

Crossover behavior of conductivity in a discontinuous percolation model

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When conducting bonds are occupied randomly in a two-dimensional square lattice, the conductivity of the system increases continuously as the density of those conducting bonds exceeds the percolation threshold. Such a behavior is well known in percolation theory; however, the conductivity behavior has not been studied yet when the percolation transition is discontinuous. Here we investigate the conductivity behavior through a discontinuous percolation model evolving under a suppressive external bias. Using effective medium theory, we analytically calculate the conductivity behavior as a function of the density of conducting bonds. The conductivity function exhibits a crossover behavior from a drastically to a smoothly increasing function beyond the percolation threshold in the thermodynamic limit. The analytic expression fits well our simulation data.

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The concept of percolation transition has played a central role as a model for the formation of a spanning cluster connecting two opposite edges of a system in Euclidean space as a control parameter p is increased beyond a certain threshold p_c [1]. This model has been used to study many phenomena such as metal-insulator transitions and sol-gel transitions. The order parameter P_∞ of percolation transition is defined as the probability that a bond belongs to a spanning cluster, which increases in the form $P_\infty(p) \sim (p - p_c)^\beta$ beyond p_c , where p is a control parameter indicating the fraction of occupied bonds and β is the critical exponent related to the order parameter. As an application of percolation model, one can construct a random resistor network in which each occupied bond is regarded as a resistor with unit resistance, and the system is in contact with two bus bars at the opposite edges of the system. When a voltage difference is applied between these two bus bars, the system is in an insulating state for $p < p_c$, but changes to a conducting state for $p > p_c$, due to the formation of several conducting paths at p_c . Above p_c , the conductivity increases continuously as $g \sim (p - p_c)^\mu$, where μ is the conductivity exponent [2].

Recently, the subject of discontinuous percolation transition (DPT) has been a central issue [3–12] with, for example, applicability to cascading failures in complex networks [13]. Among others [14–19], a model called spanning cluster avoiding (SCA) was introduced [20] aiming to generate a DPT. The DPT of the SCA model is rather trivial, for the percolation threshold is placed at $p_c = 1$ in the thermodynamic limit, but for finite-sized systems $p_c < 1$. Here, we study the conductivity as a function of p in finite-sized systems for the SCA model. Indeed, we find that the conductivity increases drastically just after the percolation threshold and then exhibits a crossover to a smoothly increasing behavior. Such crossover has never been reported, though it is meaningful as a drastic change of conductivity in random resistor networks can find application, for example, on resistance switching phenomena in nonvolatile memory devices [21]. From a theoretical perspective, the understanding of conductivity becomes comple-

mentary to the result on the percolation transition for the SCA model.

We first recall the SCA model. In this model, we take a two-dimensional regular square lattice of linear size L . Initially, the system consists of $N = L^2$ nodes and $2N$ unoccupied bonds. At each time step, one randomly chooses m unoccupied bonds, and those potential bonds are classified into two types: bridge and nonbridge bonds. Bridge bonds are those that would form a spanning cluster if any of them is occupied [22,23]. One takes a nonbridge bond randomly among those m candidates if it exists. This choice suppresses the formation of a spanning cluster. As the number of occupied bonds is increased, the total number of bridge bonds $N_{\text{BB}}(p)$ increases and thus the probability that those m bonds are all bridge bonds is also increased. If such a case happens, a bridge bond is inevitably occupied and a spanning cluster is formed. Once a spanning cluster is formed, no more restrictions are imposed on the occupation of bonds. It was found that when m is greater than a tricritical point $m_c \approx 2.55$ in two dimensions, the percolation transition is discontinuous and the percolation threshold p_{cm} approaches unity as the system size is increased [20]. In finite-sized systems, the percolation threshold p_{cm} depends on the number of candidate bonds m . In this paper, we present an analytic formula for the conductivity based on effective medium theory [24]. The analytic prediction of the conductivity function is in agreement with our numerical data.

In the SCA model, the percolation threshold is delayed by suppressing the formation of a spanning cluster. While the percolation threshold is delayed, two large clusters form independently, which are separated by bridge bonds. Bridge bonds form a fractal set of fractal dimension $d_{\text{BB}} \approx 1.215$ [22]. Moreover, for $m > m_c$, in those two separated clusters, the density of occupied bonds is extremely high, for p close to and above p_{cm} . These facts enable us to apply effective medium theory to calculate the conductivity function near the percolation threshold for the SCA model. We recall the conductivity function for ordinary percolation obtained from effective medium theory near $p = 1$, which is $g_{\text{eff}} = 2p - 1$ in two dimensions [24].

