Nonequilibrium entropic temperature and its lower bound for quantum stochastic processes

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In this paper, we have studied the Shannon "entropic" nonequilibrium temperature (NET) of quantum Brownian systems. The Brownian particle is attached to either a bosonic or fermionic bath. Based on the Fokker-Planck description of the c-number quantum Langevin equation, we have calculated entropy production, NET, and their bounds. Entropy production (EP), the upper bound of entropy production (UBEP), and the deviation of the UBEP from EP monotonically decrease as functions of time to equilibrium value for both of the thermal baths. The deviation decreases with increase of temperature of the bosonic thermal bath, but it becomes larger as the temperature of the fermionic bath grows. We also observe that nonequilibrium temperature and its lower bound monotonically increase to equilibrium value with the progression of time. But their difference as a function of time shows an optimum behavior in most cases. Finally, we have observed that at long time, the entropic temperature (for a bosonic thermal bath) first increases nonlinearly as a function of thermodynamic temperature (TT) and, if the TT is appreciably large, then it grows linearly. But for the fermionic thermal bath, the entropic temperature decreases monotonically as a nonlinear function of thermodynamic temperature to zero value.

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I. INTRODUCTION

One of the key issues in nonequilibrium statistical mechanics is understanding the interaction of a system with its surroundings during the journey towards the equilibrium state [1–6]. The evolution of entropy of the system may carry the signature of the interaction. If it is possible to define a temperature of the system, then it may also be a good observable in the present context. Very recently, the Shannon-entropy-based nonequilibrium temperature (NET) was defined in Ref. [7]. Here authors show that their definition subsumes many other attempts at defining entropic temperatures for nonequilibrium systems and is not restricted to equilibrium or near equilibrium systems. Based on this definition, we have studied the NET of quantum Brownian systems. We are motivated to study this issue from the following point of view. The diffusion constant for quantum Brownian motion is a nonlinear function of thermodynamic temperature. This is in sharp contrast to classical Brownian motion where the constant is a linear function of temperature. Therefore, the thermodynamic temperature dependence of the entropic temperature for a quantum system may be interesting. We have studied this issue using the Fokker-Planck description of the Brownian motion corresponding to the c-number quantum Langevin equation. We have calculated the nonequilibrium temperature, entropy production, and its upper bound. We have compared the effect of the quantum thermal baths on these quantities. Entropy production (EP), the upper bound of entropy production (UBEP), and the deviation of UBEP from EP monotonically decrease as functions of time to equilibrium value. It is true for both of the thermal baths. At a given time, the deviation becomes smaller as the temperature of the bosonic bath (BB) grows. But this trend reverses for the fermionic bath. Our other observation is that the nonequilibrium temperature and its lower bound monotonically increase to equilibrium value with the progression of time. But their difference as a function of time shows an optimum behavior in most cases. This may decrease monotonically, as we have demonstrated for a fermionic thermal bath. Finally, we have observed that at equilibrium, the entropic temperature (for a bosonic thermal bath) first increases nonlinearly as a function of thermodynamic temperature (TT) and, if the TT is appreciably large, then it grows linearly. But, for the fermionic thermal bath, the entropic temperature decreases as a nonlinear function of thermodynamic temperature to zero value.

The outline of the paper is as follows: In Sec. II, we have calculated the nonequilibrium temperature, entropy production, and their bound for a quantum stochastic process in which the Brownian particle is attached to the bosonic thermal bath. In the next section (i.e. in Sec. III), the same quantities are calculated for the fermionic thermal bath. In Sec. IV, we have compared the effect of baths on nonequilibrium properties. The paper is concluded in Sec. V.

II. CALCULATION OF NONEQUILIBRIUM TEMPERATURE AND ITS LOWER BOUND FOR QUANTUM STOCHASTIC DYNAMICS IN THE PRESENCE OF THE BOSONIC BATH

We consider quantum Brownian motion of a free particle of unit mass which is coupled to a bosonic thermal bath, e.g., movement of a dye molecule in a liquid fluid at low temperature. Here Brownian motion is due to collision of the dye molecule with the fluid particles. This picture can be described with the following system-bath Hamiltonian [8]:

$$\hat{H} = \frac{\hat{p}^2}{2} + \sum_j \left[\frac{\hat{p}_j^2}{2} + \frac{1}{2}\left(\omega_j \hat{x}_j - \frac{c_j}{\omega_j}\hat{q} \right)^2 \right]. \qquad (1)$$

Here $\hat{q}$ and $\hat{p}$ are the coordinate and momentum operators of the particle. The $\{\hat{x}_j, \hat{p}_j\}$ are the set of coordinate and momentum operators for the bath oscillators with unit mass. ω_j is the frequency of the jth bath oscillator. The system particle is coupled to the bath oscillators linearly through the general coupling terms $\frac{c_j}{\omega_j}\hat{q}$, where c_j is the coupling

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strength. The coordinate and momentum operators follow the usual commutation relations $[\hat{q},\hat{p}] = i\hbar$ and $[\hat{x}_j,\hat{p}_k] = i\hbar\delta_{jk}$. Eliminating the bath degrees of freedom in the usual way, we obtain the operator Langevin equation [9,10] for the particle as

$$\dot{\hat{p}}(t) = -\int_0^t \gamma(t-t')\hat{p}(t')dt' + \hat{\eta}_B(t), \tag{2}$$

where the noise operator $\hat{\eta}_B(t)$ and the memory kernel $\gamma(t)$ are given by

$$\hat{\eta}_B(t) = \sum_j \left\{ \frac{\omega_j^2}{c_j}\, \hat{x}_j(0) - \hat{q}(0) \right\} \frac{c_j^2}{\omega_j^2} \cos\omega_j t + \sum_j \frac{c_j}{\omega_j}\, \hat{p}_j(0) \sin\omega_j t, \tag{3}$$

and

$$\gamma(t) = \sum_j \frac{c_j^2}{\omega_j^2}\, \cos\omega_j t. \tag{4}$$

In the Markovian limit, the generalized quantum Langevin equation (2) reduces to the following form:

$$\dot{\hat{p}}(t) = -\gamma_0 \hat{p}(t) + \hat{\eta}_B(t), \tag{5}$$

where γ_0 is the dissipation constant in the Markovian limit. We now carry out a quantum mechanical average following Ref. [9] over the product separable bath modes with coherent states and system mode with an arbitrary state at $t = 0$ in Eq. (5) to obtain the Langevin equation as

$$\dot{p} = -\gamma_0 p + \eta_B(t), \tag{6}$$

where the quantum mechanical mean value of the momentum operator is represented by p. The quantum mechanical mean Langevin force, $\eta_B(t)$, is given by

$$\eta_B(t) = \sum_j \left\{ \frac{\omega_j^2}{c_j}\, \langle\hat{x}_j(0)\rangle - \langle\hat{q}(0)\rangle \right\} \frac{c_j^2}{\omega_j^2} \cos\omega_j t + \sum_j \frac{c_j}{\omega_j}\, \langle\hat{p}_j(0)\rangle \sin\omega_j t. \tag{7}$$

