Dynamical systems theory for the Gardner equation

Aparna Saha,[1] B. Talukdar,[1,*] and Supriya Chatterjee[2]
[1]*Department of Physics, Visva-Bharati University, Santiniketan 731235, India*
[2]*Department of Physics, Bidhannagar College, EB-2, Sector-1, Kolkata 700064, India*
(Received 22 September 2013; revised manuscript received 8 January 2014; published 14 February 2014)

The Gardner equation $u_t + auu_x + bu^2u_x + \mu u_{xxx} = 0$ is a generic mathematical model for weakly nonlinear and weakly dispersive wave propagation when the effects of higher-order nonlinearity become significant. Using the so-called traveling wave ansatz $u(x,t) = \varphi(\xi)$, $\xi = x - vt$ (where v is the velocity of the wave) we convert the (1+1)-dimensional partial differential equation to a second-order ordinary differential equation in ϕ with an arbitrary constant and treat the latter equation by the methods of the dynamical systems theory. With some special attention on the equilibrium points of the equation, we derive an analytical constraint for admissible values of the parameters a, b, and μ. From the Hamiltonian form of the system we confirm that, in addition to the usual bright soliton solution, the equation can be used to generate three different varieties of internal waves of which one is a dark soliton recently observed in water [A. Chabchoub *et al.*, Phys. Rev. Lett. **110**, 124101 (2013)].

DOI: 10.1103/PhysRevE.89.023204 PACS number(s): 05.45.Yv, 42.81.Dp, 42.25.Lc, 02.70.Wz

I. INTRODUCTION

The Gardner equation

$$u_t + auu_x + buu_x^2 + \mu u_{xxx} = 0, \quad u = u(x,t), \tag{1}$$

for some specific values of the parameters a, b, and μ, first arose as an auxiliary mathematical equation in the derivation of the infinite set of conservation laws of the Korteweg–de Vries (KdV) equation [1]. Many important properties of (1), including its integrability by the inverse spectral method, can be realized in terms of the invertible transformation given by [2]

$$t' = \mu t, \quad x' = x + \frac{a^3}{4b}t,$$
$$u'(x',t') = \left(\frac{b}{4\mu}\right)^{1/2}\left(u(x,t) + \frac{a}{2b}\right). \tag{2}$$

Using (2) in (1), we obtain the modified KdV equation

$$u_t + 6u^2u_x + u_{xxx} = 0. \tag{3}$$

We have omitted the primes in writing (3). The spectral problem for the modified KdV equation was solved by Wadati [3] and we now know all the remarkable properties of it including the Lie symmetries, Lax pair, rational solutions, and soliton solutions [4]. Despite that, there exists a vast amount of literature [5] to obtain particular traveling wave solutions of (1) by the use of ansatz methods such as the tanh method and Jacobi elliptic function method. Such studies have mainly been motivated by the applicative relevance [6–8] of the equation. In particular, the Gardner equation has played a pivotal role in the description of large-amplitude internal waves observed in coastal oceans [9]. Absorption of solar radiation in the near surface layers in an ocean results in warmer water and lower density in that region and leads to stratified fluid. Any excitation in a zone within which sea water density changes maximally tends to propagate away from the region as an internal wave. During the past few years Grimshaw *et al.* [10] envisaged a large number of studies to generate physically important traveling internal waves by taking recourse to the use of different nonlinear evolution equations. Grimshaw *et al.* [11] discovered the inverse spectral method for (1) with positive cubic nonlinearity and made use of it to generate solitons and breathers.

In the present work we employ certain concepts from the dynamical systems theory [12] to investigate the physico-mathematical origin of the traveling wave solutions that the Gardner equation can support and thereby provide results for internal waves that might be physically relevant to different density stratification in the sea water. In addition to the usual bright soliton, the results presented by us include a dark soliton and some other traveling wave solutions that involve doubly periodic Jacobi elliptic functions and trigonometric functions.

