du_1 du_2 = 2x\, d\phi\, du^* dz_1 dz_2$.

This formula generalizes to arbitrary $f(\phi)$ the one given in [25] for a particular function $f(\phi)$. There it is given in terms of the (scaled) Burgers velocity field $\mathsf{v}(r) = r - u(r)$. One easily checks the normalization, i.e., that for $\lambda = 0$ Eq. (E6) is a total derivative and integrates to unity.

It is now rather straightforward to expand this formula to $O(x)$ and to recover the expression for the shock density $\rho(s)$ given in the text using the identification (34).

APPENDIX F: ASYMPTOTICS OF THE SHOCK DENSITY

The constant C_μ in the text can be obtained as

$$C_\mu = \frac{\mu(2\mu-1)(2\pi)^{\frac{\mu+1}{1-2\mu}}}{3(4\mu+1)} \times \frac{\left(\frac{\Gamma(\mu-\frac{1}{2})}{\Gamma(\mu)}\right)^{\frac{4\mu+1}{1-2\mu}} \Gamma\left(2\mu+\frac{3}{2}\right)\Gamma\left(4+\frac{3}{2\mu-1}\right)}{\Gamma(2\mu+2)}, \quad \text{(F1)}$$

where C_μ is an increasing function which vanishes at $\mu = 1/2^+$ with an essential singularity $C_\mu \simeq \exp(-\frac{3}{4}\frac{2-\ln(9/8)}{\mu-\frac{1}{2}})$ and grows as $C_\mu \simeq \frac{\mu^{3/2}}{2\sqrt{\pi}}$ at large μ.

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