Critical behavior of a relativistic Bose gas

P. N. Pandita
P-30, North Eastern Hill University, Shillong 793 022, India

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We show that the thermodynamic behavior of relativistic ideal Bose gas, recently studied numerically by Grether *et al.*, can be obtained analytically. Using the analytical results, we obtain the critical behavior of the relativistic Bose gas exactly for all the regimes. We show that these analytical results reduce to those of Grether *et al.* in different regimes of the Bose gas. Furthermore, we also obtain an analytically closed-form expression for the energy density for the Bose gas that is valid in all regimes.

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Bose-Einstein condensation (BEC) of an ideal Bose gas has been a subject of extensive studies. In particular, Bose-Einstein condensation in a relativistic ideal Bose gas with nonzero chemical potential has been studied by several authors [1–5]. In earlier papers [1,6] on relativistic ideal Bose condensation, antiboson production was not taken into account. The necessity of the antiboson contribution to the thermodynamics of Bose gas at relativistic energies was pointed out in [3], and a high-temperature expansion for various thermodynamic quantities was established and studied. At sufficiently high temperatures, antibosons are expected to be pair-produced in sufficient numbers so that their contribution cannot be neglected.

In a recent paper, Grether *et al.* studied Bose-Einstein condensation in a relativistic ideal Bose gas, and calculated its thermodynamic properties numerically [7]. In this paper, we point out that the model can be solved exactly in a closed form, including antibosons, without making any approximations, and we show that an analytic expression for thermodynamic quantities can be obtained at all temperatures and the critical behavior studied in general. The analytical study is physically more transparent, and sheds light on the connection between relativistic and the nonrelativistic ideal Bose gas by showing that the critical behavior is the same for the two systems. It is only the overall amplitude that is different for the two systems. We also show that our exact results reduce to the results of Ref. [7] in different regions studied by these authors.

The ideal Bose gas is characterized by three basic length scales, namely the thermal wavelength λ_T, the mean interparticle spacing $\overline{\lambda}$, and the Compton wavelength λ_C. From these, one can obtain two independent ratios which we consider to be $R_1 = \overline{\lambda}/\lambda_T$ and $R_2 = \lambda_C/\lambda_T$. For quantum (classical) gas, we have $R_1 \ll 1 (\gg 1)$, whereas for $R_2 \ll 1$ and $R_2 \gg 1$ we will have nonrelativistic or ultrarelativistic gas, respectively. Clearly, it is important to understand the connection between different regions, characterized by different length scales, and have a unified treatment for all the regions involved. In this paper, we solve the model exactly in general, which includes all the cases corresponding to all the length scales and without making a high-temperature expansion. We show analytically that the critical behavior of the model is the same as that of standard nonrelativistic Bose gas. The difference between nonrelativistic and the ultrarelativistic gas, where antiparticles are important, arises only in the values of critical amplitudes.

The net "charge" or number density, assumed positive without loss of generality, for a relativistic Bose gas of N bosons and $\overline{N}$ antibosons, each of mass m, can be written as (the notation is standard, and we choose units $\hbar = c = k_B = 1, \beta = T^{-1}$; g is the spin degeneracy factor)

$$n = g \int \frac{d^3k}{(2\pi)^3}\left[\frac{1}{\exp[\beta(k^2+m^2)^{1/2} - \beta\mu] - 1} - \frac{1}{\exp[\beta(k^2+m^2)^{1/2} + \beta\mu] - 1}\right], \tag{1}$$

where we have enclosed the system in a cubic box of volume L^3, with L being the length of an edge in each of the three spatial dimensions. This can be further written as

$$n = \frac{m^3 g}{2\pi^2}\int_0^\infty dx\, x^2 \frac{\sinh(\alpha-\phi)}{\cosh a(x) - \cosh(\alpha-\phi)} \equiv \frac{m^3 g}{2\pi^2}\mathcal{W}_3(\alpha,\phi), \tag{2}$$

where $\alpha = \beta m$, $a(x) = \alpha(1+x^2)^{1/2}$, and $\phi = -\beta(\mu - m)$. The parameter ϕ is defined so as to reduce to its nonrelativistic counterpart in that limit. For d spatial dimensions, this generalizes to

