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# Nonlinear dynamics of trapped waves on jet currents and rogue waves

V. I. Shrira[1] and A. V. Slunyaev[2,3]

[1]*Department of Mathematics, EPSAM, Keele University, Keele ST5 5BG, United Kingdom*
[2]*Institute of Applied Physics, 46 Ulyanova Street, N. Novgorod 603950, Russia*
[3]*Nizhny Novgorod State Technical University, 24 Minina Street, N. Novgorod 603950, Russia*

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Nonlinear dynamics of surface gravity waves trapped by an opposing jet current is studied analytically and numerically. For wave fields narrow band in frequency but not necessarily with narrow angular distributions the developed asymptotic weakly nonlinear theory based on the modal approach of Shrira and Slunyaev [J. Fluid. Mech. **738**, 65 (2014)] leads to the one-dimensional modified nonlinear Schrödinger equation of self-focusing type for a single mode. Its solutions such as envelope solitons and breathers are considered to be prototypes of rogue waves; these solutions, in contrast to waves in the absence of currents, are robust with respect to transverse perturbations, which suggests a potentially higher probability of rogue waves. Robustness of the long-lived analytical solutions describing modulated trapped waves and solitary wave groups is verified by direct numerical simulations of potential Euler equations.

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## I. INTRODUCTION

In the last two decades there was a surge of interest in the phenomenon of extreme or rogue waves in various areas of physics, e.g., [1–3]. In the most studied area of water waves in the ocean the main thrust of the studies was the search for mechanisms of rogue waves in the absence of currents [3]. The modulational (or Benjamin-Feir, BF) instability of narrow-band wave fields has been identified as a likely mechanism leading to formation of anomalously high waves and a significant increase of their probability. This route has been most intensively studied and is most effective for one-dimensional (1D) wave propagation; the theoretical and experimental modeling is the simplest. However, in reality the 1D patterns are transversally unstable and hence, short lived; there is a dramatic difference in the probability of rogue events due to the BF instability for the strictly 1D and two-dimensional (2D) wave propagation: The likelihood of rogue events is much higher for the 1D propagation, moreover it completely vanishes for wave fields with the angular spectra width exceeding a certain threshold [3,4]. For most of the wave fields in the ocean the angular spectra are not narrow [5], and hence the 1D theory is not valid even qualitatively. On the other hand, it is known that the rogue waves are much more frequent on currents; the Agulhas current gained notoriety in this respect [3,6].

To explain an increased probability of rogue waves on currents the prevailing approach exploits separation of scales between the typical wavelength and current, which leads to the WKB or ray description with a special consideration of caustics (see the literature reviews in [3,7,8]).

With the focus on nonlinear dynamics, various versions of nonlinear Schrödinger equations (NLSE) were derived and analyzed under general assumptions of slow current, weak nonlinearity, and narrow-banded spectrum (see [7], and references therein). In these works the BF instability was found to be strengthened for waves on adverse intensifying currents. The triggering of the BF instability of a narrow-band field due to intensification of the current was considered in [9]. However there is an essential feature not captured by the existing NLS-type models: Waves propagating upstream can be trapped by the current; there are multiple caustics. Such trapped waves have been observed on the Gulf Stream and were found to have considerably higher steepness than free waves on current [10]. A more general and profound difficulty is that there is no technique enabling one to describe wave resonant interactions on currents; the waves refract on currents and hence vary in space, while the resonant interactions and, in particular, the resonance conditions have to be described in the wave-vector space. A new approach suggested in [8] allows one to overcome these obstacles. Instead of operating with rays we deal with the modes propagating on jet currents for which the standard nonlinear theory applies.

Here, based on the modal approach and weakly nonlinear asymptotic expansions, we derive equations governing one-dimensional wave evolution along the current; the transverse structure of the field is being provided by the modes. For a one mode we derive 1D NLSE without the constraint of a too narrow angular spectra. In contrast to unguided waves, here the NLSE solitary wave type solutions are robust. The robustness of such wave patterns suggests a dramatic increase in the probability of rogue waves. The predictions of the asymptotic model are validated by direct numerical simulations of the Euler equations.

## II. MODAL REPRESENTATION FOR WAVES ON JET CURRENTS

We consider wave motions on the free surface of an ideal incompressible fluid of unit density; waves are propagating along the $Ox$ direction on a given vertically uniform unidirectional steady current $U = \{U(y),0,0\}$.