To implement the methods of dynamical systems theory we consider the so-called traveling coordinate $\xi = x - vt$ and convert (1) to an ordinary differential equation by introducing

$$u(x,t) = \phi(\xi). \tag{4}$$

Here v stands for the velocity of the wave. From (1) and (4) we can write

$$a\varphi\phi' + b\phi^2\phi' + \mu\phi''' - v\phi' = 0, \tag{5}$$

where primes over ϕ denote appropriate derivatives with respect to ξ. In Sec. II we first try to solve (5) by imposing boundary conditions for the existence of a localized solution [13] and thus obtain the result for the bright soliton solution as reported in Ref. [11]. We then relax these conditions, find the equilibrium points of the equation, and study their nature using the method for linear stability analysis. The treatment presented here, on the one hand, leads to a useful constraint for the choice of the parameters of the equation and, on the other hand, provides a natural framework to visualize the relationship between the equilibrium points of (5) and solutions of the Gardner equation. We devote Sec. III to converting (5) to a Hamiltonian system and obtaining traveling wave solutions of the Gardner equation corresponding to different values of the explicitly-time-independent Hamiltonian or total energy of the system. Finally, in Sec. IV we summarize our outlook on the present work and make some concluding remarks.

*binoy123@bsnl.in

II. EQUILIBRIUM POINTS OF EQ. (5) AND THEIR STABILITY

Dividing (5) by μ, we integrate the resulting equation to write

$$\phi'' = \frac{c_1}{2} + \frac{v}{\mu}\phi - \frac{1}{2\mu}a\phi^2 - \frac{1}{3\mu}b\phi^3, \tag{6}$$

where c_1 is a constant of integration. On multiplication by ϕ', (6) can again be integrated to obtain

$$\phi'^2 = c_1\phi + \frac{v}{\mu}\phi^2 - \frac{1}{3\mu}a\phi^3 - \frac{1}{6\mu}b\phi^4 + c \tag{7}$$

with c a new constant of integration. If we now impose on (7) the boundary conditions for the existence of a localized solution [13], both constants c_1 and c are equal to zero. In this case (7) can be solved by simple quadrature. For values of a, b, and μ used in Ref. [11] we obtain

$$\phi(x,t) = \frac{v}{1 + \sqrt{1+v}\cosh[\sqrt{v}(x - vt)]}. \tag{8}$$

If we now take $v = 4\lambda^2$ with λ the eigenvalue of the spectral problem [11] used to solve the Gardner equation, we find that the result in (8) is in exact agreement with that given by Grimshaw *et al.*

If we do not restrict ourselves to the case of localized solutions, then $c_1 \neq 0$ in (6). In this case we can rewrite it as a system of two first-order differential equations given by, say,

$$\phi' = \psi = P(\phi,\psi), \quad \psi = \psi(\xi), \tag{9a}$$

$$\psi' = \frac{c_1}{2} + \frac{v}{\mu}\phi - \frac{1}{2\mu}a\phi^2 - \frac{1}{3\mu}b\phi^3 = Q(\phi,\psi). \tag{9b}$$

The equilibrium points of (6) can now be found by equating the right-hand side of each equation in (9) to zero. From (9a) we have $\psi = 0$ for all values of ϕ. As apparent from (9b), the values of ϕ can be obtained by solving an algebraic equation of the form

$$x^3 + \alpha x^2 + \beta x + \gamma = 0. \tag{10}$$

Three roots of the cubic equation in (10) will be real only if

$$(2\alpha^3 - 9\alpha\beta + 27\gamma^2)^2 \leqslant 4(\alpha^2 - 3\beta^2)^3. \tag{11}$$

Comparing (10) with $\psi' = 0$ in (9b), the condition in (11) can be rewritten as

$$\frac{b^4c_1^2\mu^2}{1296} - \frac{a^3b^2c_1\mu}{3888} - \frac{ab^3c_1\mu v}{648} - \frac{a^2b^2v^2}{3888} - \frac{b^3v^3}{729} \leqslant 0. \tag{12}$$

Henceforth we shall work with $a = 6$, $b = 6$, and $\mu = 1$, as used in Ref. [11]. With this choice for the parameters, the Gardner equation and inequality in (12) reduce to

$$u_t + 6uu_x + 6uu_x^2 + u_{xxx} = 0, \tag{13}$$

$$(c_1 - v)^2 - 2c_1 - \frac{4v^2}{3}\left(1 + \frac{2v}{9}\right) \leqslant 0. \tag{14}$$