$$n = g \int \frac{d^dk}{(2\pi)^d}\left[\frac{1}{\exp[\beta(k^2+m^2)^{1/2} - \beta\mu] - 1} - \frac{1}{\exp[\beta(k^2+m^2)^{1/2} + \beta\mu] - 1}\right], \tag{3}$$

which can be written as

$$n = a_d^{-1}\mathcal{W}_d(\alpha,\phi), \tag{4}$$

where

$$a_d = 2^{d-1}\pi^{d/2}\Gamma(d/2)m^{-d}g^{-1}, \tag{5}$$

$$\mathcal{W}_d(\alpha,\phi) = \int_0^\infty dx\, x^{d-1}\frac{\sinh(\alpha-\phi)}{\cosh a(x) - \cosh(\alpha-\phi)}. \tag{6}$$

The results (4), (5), and (6) agree with the result (8) of Ref. [7]. Similarly, we can write the energy density as

$$u = g \int \frac{d^3k}{(2\pi)^3}\left[\frac{(k^2+m^2)^{1/2}}{\exp[\beta(k^2+m^2)^{1/2} - \beta\mu] - 1} + \frac{(k^2+m^2)^{1/2}}{\exp[\beta(k^2+m^2)^{1/2} + \beta\mu] - 1}\right], \tag{7}$$

which in n dimensions can be written as

$$u = m a_d^{-1}\mathcal{Y}_d(\alpha,\phi), \tag{8}$$

where

$$\mathcal{Y}_d(\alpha,\phi) = \int_0^\infty dx\, x^{d-1} \frac{\alpha^{-1}a(x)\cosh(\alpha-\phi) - \sinh(\alpha-\phi) - \alpha^{-1}a(x)e^{-a(x)}}{\cosh a(x) - \cosh(\alpha-\phi)}. \quad (9)$$

To study the critical behavior of relativistic Bose gas, it is useful to perform the integrals in (3) in a closed form by expanding the integrands in infinite series and introducing modified Bessel functions in the resulting integrals [8,9]. To do this, we use the expansion

$$\frac{1}{\exp[(t^2+\alpha^2)^{1/2} - r\alpha] - 1} = \sum_{p=1}^{\infty} e^{r\alpha p} e^{-p(t^2+\alpha^2)^{1/2}}, \quad (10)$$

in the integrand of (3), where $r = \beta\mu = (\alpha - \phi)$. Inserting (10) in (3), and integrating term by term, we obtain

$$\mathcal{W}_d(\alpha,\phi) = b_d \sum_{p=1}^{\infty} p^{-(d'-1)} \sinh(p\alpha - p\phi) K_{d'}(p\alpha), \quad (11)$$

where $K_{d'}(p\alpha)$ is a modified Bessel function of order d' and the real argument $p\alpha$,

$$K_\nu(x) = \frac{(x/2)^2\Gamma(1/2)}{\Gamma(\nu+1/2)} \int_1^\infty dt (t^2-1)^{\nu-(1/2)} e^{-xt}, \quad (12)$$

and

$$b_d = \pi^{-1/2} 2^{d'} \Gamma(d/2) \overline{\alpha}^{(d'-1)}, \quad d' = \tfrac{1}{2}(d+1). \quad (13)$$

Similarly, for the energy density (8) and (9), we obtain

$$\mathcal{Y}_d(\alpha,\phi) = b_d \sum_{p=1}^{\infty} p^{-(d'-1)} [\cosh(p\alpha - p\phi) K_{d'+1}(p\alpha) - \sinh(p\alpha - p\phi) K_{d'}(p\alpha) - (p\alpha)^{-1} \times \cosh(p\alpha - p\phi) K_{d'}(p\alpha)]. \quad (14)$$