The motions are governed by the standard Euler and continuity equations in the domain occupied by the fluid $z \leqslant \eta$, where $\eta(x,y,t)$ is the water surface elevation; the water depth is assumed infinite for convenience. These equations are complemented with the standard boundary conditions for gravity waves: dynamic and kinematic boundary conditions on the surface and decay of velocities as $z \to -\infty$. We focus on the evolution of waves trapped by the current; trapped modes are selected by the boundary conditions stipulating lateral decay of velocities at $y \to \pm\infty$.

In the linear setting the problem formulated above was thoroughly examined in [8]. Making use of uniformity of the problem with respect to $x$ and $t$, the Fourier transform may be applied with respect to these variables; then the complete linear solution has the form of a superposition of traveling waves propagating collinear to the current with some structure in the $(y,z)$ plane,

$$w = \mathrm{Re}\sum_n \hat{w}_n(y,z)\exp(i\omega_n t - ikx), \tag{1}$$

where $w(x,y,z,t)$ is the vertical component of fluid velocity and index $n$ numerates the lateral modes. The modes $\hat{w}_n(y,z)$ are specified by the two-dimensional boundary value problem (BVP)

$$\frac{\partial^2 \hat{w}_n}{\partial z^2} + \frac{\partial^2 \hat{w}_n}{\partial y^2} + \left(\frac{\Omega_n''}{\Omega_n} - 2\frac{\Omega_n'^2}{\Omega_n^2} - k^2\right)\hat{w}_n = 0, \tag{2}$$
$$\hat{w}_n|_{y\to\pm\infty} \to 0, \qquad \hat{w}_n|_{z\to-\infty} \to 0,$$

with $\Omega_n(y) = \omega_n - kU$. Each mode is characterized by its cyclic frequency, $\omega_n$, and longitudinal wave number, $k$; we choose $k > 0$ with no loss of generality.

The 2D BVP (2) may be either solved numerically or reduced to a one-dimensional BVP using an asymptotic separation of variables [8]. Here we adopt the second route, which assumes the following representation:

$$\hat{w}_n = B_n Y_n(y) Z_n(z,y). \tag{3}$$

Thus, the mode is specified by two real functions: $Z_n(z,y)$ determines the vertical structure which depends on $y$ slowly, $Y_n \partial Z_n/\partial y \ll dY_n/dy Z_n$, and is equal to one on the surface; $Y_n(y)$ determines the mode transverse dependence. Constants $B_n$ may be complex.

Asymptotic 1D reductions of BVP (2) were derived in [8] for two regimes: of "weak" currents (compared to the wave phase velocity), and of "broad" currents (compared to the longitudinal wave length). For the dominant wind waves and swell we are primarily interested in, all oceanic currents are weak in this sense. The corresponding 1D BVP is of the Sturm-Liouville type,

$$\frac{d^2 Y_n}{dy^2} + 4k^2\left[\frac{\omega_n}{\omega_g} - \left(1 + \frac{kU}{\omega_g}\right)\right]Y_n = 0, \tag{4}$$
$$Y_n|_{y\to\pm\infty} \to 0.$$

Here $\omega_g = \sqrt{kg}$ denotes the frequency of linear gravity waves unaffected by the current. The wave frequency $\omega_n$ is the to-be-defined eigenvalue of the problem. When the eigenmodes of (4) are complemented by the nondecaying solutions of (4) corresponding to passing trough modes, they form a complete basis and thus provide an efficient tool for studying waves on jet currents. For single-humped currents trapped modes require $kU < 0$ to exist; waves must run against the current.

## III. WEAKLY NONLINEAR MODEL FOR MODULATED WAVES ON JET CURRENTS

Now we concentrate on the regime where nonlinear interactions only between the trapped modes are essential. Employing a standard asymptotic procedure based upon a small parameter characterizing smallness of wave steepness: $\epsilon \sim k \max|\eta|$ or, equivalently, smallness of fluid velocities, $\epsilon \sim k \max|w|/\omega_g$; it is straightforward to derive a variety of evolution equations governing wave field nonlinear dynamics; the resulting equations for constants $B_n$ now being slow functions of time and coordinate $x$ are determined by the choice of initial configurations of the field. In view of our interest in rogue wave formation we focus on narrow-band spectra and consider wave fields with the longitudinal spectrum confined to $\epsilon$ vicinity of the chosen carrier wave number, $k$.