To have $\eta_B(t)$ as an effective c-number noise, we now introduce the ansatz [9,11–15] in which the momentum $\langle\hat{p}_j(0)\rangle$ and the shifted coordinates $\{\frac{\omega_j^2}{c_j}\, \langle\hat{x}_j(0)\rangle - \langle\hat{q}(0)\rangle\}$ of the bath oscillators are distributed according to a Wigner canonical thermal distribution of Gaussian form as

$$\mathcal{P}_j\left[\left\{\frac{\omega_j^2}{c_j}\, \langle\hat{x}_j(0)\rangle - \langle\hat{q}(0)\rangle\right\}, \langle\hat{p}_j(0)\rangle\right] = \mathcal{N} \exp\left\{ -\frac{\left[\langle\hat{p}_j(0)\rangle^2 + \frac{c_j^2}{\omega_j^2}\left\{\frac{\omega_j^2}{c_j}\, \langle\hat{x}_j(0)\rangle - \langle\hat{q}(0)\rangle\right\}^2\right]}{2\hbar\omega_j\left[\bar{n}_j(\omega_j) + \frac{1}{2}\right]} \right\}, \tag{8}$$

so that for any quantum mechanical mean value, $\mathcal{O}_j[\{\frac{\omega_j^2}{c_j}\, \langle\hat{x}_j(0)\rangle - \langle\hat{q}(0)\rangle\}, \langle\hat{p}_j(0)\rangle]$, which is a function of the mean value of the bath operators $\langle\hat{x}_j(0)\rangle$ and $\langle\hat{p}_j(0)\rangle$, its statistical average $\langle\cdot\rangle_S$ is

$$\langle\mathcal{O}_j\rangle_S = \int \mathcal{O}_j \mathcal{P}_j\, d\langle\hat{p}_j(0)\rangle\, d\left\{\frac{\omega_j^2}{c_j}\langle\hat{x}_j(0)\rangle - \langle\hat{q}(0)\rangle\right\}. \tag{9}$$

Here $\bar{n}_j(\omega_j)$ indicates the average thermal photon number of the jth oscillator at the temperature T and $\bar{n}_j(\omega_j) = [\exp(\frac{\hbar\omega_j}{k_B T}) - 1]^{-1}$, and $\mathcal{N}$ is the normalization constant. For a detailed discussion about the use of distribution $\mathcal{P}_j$ [Eq. (8)], we refer the reader to Ref. [16].

The above distribution $\mathcal{P}_j$ [Eq. (8)] and the definition of statistical average, given by Eq. (9), imply that c-number noise $\eta(t)$ must satisfy

$$\langle\eta_B(t)\rangle_S = 0, \tag{10}$$

$$\langle\eta_B(t)\eta_B(t')\rangle_S = \frac{1}{2}\sum_j \frac{c_j^2}{\omega_j^2}\, \hbar\omega_j \left(\coth\frac{\hbar\omega_j}{2k_B T}\right)\cos\omega_j(t-t'). \tag{11}$$

In the Markovian limit, the above noise correlation function becomes [17,18]

$$\langle\eta_B(t)\eta_B(t')\rangle_S = 2D_B\delta(t-t'), \tag{12a}$$

$$D_B = \frac{1}{2}\gamma_0\hbar\omega_0 \coth\frac{\hbar\omega_0}{2k_B T}, \tag{12b}$$

where ω_0 is the average bath frequency and the spectral density function is considered in the Ohmic limit.

The Fokker-Planck equation corresponding to the Langevin equation (6) can be written as

$$\frac{\partial\rho(p,t)}{\partial t} = \gamma_0\frac{\partial p\rho}{\partial p} + D_B\frac{\partial^2\rho}{\partial p^2}, \tag{13}$$

where $\rho(p,t)$ is the probability distribution function. This distribution function gives the Shannon information measure [19,20]. For the present problem, it can be written as

$$S_B = -k_B\int \rho(p,t)\ln\rho(p,t)dp, \tag{14}$$

which typically is not a conserved quantity. S in the above equation is called information entropy. If one considers the Boltzmann constant as the information unit and identifies the Shannon measure with the thermodynamic entropy, then the whole of statistical mechanics can be elegantly reformulated by extremization of S, subject to the constraints imposed by the *a priori* information one may possess concerning the system of interest [19,20].

Now we are in a position to calculate the Shannon-entropy-based nonequilibrium temperature (θ_B) of a quantum Brownian system coupled to the bosonic thermal bath, following Ref. [7] and making use of the solution of the Fokker-Planck equation (13). Using both the statistical or Shannon entropy of a system and the de Bruijn identity from information theory, the authors in Ref. [7] have defined the NET. It is the ratio between the average curvature of the Hamiltonian function associated with the system and the trace of the Fisher information matrix of the nonequilibrium probability distribution function. Thus, for the Brownian motion of a free particle, the phase space distribution function can be related to the entropic temperature as

$$\frac{1}{\theta_B} = k_B \int_{-\infty}^{\infty} \rho \left(\frac{\partial ln\rho}{\partial p}\right)^2 dp. \quad (15)$$

For the one-dimensional Brownian motion, the trace of the Fisher information matrix of the nonequilibrium probability distribution function has been replaced by $(\frac{\partial ln\rho}{\partial p})^2$, which is a measure of the average curvature of the probability density function. The average curvature of the Hamiltonian for the present system with unit mass is one in magnitude. However, it is apparent in the above relation that the nonequilibrium temperature increases with a decrease of the average curvature of the probability density function. In other words, the NET increases with an increase of width of the distribution function. Thus the above relation is qualitatively consistent with our expectation.