The inequality in (14) is satisfied by a wide variety of values of c_1. Interestingly, for $c_1 = v$ the relation (14) is true for any value of v. In contrast, if we take any other value of c_1, the choice will impose restriction on the velocity of the traveling wave. In view of this, we shall work with $c_1 = v$ to obtain solutions of (13). In this context one may like to know our motivation for using the constraint in (12) or (14). The reason is fairly simple. If the condition (14) is satisfied, then all three roots of $\psi' = 0$ will be real to give all physically possible equilibrium points of the equation. We shall thus be able to obtain maximum information about the system modeled by the equation.

For $c_1 = v = 10$ we found (−2.919 37,0), (1.875 87,0), and (−0.456 51,0) as the stable or equilibrium points of the equation. Our choice for the value of c_1 is entirely arbitrary. It is of interest to examine how a perturbation grows if the system is disturbed infinitesimally near an equilibrium point. This can be done by calculating the eigenvalues λ of the Hessian determinant [12]

$$\Delta = \begin{vmatrix} \alpha - \lambda & \beta \\ \gamma & \delta - \lambda \end{vmatrix} \tag{15}$$

with

$$\alpha = \left(\frac{\partial P}{\partial \phi}\right)_{(\phi_0,\varphi_0)}, \quad \beta = \left(\frac{\partial P}{\partial \psi}\right)_{(\phi_0,\varphi_0)},$$
$$\gamma = \left(\frac{\partial Q}{\partial \phi}\right)_{(\phi_0,\varphi_0)}, \quad \delta = \left(\frac{\partial Q}{\partial \psi}\right)_{(\phi_0,\varphi_0)}. \tag{16}$$

Here (ϕ_0,ψ_0) stands for the equilibrium point. We found the imaginary eigenvalues $\lambda_{1,2} = \pm 4.860\,05i$ and $\lambda_{1,2} = \pm 4.7295i$ for the first two equilibrium points, while for the third one we got $\lambda_{1,2} = \pm 3.389\,49$. This implies that (−2.919 37,0) and (1.875 87,0) are center-type (case 1) equilibrium points and (−0.456 51,0) is of hyperbolic type (case 2). In the first case the perturbation neither decays to zero nor diverges, but it varies periodically with time such that the time evolution of the perturbation corresponds to a periodic solution of the differential equation. Here phase trajectories or trajectories in the (ϕ,ψ) plane form closed curves. In contrast, trajectories reach the hyperbolic equilibrium point (case 2) along two directions only and in all other directions the trajectories diverge from it. Any phase path that joins an equilibrium point to itself is a form of separatrix known as a homoclinic path. A separatrix is generally a phase path that separates obvious distinct regions in the phase plane. In the presence of homoclinic paths, a nonlinear differential equation can have localized solutions [12]. Since for (13) we have both center-type and hyperbolic-type equilibrium points, the equation will have both periodic and solitonic traveling wave solutions.

III. EQUATIONS (9) AS A HAMILTONIAN SYSTEM AND SOLUTION OF THE GARDNER EQUATION

In analogy with the form of Hamilton's equations of classical mechanics, a system of planar equations

$$\dot{x} = X(x,y), \quad \dot{y} = Y(x,y), \quad x = x(t),$$
$$y = y(t), \quad \dot{x} = \frac{dx}{dt}, \quad \dot{y} = \frac{dy}{dt} \tag{17}$$

is called a Hamiltonian system provided there exists a function $H(x,y)$ such that

$$X = \frac{\partial H}{\partial y}, \quad Y = -\frac{\partial H}{\partial x}. \tag{18}$$

Then H is called a Hamiltonian of the system. A necessary and sufficient condition for (17) to be Hamiltonian is that