We stress that jet currents profoundly modify the picture of wave resonances as compared to the freely propagating deep-water gravity waves; crucially, three-wave resonances become possible, which results in dynamical equations of the three-wave-interaction type. However, it can be shown that in the limit of weak currents, in which we are primarily interested, the four-wave interactions dominate. A key feature of the four-wave regimes is that to leading order wave dynamics is potential. A detailed analysis of all the regimes will be reported elsewhere. Here we provide and briefly discuss the new version of the NLSE which we obtain for the wave fields belonging to a single, say $n$th, trapped mode with spectra narrow in longitudinal wave numbers $k$; it reads

$$\begin{aligned} &-i\left(\frac{\partial A}{\partial t} + V\frac{\partial A}{\partial x}\right) + \frac{\omega_n}{8k^2}\frac{\partial^2 A}{\partial x^2} + \kappa\frac{\omega_n k^2}{2}A|A|^2 \\ &\quad = 0, \\ &V = \frac{1}{\int_{-\infty}^{\infty} Y_n^2 dy}\int_{-\infty}^{\infty}\left(\frac{kg^2}{2\Omega_n^3} + U\right)Y_n^2 dy, \\ &\kappa = \frac{\int_{-\infty}^{\infty} Y_n^4 dy}{\int_{-\infty}^{\infty} Y_n^2 dy}. \end{aligned} \tag{5}$$

Equation (5) describes evolution of the surface elevation,

$$\eta = \mathrm{Re}[A(x,t)Y_n(y)\exp(i\omega_n t - ikx)]. \tag{6}$$

$A$ is linked to the vertical velocity component as $B = i\omega_n A$; the subscripts for $A$ and $B$ are omitted for brevity. Sets of similar coupled equations appear when more than one mode is initially excited. The detailed derivation of (5) and its coupled generalizations will be reported elsewhere.

Equation (5) differs from the classical NLSE in the still water by the account for the Doppler frequency shift and a reduced nonlinear coefficient due to factor $\kappa$. By virtue of the Cauchy-Schwarz inequality $\kappa < 1$.

The NLSE (5) is of focusing type, hence it supports the BF instability and being integrable it admits a wide class of well studied exact solutions of variable degrees of complexity [11]. The basic solutions are uniform waves, localized solitary wave groups (envelope solitons), and breathers. Being a one-dimensional evolution equation, (5) provides a dramatic simplification of a description of the complicated dynamics of 2D nonlinear trapped wave patterns. Crucially, in contrast to the situations involving unguided waves these 1D patterns are stable with respect to 2D perturbations and hence robust. In particular, the 1D NLSE envelope solitons are asymptotics of the initial problem with generic localized initial data, which is not the case in the absence of a waveguide.

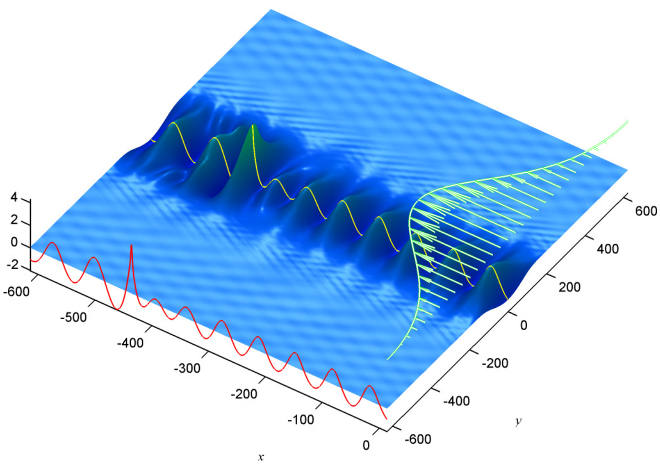


FIG. 1. (Color online) Simulation of propagation of one trapped mode with $k = 0.1$ rad/m and $kH/2 = 0.15$ with an initial 5% longitudinal modulation. Surface elevation at time $t = 829.4$ s; the maximal wave is characterized by $kH/2 \approx 0.32$. The longitudinal cross section through the peak of the maximal wave is shown by lines above the current maximum and above the $x$ axis. The profile of the current is shown above the $y$ axis.