To find the explicit value of θ_B at the nonequilibrium state, we have solved the Fokker-Planck equation (15) following Ref. [21]. The nonequilibrium distribution function is given by

$$\rho(p,t) = \nu \exp\left\{\frac{-[p-\alpha(t)]^2}{\sigma_B(t)}\right\}. \quad (16)$$

In the above equation, $\sigma_B(t)$ measures the width of the distribution function and $\alpha(t)$ takes care of the time evolution of $\langle p\rangle$ due to deterministic force. The remaining undefined quantity ν is the normalization factor in the expression of the distribution function. Their time evolution equations are given by

$$\dot{\sigma_B}(t) = -2\gamma_0\sigma_B(t) + 4D_B, \quad (17)$$

$$\dot{\alpha}(t) = -\gamma_0\alpha(t), \quad (18)$$

and

$$\frac{1}{\nu(t)}\dot{\nu}(t) = -\frac{1}{2\sigma_B(t)}\dot{\sigma_B}(t). \quad (19)$$

The relevant solutions of $\sigma_B(t)$ and $\alpha(t)$ for the present problem are given by

$$\sigma_B(t) = \frac{2D_B}{\gamma_0}[1-\exp(-2\gamma_0 t)] + \sigma_B(0)\exp(-2\gamma_0 t) \quad (20)$$

and

$$\alpha(t) = \alpha(0)\exp(-\gamma_0 t). \quad (21)$$

Now using Eq. (16) along with Eqs. (20) and (21) in Eq. (15), one obtains the following relation:

$$\theta_B(t) = \frac{\sigma_B(t)}{2k_B}. \quad (22)$$

Thus the nonequilibrium temperature is proportional to the width of the distribution function. It depends on the temperature as well as damping induced by the thermal bath. As the quantum diffusion coefficient is not proportional to temperature, $\theta_B(t)$ depends on the temperature of the bosonic thermal bath in a quite different way compared to a classical bath.

Now we check whether or not the above relation reduces to the standard result. At equilibrium, the above relation becomes

$$\theta_B(\infty) = \frac{\hbar\omega_0 \coth\frac{\hbar\omega_0}{2k_BT}}{2k_B}. \quad (23)$$

The above equation suggests that at the thermodynamic low temperature regime, the "entropic temperature" is different from the temperature of the bosonic thermal bath. At the high temperature limit, the above equation further reduces to the following relation:

$$\theta_B = T. \quad (24)$$

Thus, the present calculation is consistent with the expected limiting result. Now to relate the nonequilibrium temperature with the Shannon entropy (S), we come back to the Fokker-Planck equation (13), which can be rearranged into the following general form of continuity equation:

$$\frac{\partial\rho(p,t)}{\partial t} = -\frac{\partial j}{\partial p}, \quad (25)$$

where the current j is defined as

$$j = -\gamma_0 p\rho - D_B\frac{\partial\rho}{\partial p}. \quad (26)$$

The time evolution equation for S can be written as

$$\dot{S_B} = k_B\int dp\frac{\partial j}{\partial p}\ln\rho. \quad (27)$$

Performing partial integration on the right-hand side of the above equation and then putting the natural boundary conditions $j|_{\text{boundary}} = 0$ and $j\ln\rho|_{\text{boundary}} = 0$, we get

$$\dot{S_B} = -k_B\int dp\frac{1}{\rho}j\frac{\partial\rho}{\partial p}. \quad (28)$$

In the next step, an application of the Schwarz inequality $|\int dq AB|^2 \leqslant \int dq|A|^2\int dq|B|^2$ to the integral (28), where A and B can be appropriately identified, yields an upper bound (U_{BB}) for the rate of entropy change:

$$\dot{S_B} \leqslant U_{BB},$$
$$U_{BB} = k_B\left(\int dp\frac{j^2}{\rho}\right)^{1/2}\left[\int dp\frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)^2\right]^{1/2}. \quad (29)$$

Here, it is to be noted that the second integral is the same as the trace of the Fisher information matrix [22]. Thus the maximum rate of increase of S for an isolated system is limited by the Fisher information level. Now, making use of Eq. (16) in Eqs. (27) and (28), we obtain the explicit time dependence of the time derivative of entropy and its upper bound as

$$\dot{S_B} = k_B\left[-\gamma_0 + \frac{2D_B}{\sigma_B}\right], \quad (30)$$

and

$$U_{BB} = k_B\left[\alpha^2\gamma_0^2 + \frac{\gamma_0^2\sigma_B}{2} + \frac{2D_B^2}{\sigma_B} - 2\gamma_0 D_B\right]^{\frac{1}{2}}\left[\frac{2}{\sigma_B}\right]^{\frac{1}{2}}. \quad (31)$$

Since the information entropy is the negative of the Shannon information, the rate of change of entropy can be interpreted as the rate of information transmission. So the upper bound (31) is interesting in the sense that the amount of information transmitted per unit of time cannot exceed this quantity. The role of the bath on the deviation of the bound (ΔU_{BB}) from $\frac{dS}{dt}$ can be determined from Eqs. (30) and (31) as

$$\Delta U_{BB} = U_{BB} - \frac{dS_B}{dt} = k_B\left[\alpha^2\gamma_0^2 + \frac{\gamma_0^2\sigma_B}{2} + \frac{2D_B^2}{\sigma_B} - 2\gamma_0 D_B\right]^{\frac{1}{2}}\left[\frac{2}{\sigma_B}\right]^{\frac{1}{2}} + k_B\gamma_0 - k_B\frac{2D_B}{\sigma_B}. \quad (32)$$

Now, using Eq. (22) in Eq. (30), we have the following relation between entropy production and nonequilibrium temperature:

$$\dot{S_B} = \frac{D_B - k_B\theta_B\gamma_0}{\theta_B}. \quad (33)$$

Equation (29) and the above relation suggest a lower bound of nonequilibrium temperature as

$$\theta_B \geqslant L_{BB}, \quad (34)$$

where L_{BB} is the lower bound of the nonequilibrium temperature and is given by

$$L_{BB} = \frac{D_B}{k_B\gamma_0 + U_{BB}}. \quad (35)$$

In the Brownian motion, some events are frequently possible and some events rarely occur. Here one cannot avoid a minimum degree of randomness in the experiment. As a signature of that, one would expect a bound related to an unavoidable degree of randomness. L_{BB} in the present study is such a kind of physical quantity. It is an interesting result from the point of view of the experimentalist. For a given parameter set, the nonequilibrium temperature associated with the experimental distribution function cannot be lower than L_{BB}. In other words, if the NET is lower than the bound, then the experimentalist would search for a better distribution function. Thus, with the bound, one can check the validity of the experimental results.

From Eqs. (22) and (35), one can easily determine the deviation of the nonequilibrium temperature from its lower bound as

$$\Delta\theta_{BB} = \frac{\sigma_B(t)}{2k_B} - \frac{D_B}{k_B\gamma_0 + U_{BB}}. \quad (36)$$

We now check whether the relations (22) and (36) reduce to the standard results. At high temperature, the above bound (35) becomes

$$L_{BB} = \frac{\gamma_0 k_B T}{k_B\gamma_0 + U_{BB}}. \quad (37)$$

Using the equilibrium condition, $\dot{S_B} = U_{BB} = 0$, in Eq. (36), we have the same earlier Eq. (24). Using this condition in Eq. (35), we also have $\Delta\theta_B = 0$. These are important checks of our calculation. In the next section, we shall explore how the lower bound depends on the characteristics of the fermionic thermal bath.