We can study the critical behavior of the relativistic Bose gas by studying the variation of μ, or equivalently ϕ, as a function of T. By standard argument of keeping charge density positive, μ must satisfy the condition $-m \leqslant \mu \leqslant m$. The parameter ϕ is, therefore, defined to be positive. From Eqs. (6) or (11), it is clear that as $\beta \to 0, \mu \to 0$, or $\phi \to \beta m$. Decreasing β increases μ until it reaches its limiting value m. The critical temperature is thus obtained from (4) for $\phi = 0$. Thus,

$$a_d n = \mathcal{W}_d(\alpha_c,0) = b_{dc} \sum_{p=1}^{\infty} p^{-(d'-1)} \sinh(p\alpha_c) K_{d'}(p\alpha_c), \quad (15)$$

where b_{dc} is the value of b_d at $\alpha_c = \beta_c m = m/T_c$. Using the large argument form of the function $K_\nu(z)$,

$$K_\nu(z) \approx \left[\frac{\pi}{2z}\right]^{1/2} e^{-z} \left[1 + \frac{4\nu^2 - 1}{8z} + \cdots\right], \quad (16)$$

we see that the ratio of the successive terms of the series (15) can be written as

$$\frac{u_{r+1}}{u_r} = 1 - \frac{d}{2r} + O\left(\frac{1}{r^2}\right), \quad r \to \infty, \quad (17)$$

so that by ratio test the series converges for $d > 2$. We therefore conclude that a nonzero T_c exists. *It is important to note that this result is independent of* α_c [$\equiv mc^2/(k_B T_c)$].

Using (4) and (6), we can determine the behavior of ϕ in the critical region. To do so, we expand $\mathcal{W}_d(\alpha,\phi)$ near $\phi = 0$, which can be obtained by calculating $\partial\mathcal{W}_d(\alpha,\phi)/\partial\phi$ at $\phi = 0$. We obtain

$$\partial\mathcal{W}_d(\alpha,\phi)/\partial\phi|_{\phi=0} = -b_d \sum_{p=1}^{\infty} p^{-(d'-2)} \cosh(p\alpha) K'_d(p\alpha). \quad (18)$$

This sum converges only for $d > 4$. The behavior for $2 < d < 4$ as $\phi \to 0$ can be obtained by calculating the derivative

$$\partial\mathcal{W}_d(\alpha,\phi)/\partial\phi = -b_d \sum_{p=1}^{\infty} p^{-(d'-2)} \cosh(p\alpha - p\phi) K'_d(p\alpha), \quad (19)$$

from which we get, asymptotically,

$$\partial\mathcal{W}_d(\alpha,\phi)/\partial\phi \approx -\frac{1}{2}(2/\alpha)^{d/2}\Gamma(d/2) F_{(d-2)/2}(\phi), \quad (20)$$

$$F_n(\phi) = \sum_{p=1}^{\infty} p^{-n} e^{-p\phi}, \quad (21)$$

$$F_n(\phi) \approx \Gamma(1-n)\phi^{n-1}, \quad n < 1, \phi \to 0. \quad (22)$$

Putting these results together, we can write

$$\partial\mathcal{W}_d(\alpha,\phi)/\partial\phi| \approx -\frac{1}{2}(2/\alpha)^{d/2}\Gamma(d/2)\Gamma(2-d/2)\phi^{(d-4)/2}, \quad (23)$$

for $2 < d < 4$. However, the quantity $\partial\mathcal{W}_d(\alpha,0)/\partial\alpha$ is calculated to be finite and negative at α_c. We can therefore write

$$\mathcal{W}_d(\alpha,0) \approx \mathcal{W}(\alpha_c,0) - (\alpha - \alpha_c)\mathcal{W}', \quad (24)$$

$$\mathcal{W}' \equiv -[d\mathcal{W}_d(\alpha,0)/d\alpha]_{\alpha=\alpha_c}. \quad (25)$$

Using (23) and (24) in (4), we can write

$$\phi \approx (C^+)^{-1} t^{2/(d-2)}, \quad 2 < d < 4, \quad t > 0, \quad (26)$$

$$C^+ = \left[\frac{2^{d/2}\Gamma(d/2)\Gamma(2-d/2)}{(d-2)\alpha_c^{(d+2)/2}\mathcal{W}'}\right]^{2/(d-2)}, \quad (27)$$

where we have written

$$t = (T - T_c)/T_c. \quad (28)$$

From the result (26) we note that the critical exponent is $2/(2-d)$, which is the same as that of the nonrelativistic Bose gas.