Next, we derive a formula for the conductivity using heuristic arguments. To proceed, we examine the structure of

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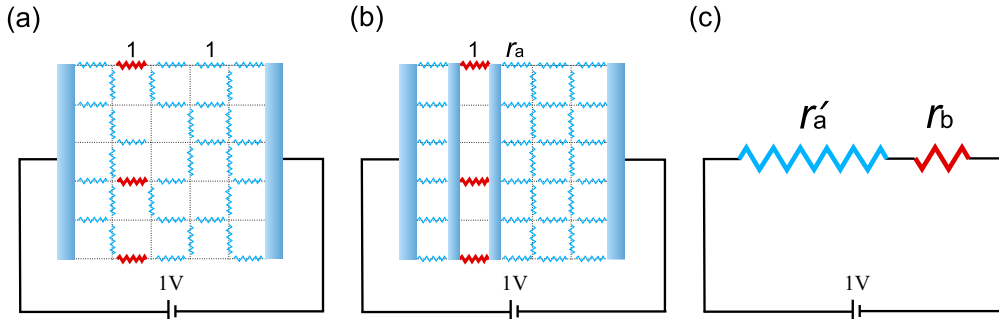


FIG. 1. (Color online) (a) Schematic diagram of circuit structure for $p \geq p_{cm}$ for the SCA model that consists of bonds of unit resistance. The occupied bonds are classified into original bridge bonds (thick red resistors) and original nonbridge bonds (thin blue resistors). (b) We simplify the whole circuit as series connection of a bundle of original bridge bonds of unit resistance and two compact clusters consisting of bonds of resistance r_a by applying effective medium theory. (c) The combined resistance of two compact clusters is calculated as $r'_a = r_a(L - 1)/L \approx r_a$ for large L , and the combined resistance of original bridge bonds r_b is calculated as $r_b = 1/Lp_b$, where the derivations are shown in the main text. We can calculate the conductivity as $g_m(p) = 1/(r'_a + r_b)$.

the system at the onset of the percolation transition, denoted as p_{cm}^- . As shown in Fig. 1(a), the system consists of two disconnected clusters separated by unoccupied bridge bonds. Due to the unoccupied separatrix, the conductivity of the system is zero at p_{cm}^- , but becomes nonzero once a bond among those bridge bonds is occupied as shown in Fig. 2. Since the number of bridge bonds increases as p increases for $p < p_{cm}$, from now on, we use the phrase “original bridge bonds” to refer to those bonds that were bridge bonds at p_{cm}^- . Similarly, “original nonbridge bonds” are all the other bonds. The densities of occupied bonds of original nonbridge bonds and original bridge bonds are denoted as p_a and p_b , respectively. Those two densities depend on p , and $p_b = 0$ at p_{cm}^- . Then, the following relation holds:

$$2L^2 p \approx p_a(2L^2 - L^{d_{BB}}) + p_b L^{d_{BB}}, \quad (1)$$

where we use the number of bridge bonds $N_{BB}(p_{cm}^-) \approx L^{d_{BB}}$ and $L^2 \gg L^{d_{BB}}$ for large L . Then $p_a \approx p + O(1/L^{2-d_{BB}})$. For $p_b(p)$, we use the fact that the occupation of original bridge

bonds increases linearly with increasing p for $p > p_{cm}$ and $p_b(p_{cm}) = 0$. Then, one obtains

$$p_b(p) = \frac{p - p_{cm}}{1 - p_{cm}}. \quad (2)$$

In the spirit of effective medium theory, we assume that original nonbridge bonds are fully occupied but we consider that each bond has resistance $r_a \neq 1$. Next, we make a more crude assumption. Due to the fractal nature of the set of bridge bonds, the separatrix is not linear in its shape, and the density of original nonbridge and original bridge bonds are different. Thereby, the current can flow along the boundary between the original nonbridge and original bridge bonds. However, this current contribution to the conductivity of the system can be negligible when the system size is sufficiently large. Based on such facts, we simplify the system as shown in Figs. 1(b) and 1(c). That is, the system consists of two parts, a rectangular-shape regular lattice of size $(L - 1) \times L$ in which original all nonbridge bonds are all occupied with resistance $r_a = 2p_a - 1$, and one-dimensional columnar lattice of size L in which original bridge bonds are occupied with probability p_b and unit resistance. We also assume that there exists a busbar between the two parts, and thereby there is no net current on each vertical bond.

This simplified picture enables us to calculate the overall conductivity. The resistivity (the inverse of conductivity) is obtained as

$$\frac{1}{g_m(p)} \approx \frac{1}{2p_a - 1} + \frac{1}{Lp_b}, \quad (3)$$

where $g_m(p)$ denotes the conductivity at p of the SCA model with the control parameter m . We compare the analytic result with our simulation data for different $m = 2, 3, 4$, and 5 . As can be seen in Fig. 3, the data for $L \times L = 300 \times 300$ is in good agreement with the analytic expression for $m \geq 3$. Since for these cases p_{cm} is close to unity (for example, $p_{c3} \approx 0.84$, $p_{c4} \approx 0.94$, and $p_{c5} \approx 0.97$), the approximation based on effective medium theory is more accurate. For $m = 2$, the data clearly differs from the analytic expression. Actually, the percolation threshold for $m = 2$ reduces to the one of the ordinary percolation in the thermodynamic limit and, therefore, the two clusters connected through original bridge