III. CALCULATION OF NONEQUILIBRIUM TEMPERATURE AND ITS LOWER BOUND FOR QUANTUM STOCHASTIC DYNAMICS IN THE PRESENCE OF THE FERMIONIC BATH

In this section, we consider the quantum Brownian motion of a free particle which is coupled to a spin bath, e.g., the motion of an ion in liquid ^{3}He [23]. The spin state of the He atom may change during its interaction with the ion. To study this kind of Brownian motion, we start with a system-reservoir model described by the following Hamiltonian:

$$\hat{H} = \frac{\hat{p}^2}{2} + \hbar\sum_k \omega_k\hat{\sigma}_k^\dagger\hat{\sigma}_k + \hbar\sum_k g_k\hat{q}(\hat{\sigma}_k^\dagger + \hat{\sigma}_k). \quad (38)$$

Here a particle of unit mass is coupled to a set of spin-$\frac{1}{2}$ particles (two-level atoms) with characteristic frequencies ω_k. $\hat{q}$ and $\hat{p}$ are coordinate and momentum operators of the particle. The two-level bath atoms are described by a set of Pauli operators $\{\hat{\sigma}_k,\hat{\sigma}_k^\dagger$, and $\hat{\sigma}_{zk}\}$. $\hat{\sigma}_k^\dagger$ ($\hat{\sigma}_k$) is the creation (annihilation) operator for the kth two-level atom coupled linearly to the particle through the coupling constant g_k. $\hat{q}$ and $\hat{p}$ follow the usual commutation relation $[\hat{q},\hat{p}] = i\hbar$. The kth spin-$\frac{1}{2}$ particle or two-level atom obeys the anticommutation rule $\{\hat{\sigma}_k,\hat{\sigma}_k^\dagger\} = 1$, the associated algebra $\hat{\sigma}_k^2 = \hat{\sigma}_k^{\dagger 2} = 0$, and the commutation relations are $[\hat{\sigma}_k^\dagger,\hat{n}_k] = -\hat{\sigma}_k^\dagger$, $[\hat{\sigma}_k,\hat{n}_k] = \hat{\sigma}_k$, and $[\hat{\sigma}_k^\dagger,\hat{\sigma}_k] = \hat{\sigma}_{zk}$, where $\hat{n}_k = \hat{\sigma}_k^\dagger\hat{\sigma}_k$ is the number operator for the kth spin bath. These relations also imply $\hat{\sigma}_{zk} = 2\hat{n}_k - 1$. Thus the present model is basically a spin bath analog of the Zwanzig version of the system-harmonic bath model. Eliminating the bath degrees of freedom as carried out in [24], we obtain the operator Langevin equation for the particle,

$$\ddot{\hat{q}} + \int_0^t dt'\,\dot{\hat{q}}(t')\kappa(t-t') = \hat{f}(t), \quad (39)$$

where the memory kernel and the noise operator are given by

$$\kappa(t-t') = 2\hbar\sum_k \frac{g_k^2}{\omega_k}\cos\omega_k(t-t') \quad (40)$$

and

$$\hat{f}(t) = -\hbar\sum_k g_k[\hat{S}_k(0)e^{-i\omega_k t} + \hat{S}_k^\dagger(0)e^{i\omega_k t}], \quad (41)$$

respectively. $\hat{S}_k^\dagger(0)$ and $\hat{S}_k(0)$ are shifted bath operators and are defined by

$$\hat{S}_k^\dagger(0) = \hat{\sigma}_k^\dagger(0) + \frac{g_k}{\omega_k}\hat{q}(0),$$
$$\hat{S}_k(0) = \hat{\sigma}_k(0) + \frac{g_k}{\omega_k}\hat{q}(0). \quad (42)$$

Now, following Ref. [25] and the earlier section, one can write the c-number Langevin equation for the present problem in the Markovian limit as

$$\dot{p} = -\gamma_0 p + \eta_F(t), \quad (43)$$

where γ_0 is the frictional coefficient at this limit and $\langle\hat{f}(t)\rangle = \eta_F(t)$ is given by

$$\langle\eta_F(t)\eta_F(t')\rangle_s = 2D_F\delta(t-t'). \quad (44)$$

D_F in the above equation is represented as

$$D_F = \gamma_0 \frac{\hbar\omega_0}{2} \tanh \frac{\hbar\omega_0}{2K_B T}, \tag{45}$$

where ω_0 is the average value of the characteristic frequency of the bath modes. The above equation suggests that the temperature dependence of the diffusion constant (DC) for the fermionic bath (FB) is quite different from the bosonic one. The DC decreases with increase in temperature for the FB, but it increases in the case of a BB. This can be understood considering the number of accessible states for the bath modes. For the bosonic bath, the number of accessible states increases for each bath mode as the temperature of the bath grows. There is no upper limit of the number of accessible states. Thus the distribution of initial conditions for bath modes becomes wider [Eq. (8)] with an increase in temperature. As a result, the diffusion constant as well as the degree of randomness in the Brownian system are enhanced for the bosonic thermal bath as temperature rises. But in the case of a fermionic bath, the opposite scenario occurs because of the finite number (only two) of accessible states for each bath component. With an increase in temperature, the bath mode is arrested more strongly at the excited state, with a decreasing probability of transition to the ground state. As a signature of that, the distribution of initial conditions in c number for bath modes becomes narrower as the temperature of the fermionic bath grows [24]. Therefore, the diffusion behavior of the Brownian system, which is coupled to the fermionic thermal bath, is suppressed as the temperature of the bath grows. In other words, with increase in temperature, the population difference between the two levels of the bath modes decreases and the FB becomes inert.