Thus, the critical behavior of the gas is the same in all regions, whether relativistic or nonrelativistic [10]. The only difference relates to the amplitude, the overall factor multiplying $[(T - T_c)/T_c]^{2/(d-2)}$ in (26). Finally, turning to the behavior for $T < T_c$, we note that $\phi = 0$ for $T < T_c$.

To see the connection of our general analytical results with the results of Ref. [7], we consider the nonrelativistic (NR) and the ultrarelativistic (UR) limit for the number density and the energy density of the Bose gas. These correspond to $\alpha_c \gg 1$ and $\alpha_c \ll 1$, respectively. From Eq. (15), we have

$$\mathcal{W}_d(\alpha_c,0) = b_{dc} \sum_{p=1}^{\infty} p^{-(d'-1)} \sinh(p\alpha_c) K_{d'}(p\alpha_c). \quad (29)$$

Using (16), we get for the nonrelativistic case

$$\mathcal{W}_d(\alpha_c,0) = \frac{1}{2}\left[\frac{2}{\alpha_c}\right]^{d/2} \Gamma(d/2)\zeta(d/2), \quad \alpha_c \gg 1. \quad (30)$$

Using the fact that $\mathcal{W}_d(\alpha_c,0) = a_d n$, and solving (30) for the critical temperature $\alpha_c = m/T_c$, we get

$$T_c = \frac{2\pi}{m}\left[\frac{n}{g\zeta(d/2)}\right]^{2/d}, \quad (31)$$

in the nonrelativistic region, which is the result (5) of Ref. [7].

We now consider the behavior of energy density in the nonrelativistic region. Using (8), (9), and (14), we can write for the energy density in the nonrelativistic region

$$u = m a_d^{-1} \mathcal{Y}_d(\alpha_c,0), \quad (32)$$

where

$$\mathcal{Y}_d(\alpha_c,0) = \frac{1}{2\alpha_c}\left[\frac{2}{\alpha_c}\right]^{d/2} \Gamma\left(\frac{d+2}{2}\right)\zeta(d/2), \quad \alpha_c \gg 1. \quad (33)$$

On the other hand, for the ultrarelativistic limit $\alpha_c \ll 1$, we can use the expansion [9]

$$K_\nu(z) \approx \frac{2^{\nu-1}\Gamma(\nu)}{z^\nu} \quad (34)$$

for $z \ll 1$ to obtain

$$\mathcal{W}_d(\alpha_c,0) = \frac{2}{\alpha_c^{d-1}} \Gamma(d)\zeta(d-1), \quad \alpha_c \ll 1, \quad (35)$$

which, together with the result (15), can be solved for the critical temperature to obtain

$$T_c = \left[\frac{n(2\pi)^d \Gamma(d/2)}{4gm\pi^{d/2}\Gamma(d)\zeta(d-1)}\right]^{1/(d-1)}, \quad \alpha_c \ll 1, \quad (36)$$

which is the same as the result (9) of Ref. [7]. Finally, in the UR limit the energy density can be written as

$$u = m a_d^{-1} \mathcal{Y}_d(\alpha_c,0), \quad (37)$$

$$\mathcal{Y}_d(\alpha_c,0) = \frac{2}{\alpha_c}(d-1)\Gamma(d)\zeta(d-1), \quad \alpha_c \ll 1. \quad (38)$$

In summary, we have solved the ideal, relativistic Bose gas in a closed form, and we have shown that the critical behavior of the gas is the same as for the standard nonrelativistic Bose gas. Our general results are applicable to all the regions of the Bose gas. We have shown that our general results reduce to the known results in the nonrelativistic and the ultrarelativistic regions. Furthermore, we have also obtained an analytical result for the energy density of the Bose gas in a closed form. We have thus provided a unified treatment of the Bose gas that is applicable in all regions of parameter space.

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