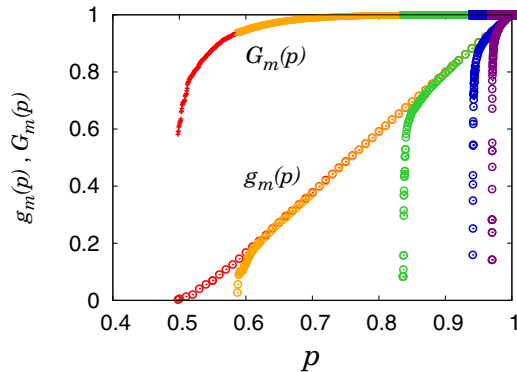


FIG. 2. (Color online) Plot of $G_m(p)$ and $g_m(p)$ vs p . $G_m(p)$ is the fraction of nodes belonging to the spanning cluster. $G_m(p)$ jumps to $G_1(p)$ from zero at p_{cm} and follows the envelope of $G_1(p)$ after that. $g_m(p)$ is the conductivity; it becomes positive at p_{cm} and grows drastically after that. As m is increased, p_{cm} is delayed. Data are shown for $m = 1$ (red), $m = 2$ (yellow), $m = 3$ (green), $m = 4$ (blue), and $m = 5$ (purple) from left to right. $L = 300$ are considered. Results are for a single sample.

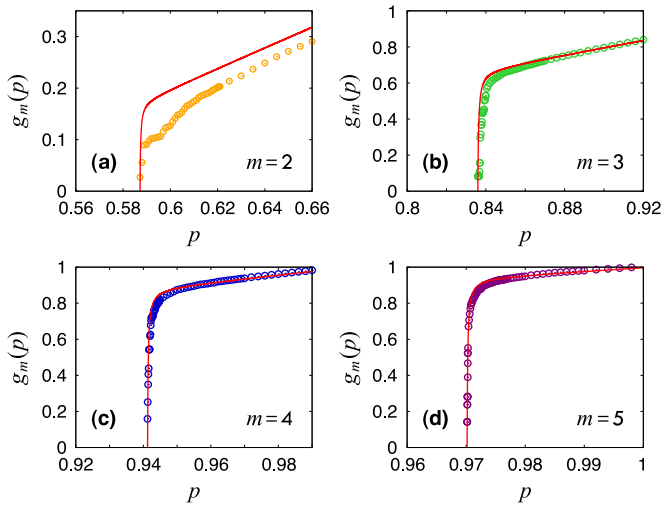


FIG. 3. (Color online) Plot of $g_m(p)$ vs p for one sample with $L = 300$. Just after p_{cm} , $g_m(p)$ becomes positive and grows drastically. Red solid lines are obtained from Eq. (3) for $m = 2$ (a), $m = 3$ (b), $m = 4$ (c), and $m = 5$ (d). We can find that the theoretical formula fits the simulation data well when m is larger than the tricritical point $m_c \approx 2.55$.

bonds cannot be considered compact, as is necessary to apply effective medium theory.

Finally, we recall the previous result [20] that p_{cm} approaches one as L is increased as

$$1 - p_{cm} \sim L^{-\frac{2}{m-1}(\frac{m}{m_c}-1)} \quad \text{for } m > m_c, \quad (4)$$

where $m_c \approx 2.55$ in two dimensions. Then, $Lp_b \equiv L^\alpha(p - p_{cm})$, where

$$\alpha = 1 + \frac{2}{m-1} \left(\frac{m}{m_c} - 1 \right). \quad (5)$$

Numerically, $\alpha \approx 1.18, 1.38,$ and 1.48 for $m = 3, 4,$ and 5 in two dimensions, respectively. Depending on the magnitude of $p - p_{cm}$, the conductivity behaves as follows:

$$g_m(p) \approx \begin{cases} L^\alpha(p - p_{cm}) & \text{for } \delta \ll 1/L^\alpha, \\ 2p - 1 & \text{for } \delta \gg 1/L^\alpha, \end{cases}$$

where $\delta = p - p_{cm}$. Thus there exists a crossover in the conductivity for $\delta_c \approx 1/L^\alpha$. We remark that the conductivity increases more rapidly to $2p_{cm} - 1$ for larger systems due to the prefactor L^α .

In summary, we studied the conductivity transition of the two-dimensional SCA model. In this model, p_{cm} increases to 1 for $m > m_c \approx 2.55$, but otherwise it decreases to $p_{c1} = 0.5$ as the system size increases. We used effective medium theory which is valid for $p \gg p_{c1}$ to calculate the analytic expression of conductivity in this model. We numerically confirmed the validity of this expression for $m = 2, 3, 4,$ and 5 in finite-sized systems and found that the data is well fitted for $m = 3, 4,$ and 5 . However, the case $m = 2$ cannot be described by our theory.

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