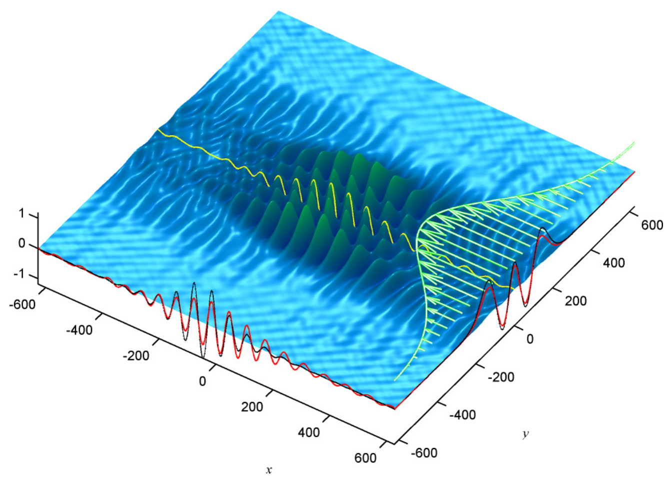


FIG. 2. (Color online) Simulation of a solitary group of trapped waves belonging to the fifth mode (the surface at $t \approx 868$ s is shown; see videoclip [13]). The snapshot longitudinal cross section is shown by red solid lines above the current maximum and above the $x$ axis; the corresponding sections of the initial condition are shown by thin black curves. The snapshot transverse cross section is shown in front of the surface above the $y$ axis.

## IV. NONLINEAR DYNAMICS OF TRAPPED WAVES IN SIMULATIONS OF THE PRIMITIVE EQUATIONS

To verify the obtained asymptotic description a few key solutions of (5) are tested below by means of strongly nonlinear numerical simulations of the primitive hydrodynamic equations. The high order spectral method (HOSM, [12]) to solve the potential Euler equations is adopted for the situations when the four-wave interactions dominate. The computational domain is periodic in both $x$ and $y$ coordinates. The current is chosen to be close to $\mathrm{sech}^2 y$, and it is taken to be periodical in $y$ with widely separated humps; it is specified as $U = U_{\max} cn^2(2K \frac{y}{L_y}, s^2)$, where $K(s^2)$ is the complete elliptic integral of the first kind, $L_y$ is the computational domain size in the $Oy$ direction, and $U_{\max} = -2$ m/s and $s = 0.9$ are used. The current varies from zero to $U_{\max}$ at $y = 0$; it is shown with arrows in Figs. 1 and 2.

### A. Single trapped mode

In the first experiment we verify the ability of trapped waves belonging to a single mode to propagate with no noticeable radiation in the fully nonlinear system. The initial condition has the form of a uniform train of ten Stokes waves with the wave number $k = 0.1$ rad/m and steepness $kH/2 = 0.15$ (where $H$ is the trough-to-crest wave height), modulated in the transverse direction according to the fundamental ($n = 0$) mode function, $Y_0(y)$. The function $Y_0$ is found by solving (2) numerically (see details in [8]).

The evolution of an initially uniform wave train of trapped waves is simulated for about 80 wave periods. It propagates steadily with no evidence of significant radiation or structural deformation [13]. The presence of some small-amplitude ripples is natural since the initial condition is not exactly a one-mode solution. The examination of the instantaneous wave height record gives some clues of two processes which lead to a slow decrease of the trapped wave height: (i) about 10% of wave height is lost during the first ≈10 wave periods (we attribute this to the imperfect initial conditions); (ii) a longer-term slow trend resulting in the total loss of about 2% of energy over the simulated 80 wave periods is apparently caused by interaction with noise. In other respects the train of trapped waves exhibits robustness. The peak frequency of the numerical solution corresponds to the eigenfrequency of the boundary value problem.

### B. Modulational instability of a single trapped mode leading to a rogue wave pattern

An initial 5% modulation along $Ox$ was applied to the wave train used in the previous simulation to initiate the modulational instability. Also, the reference simulation was performed when the current was set equal to zero, and the train had no modulation in the transverse direction. Supporting the NLSE prediction, the initial modulations grow in both cases with the maximal waves eventually reaching the breaking limit.

The initially modulated train of trapped waves undergoes further localization of wave energy, and the emerging large wave breaks at some instant, which leads to blowing up of the numerical iterations in time. The picture of surface elevation at the moment close to the wave breaking is given in Fig. 1. Due to the factor $\kappa$ in (5), the evolution governed by the NLSE for trapped waves on currents is slower compared with the 1D free gravity waves. Indeed, the curves of instantaneous maximal wave heights versus time in the simulations discussed above may be fitted onto each other, when the time is scaled with factor 0.65 in the case of trapped waves. This value only slightly differs from $\kappa \approx 0.71$ calculated for the chosen profile of the current; the discrepancy is most likely due to the inaccuracy in prescribing the single-mode initial conditions for the trapped wave simulation.