The Langevin equations (6) and (43) are similar in structure and therefore, following the previous section, one can write the nonequilibrium temperature $[\theta_F(t)]$ and entropy production $(\dot{S_F})$ for the quantum system which is coupled to the fermionic bath as

$$\theta_F(t) = \frac{\sigma_F(t)}{2k_B}, \tag{46}$$

and

$$\dot{S_F} = k_B \left[-\gamma_0 + \frac{2D_F}{\sigma_F} \right]. \tag{47}$$

In the above equations, σ_F measures the width of the phase space distribution function and is given by

$$\sigma_F(t) = \frac{2D_F}{\gamma_0}[1 - \exp(-2\gamma_0 t)] + \sigma_F(0)\exp(-2\gamma_0 t). \tag{48}$$

We now write other relevant quantities, such as the lower bound of $\theta_F(t)$ and the upper bound of $\dot{S_F}$. The dependence of these quantities on the characteristics of the thermal bath obey the following relations:

$$L_{BF} = \frac{D_F}{k_B\gamma_0 + U_{BF}}, \tag{49}$$

and

$$U_{BF} = k_B \left[\alpha^2\gamma_0^2 + \frac{\gamma_0^2\sigma_F}{2} + \frac{2D_F^2}{\sigma_F} - 2\gamma_0 D_F \right]^{\frac{1}{2}} \left[\frac{2}{\sigma_F} \right]^{\frac{1}{2}}. \tag{50}$$

In the above two equations, L_{BF} and $\dot{S_F}$ are the lower bound of nonequilibrium temperature and upper bound of entropy production, respectively.

From Eqs. (46) and (49), one can easily determine the deviation of the nonequilibrium temperature from its lower bound as

$$\Delta\theta_{BF} = \frac{\sigma_F(t)}{2k_B} - \frac{D_F}{k_B\gamma_0 + U_{BF}}. \tag{51}$$

Similarly one can also easily determine the difference between entropy production and its upper bound from Eqs. (47) and (50) and it is given by the following relation:

$$\begin{aligned}\Delta U_{BF} &= U_{BF} - \frac{dS_F}{dt} \\ &= k_B \left[\alpha^2\gamma_0^2 + \frac{\gamma_0^2\sigma_F}{2} + \frac{2D_F^2}{\sigma_F} - 2\gamma_0 D_F \right]^{\frac{1}{2}} \left[\frac{2}{\sigma_F} \right]^{\frac{1}{2}} \\ &\quad + k_B\gamma_0 - k_B \frac{2D_F}{\sigma_F}. \end{aligned} \tag{52}$$

In the next section, we will compare the results of the present section with that of the previous section.

IV. COMPARATIVE STUDY OF THE EFFECT OF BOSONIC AND FERMIONIC BATHS ON NONEQUILIBRIUM TEMPERATURE, ENTROPY PRODUCTION, AND THEIR BOUNDS

To demonstrate the time dependence of the entropy production (EP) and its upper bound (UB), we have calculated these quantities as a function of time and plotted them in Fig. 1 for the bosonic thermal bath. In the same figure, we have also demonstrated the time dependence of the deviation of the bound from the entropy production. Figure 1 shows that the EP, UB, and the deviation monotonically decrease (as functions of time) to their equilibrium values. We now explain the monotonic decrease of entropy production and its upper bound. At the initial time, phase space volume is small and noise starts to work to expand the space against the damping. The signature of noise should be significant at this regime compared to the long time situation when the system already has large phase space volume. Thus, at $t \to 0$, the phase space expansion rate, entropy production, and its upper bound are maximum. With the progression of time, the phase space volume increases monotonically. Consequently, the above-mentioned quantities decrease regularly to their equilibrium values with the progression of time. This signature is implied in Eq. (30). As the system approaches the equilibrium state, the width of the distribution function increases and, according to that, the entropy production decreases. Similarly the entropy production and the other related quantities also monotonically decrease for the fermionic thermal bath. This has been demonstrated in Fig. 2. Inspecting Figs. 1 and 2, one may conclude that the rates of decrease of entropy production and its upper bound increase as the temperature of the bosonic bath (BB) grows. This order reverses for the fermionic bath. One can account for these facts considering the temperature dependence of the diffusion constant (noise strength) for both cases. We have already discussed the temperature dependence of D_B and D_F in detail after Eq. (45).

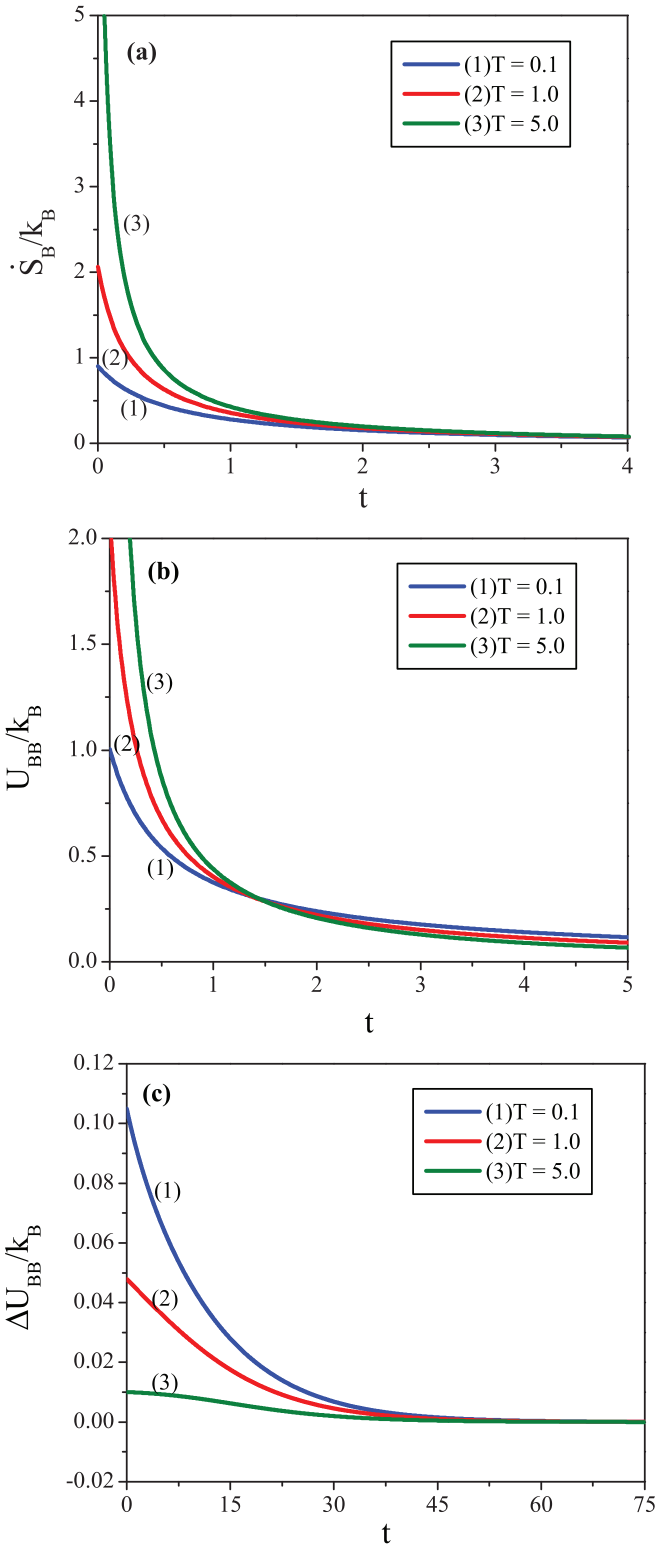

FIG. 1. (Color online) Demonstration of the variation of information entropy production, its upper bound, and their difference as a function of time for the given parameter set $\sigma_B(0) = 0.1$, $\alpha(0) = 1.0$, $\frac{\hbar\omega_0}{k_B} = 1.0$, and $\gamma_0 = 0.1$ for bosonic thermal baths. (a) Entropy production $\dot{S}_B$ for the bosonic thermal bath, (b) upper bound (U_{BB}) of entropy production for the bosonic thermal bath, and (c) deviation of the bound from the entropy production (ΔU_{BB}) for the bosonic thermal bath (arbitrary units).

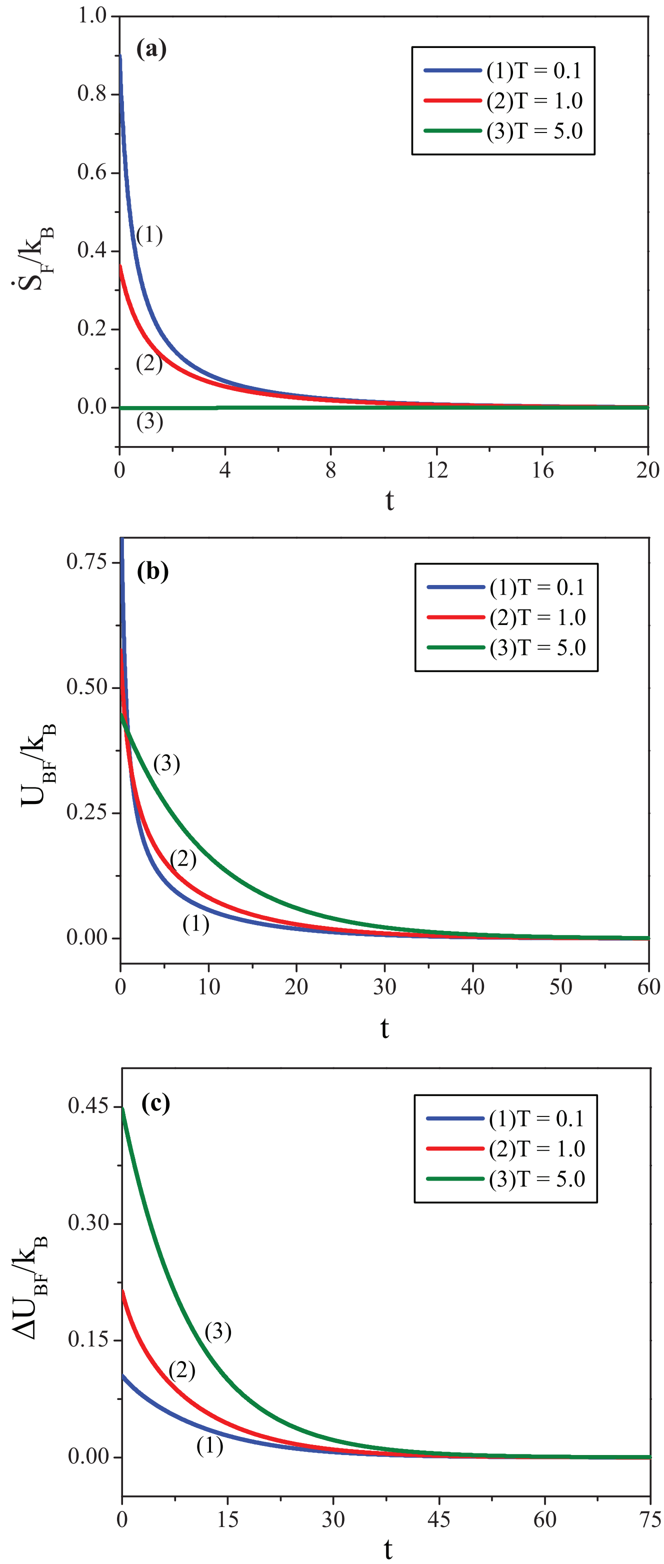

FIG. 2. (Color online) Demonstration of the variation of information entropy production, its upper bound, and their difference as a function of time for the given parameter set $\sigma_F(0) = 0.1$, $\alpha(0) = 1.0$, $\frac{\hbar\omega_0}{k_B} = 1.0$, and $\gamma_0 = 0.1$ for the fermionic thermal bath. (a) Entropy production $\dot{S}_F$ for the fermionic thermal bath, (b) upper bound (U_{BF}) of entropy production for the fermionic thermal bath, and (c) deviation of the bound from the entropy production (ΔU_{BF}) for the fermionic thermal bath (arbitrary units).

However, for higher noise strength, the equilibrium phase space volume is greater. It implies that for strong random force [corresponding to a large diffusion constant (DC)], the phase space volume varies rapidly and thereby entropy production as well as its bound change at faster rates with an increase of the diffusion constant (DC) when damping strength (relaxation time) is fixed. Therefore, an increase in temperature results in an enhancement of the DC as well as the rates of decrease of entropy production and its bound in the case of a BB. But, in the case of the FB, the rising of temperature suppresses the diffusion process as the two-level bath becomes inert as a signature of the decrease of the population difference. This accounts for the above-mentioned reverse order.

We now consider the difference between entropy production (EP) and its upper bound. It is apparent in Figs. 1 and 2 that the difference is higher for a lower diffusion constant. In other words, if the noise is weak against the phase space contracting damping force, then the deviation is significant. It is supported by the fact that with the progression of time, noise becomes more effective in the dynamics, and the deviation of the upper bound from entropy production decreases as the system approaches the equilibrium state. It suggests that the deviation grows as the temperature of the bosonic thermal bath decreases and the trend reverses for the fermionic bath. This observation is a clear signature of the amount of relative fluctuation in phase space volume. It is obvious that if the relative fluctuation is higher, then the deviation of the bound from the entropy production is greater. For small phase space volume, the relative fluctuations in volume are large. Thus the deviation at the initial time of the dynamics is highest and it decreases regularly with the progression of time as the phase space volume grows. Using the same argument, one can easily explain the temperature dependence of the deviation of the upper bound from the entropy production.