### C. Solitary groups of trapped waves

Envelope solitons are the most fundamental solutions supported by the 1D NLSE; they represent asymptotics of the initial problem with generic localized initial data. The 2D generalizations of the NLSE for deep-water waves also admit planar envelope soliton solutions, but they are no longer asymptotics of the initial problem and are known to be unstable with respect to long transverse perturbations [14] and, hence, relatively short lived.

To specify the initial condition for the strongly nonlinear simulations we use the exact analytic solution to Eq. (5) in the form of an envelope soliton, $A(x,t=0) = A_s\ \mathrm{sech}(\sqrt{2}k^2 A_s x)$, with the transverse shape prescribed by the modal function, $Y_n(y)$; two cases ($n=0$ and $n=4$) were considered. In the runs the carrier wave has the same longitudinal wave number, $k = 0.1$ rad/m, though the intensity is smaller, $kA_s \approx 0.12$.

Figure 2 shows the result of simulation of a solitary wave which corresponds to the fifth mode. The surface snapshot corresponds to the moment when the solitary group has passed the computational domain twice, that is. about 40 wave lengths. The longitudinal section of the solitary group is shown by lines above the maximum of the current and above the $x$ axis (red solid line); the transversal section of the group is shown in front of the surface above the $y$ axis (red solid line). These sections are compared with the corresponding sections of the initial condition (thin black curves). The amplitude of the solitary wave group ends up somewhat reduced compared to the initial condition; the radiated wave patterns are discernible in the figure. The survived intense solitary wave group in Fig. 2 is an indication of the robustness of such groups of trapped waves, although the solitary group produces some radiation each time it interacts with other wave patterns which exist in the simulation domain [13]. In the course of evolution the group is slowly losing energy; the total drop of the maximum wave height over 120 wave periods is about 20%–25%. The solitary group of waves belonging to the fundamental mode preserves energy better, though some radiation is observed as well [13]. The decrease of maximum wave height over the same time of simulation is about 10%–15%. The patterns causing radiation by the groups can be viewed as an artifact of the imperfect choice of the initial conditions, or an unavoidable element of the imperfect reality the solitary waves are likely to encounter in nature.

## V. CONCLUDING REMARKS

In the context of oceanic waves the robust solitary groups of trapped waves found to be possible on jet currents represent a unique case of intense patterns of deep-water gravity waves localized in both dimensions; these long-lived groups can emerge as asymptotics of generic initial conditions. The employed asymptotic theory assumes that the characteristic spatial scale of nonlinear evolution along $Ox$ far exceeds the characteristic width of the trapped mode, which is often the case in the oceanic conditions.

We stress that the exploited one-dimensionalization of wave dynamics occurs without a narrow angular spectrum assumption; in the example shown in Fig. 2 the width of the spectrum is $O(1)$. It makes relevant for oceanic conditions a huge corpus of theoretical and laboratory studies concerned with strictly one-dimensional wave dynamics. Waves in the ocean are typically characterized by a relatively broad angular spectra, so that nonlinearity is hardly able to balance dispersive effects and support long-lived coherent patterns; even for swells with narrow angular spectra about 10° [5] this narrowness is seemingly not enough to balance dispersive terms with much weaker nonlinearity. This makes the applicability of 1D and even 2D NLSE models to real sea states doubtful. In our setting the 1D NLSE model is applicable wherever a single trapped mode is dominant and as long as its frequency spectrum is narrow.

As long as to the leading order the trapped waves are described by the integrable 1D NLSE, all powerful mathematical techniques and analytic solutions obtained since the 1970s may be applied (e.g., [11], and references therein). Correspondingly, the same well studied dynamics resulting in extreme waves for planar waves in other contexts (e.g., [1,2]) holds for the trapped waves. In particular, a higher likelihood of rogue waves in the field of trapped waves is expected due to the existence of long-lived coherent nonlinear wave patterns. Exploiting integrability of the NLSE, elements of deterministic forecasting of oceanic rogue waves might be possible to elaborate for particular conditions.

The situations when several or many trapped modes and/or passing modes are present and interact with the trapped waves require a dedicated study. Here we note that the phenomenon of one dimensionalization of wave dynamics and its implications we discussed are not confined to water waves on currents; similar effects are likely in all branches of physics wherever there are nonlinear guided waves.

## ACKNOWLEDGMENTS

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[13] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevE.89.041002 for video clips showing (a) the evolution of one trapped mode, and (b), (c) the evolution of solitary groups which belong to the fundamental and fifth modes.

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