In the next step, we have demonstrated the time dependence of the nonequilibrium temperature (NET) and its lower bound (LB) in Figs. 3 and 4 for the bosonic and the fermionic thermal baths, respectively. In these figures, we have also demonstrated the deviation of the NET from the lower bound. Monotonic increase of the nonequilibrium temperature and its LB is a signature of regular growth of the phase space volume of the system. At an early stage of dynamics, rapid growth of the entropic temperature suggests that the Brownian force induced phase space expansion rate is high at this regime and the rate monotonically decreases to zero at equilibrium. One can account for the temperature-dependent growth rate based on the earlier discussion.

We now consider the deviation of the NET from its lower bound. The nonequilibrium temperature is proportional to the width of the distribution function, and the LB depends on the upper bound of entropy production, which is also related to the width in a complex manner. It is apparent in Figs. 3 and 4 that the interplay of the width of the distribution function and the upper limit of the information entropy production, in general, produces a maximum in the variation of the deviation ($\Delta\theta_{BB}$, $\Delta\theta_{BF}$) as a function of time. Figure 4(c) suggests that the interplay may also result in a monotonic decrease of the deviation, particularly at the regime of very small diffusion coefficient.

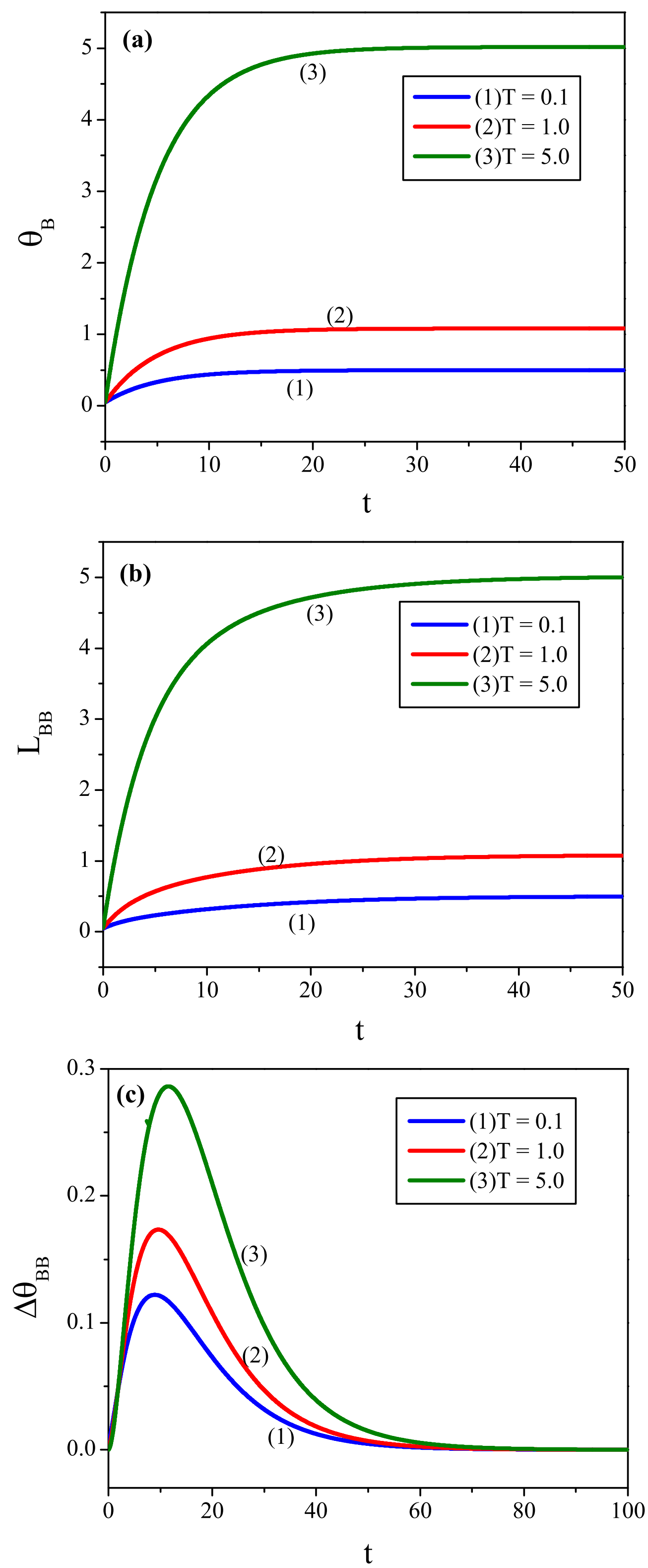

FIG. 3. (Color online) Demonstration of the variation of the nonequilibrium temperature, its lower bound, and their difference as a function of time for the parameter set $\sigma_B(0) = 0.1$, $\alpha(0) = 1.0$, $\frac{\hbar\omega_0}{k_B} = 1.0$, and $\gamma_0 = 0.1$. (a) Nonequilibrium temperature (θ_{BB}) for the bosonic thermal bath, (b) lower bound (L_{BB}) of the nonequilibrium temperature for the bosonic thermal bath, and (c) deviation of the bound from the nonequilibrium temperature ($\Delta\theta_{BB}$) for the bosonic thermal bath (arbitrary units).

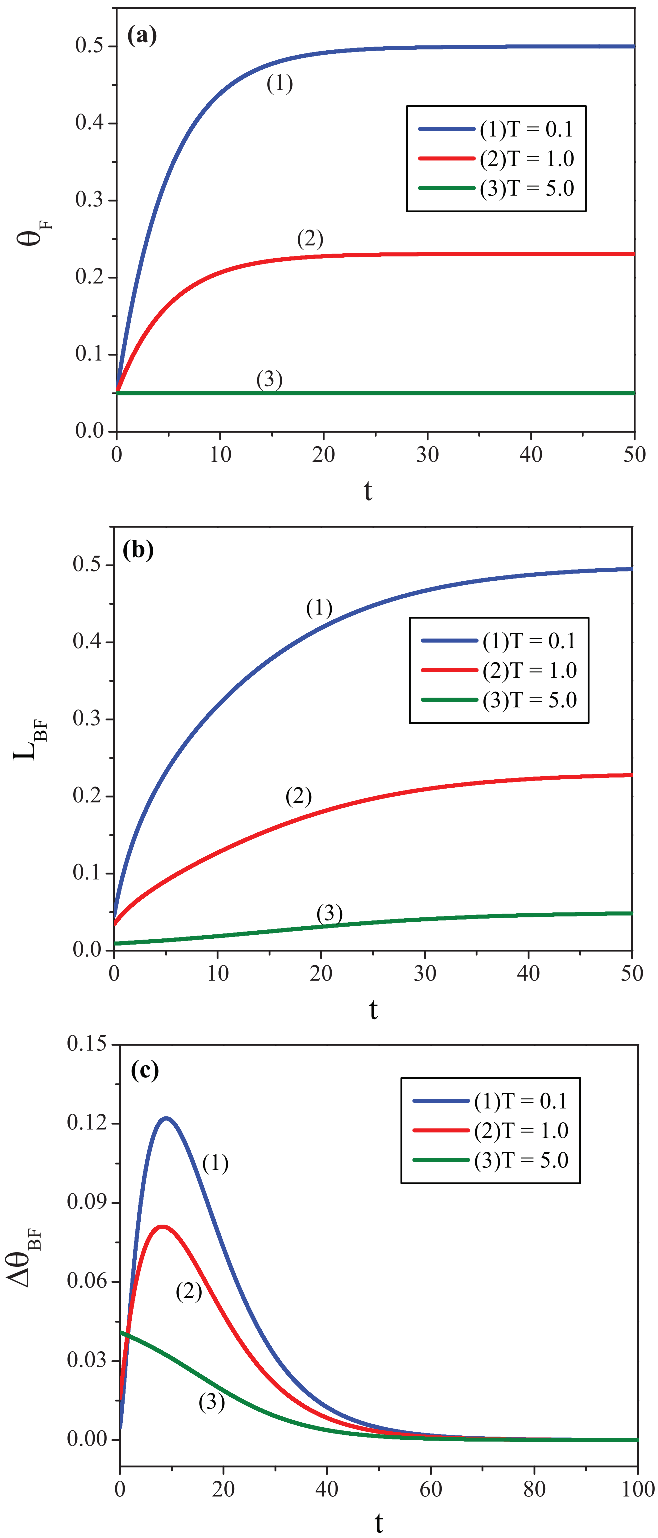

FIG. 4. (Color online) Demonstration of the variation of the nonequilibrium temperature, its lower bound, and their difference as a function of time for the parameter set $\sigma_F(0) = 0.1$, $\alpha(0) = 1.0$, $\frac{\hbar\omega_0}{k_B} = 1.0$, and $\gamma_0 = 0.1$. (a) Nonequilibrium temperature θ_{BF} for the fermionic thermal bath, (b) lower bound (L_{BF}) of the nonequilibrium temperature for the fermionic thermal bath, and (c) deviation of the bound from the nonequilibrium temperature ($\Delta\theta_{BF}$) for the fermionic thermal bath (arbitrary units).

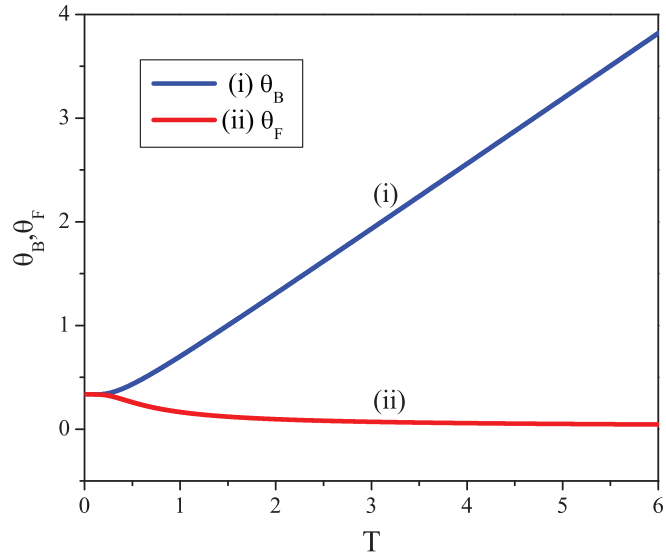

FIG. 5. (Color online) Demonstration of the variation of the nonequilibrium temperature as a function of thermodynamic temperature of the thermal bath for the parameter set $\sigma_B(0) = \sigma_F(0) = 0.1$, $\alpha(0) = 1.0$, $t = 5.0$, $\frac{\hbar\omega_0}{k_B} = 1.0$, and $\gamma_0 = 0.1$ (arbitrary units).

In Fig. 5, we have presented the variation of the nonequilibrium temperature as a function of temperature of the thermal bath. It shows that at the low temperature regime of the bosonic thermal bath, θ_B increases nonlinearly as the diffusion coefficient is not proportional to the temperature of the thermal bath. As the bath approaches the classical limit, the diffusion coefficient as well as the nonequilibrium temperature become a linear function of the thermodynamic temperature. We now consider the case of the fermionic bath. For this bath, the entropic temperature decreases as a nonlinear function of temperature. This is a consequence of the decrease of the population difference between the two levels of the thermal bath. Thus the bath becomes inert as the thermodynamic temperature (TT) grows. As a result, both the diffusion coefficient as well as the nonequilibrium temperature decrease regularly with an increase of the TT.

V. CONCLUSION

In this paper, we have studied relaxation of nonequilibrium quantum systems based on the Fokker-Planck description corresponding to c-number Langevin equations of motion. Using the distribution function, we have calculated the Shannon entropy production, entropic temperature, and their bounds. Our study includes the following major points.

(i) Entropy production and its upper bound as functions of time monotonically decrease to their equilibrium values. The deviation of the bound from the entropy production also regularly decreases to zero magnitude. At a given time, the deviation becomes smaller as the temperature of the bosonic bath grows. But, for the fermionic bath, the trend reverses with the variation of temperature.

(ii) *There is a lower bound of nonequilibrium temperature*. Both the NET and its lower bound monotonically increase to their equilibrium values with the progression of time. But their difference as a function of time shows an optimum behavior in most cases. It may decrease monotonically as shown for the fermionic thermal bath.

(iii) At low thermodynamic temperature, the entropic temperature increases nonlinearly as a function of T for the bosonic thermal bath. If the temperature is appreciably high, then it increases linearly. But for the fermionic thermal bath, the NET decreases as a nonlinear function of thermodynamic temperature to zero value.

The nonequilibrium temperature is a very important quantity as it is able to give an idea about the degree of randomness in a Brownian system at the *nonstationary state* (NSS). Therefore, one may explain the properties of Brownian particles at the NSS easily in terms of it. As it is a recently derived quantity [7], the expression of the properties of stochastic systems as a function of the NET may open a new branch in the field of nonequilibrium statistical mechanics. Following the present method, the study of the NET of a Brownian particle having charge in the presence of Lorentz force may appear elsewhere. One may also extend the present study for the non-Markovian thermal bath.

ACKNOWLEDGMENT

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