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 0. \tag{19}$$

If we now use $c_1 = v$ and values of a, b, and μ as given in (13), the planar equations (9a) and (9b) become

$$\phi' = \psi, \tag{20a}$$

$$\psi' = \frac{v}{2} + v\phi - 3\phi^2 - 2\phi^3. \tag{20b}$$

Dividing (20b) by (20a), we can integrate the resulting equation to obtain a constant of integration c_2 as

$$c_2 = \psi^2 + \phi^4 + 2\phi^3 - v\phi^2 - v\phi. \tag{21}$$

In the traveling coordinate ψ can be regarded as the velocity of ϕ. It is thus reasonable to regard $\frac{c_2}{2}$ as the Hamiltonian $H(\phi,\psi)$ of the system. One can easily verify that the expression for $H(\phi,\psi)$ via an appropriate Hamilton equation such as those in (18) leads to (20a) and (20b). This confirms that the Gardner equation is a Hamiltonian system. Since $\psi = \frac{d\phi}{d\xi}$ we can write (21) as a first-order differential equation for ϕ, which permits separation of variables so as to write

$$\frac{d\phi}{\sqrt{\chi}} = d\xi, \tag{22}$$

with, say,

$$\chi = 2H - \phi^4 - 2\phi^3 + v\phi^2 + v\phi = f(\phi). \tag{23}$$

We shall now work with $v = 10$ and integrate (22) to obtain traveling solutions of the Gardner equation for different values of the Hamiltonian H.

Case 1. First we take $H = 3$. In this case the plot of χ as a function of ϕ is shown in Fig. 1. The curve $\chi = f(\phi)$ intersects the ϕ axis only at two real points marked by A and D in the figure such that two roots of the biquadratic equation $\chi = 0$ are real while two others (middle two) are complex. The numerical solution of $\chi = 0$ gives the roots as

$$\alpha_1 = -2.7838, \quad \alpha_{2,3} = -0.7894 \pm 0.5377i, \quad \alpha_4 = 2.3627. \tag{24}$$

The phase diagram (plot of ψ versus ϕ) drawn by using (21) is given in Fig. 2. It consists of a closed curve so that $\alpha_1(A)$ and $\alpha_4(D)$ on the ϕ axis correspond to its boundary points. In the presence of such a phase diagram the corresponding evolution equation has been found to possess a triangular traveling wave solution [14]. In the following we show that this is also true in the present case.

Writing (22) in the form

$$\frac{d\phi}{\sqrt{\phi^4 + 2\phi^3 - v\phi^2 - v\phi - 2H}} = i\,d\xi, \tag{25}$$

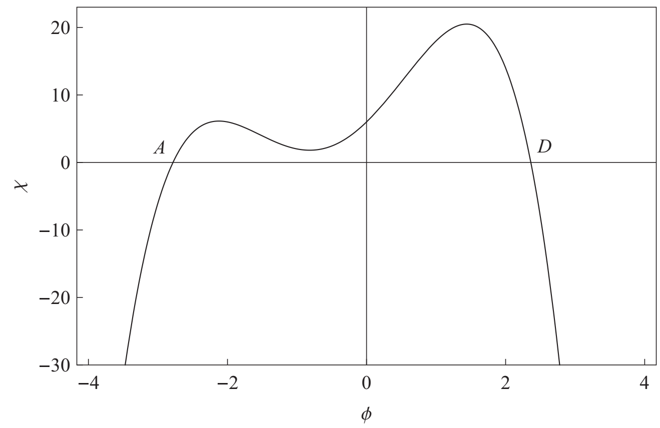

FIG. 1. Hamiltonian $H = 3$. Here χ is a function of ϕ showing that the equation $\chi = 0$ has two real roots indicated by A and D and two other middle roots are imaginary.

we can integrate the equation by using the general result [15]

$$\int \frac{dx}{\sqrt{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)}} = \frac{2}{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}\mathrm{sn}^{-1}(z,\beta), \tag{26}$$

with

$$z = \sqrt{\frac{(\alpha_2-\alpha_3)(\alpha_1-x)}{(\alpha_1-\alpha_3)(\alpha_2-x)}} \tag{27a}$$

and

$$\beta = \frac{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}{(\alpha_3-\alpha_2)(\alpha_1-\alpha_4)}. \tag{27b}$$

Here $\mathrm{sn}^{-1}(\cdot)$ stands for the inverse Jacobi sine function. From (24)–(26) we find the traveling wave solution of the Gardner equation for $H = 3$ in the form

$$\phi = \frac{\rho_1 + \rho_2 \mathrm{sn}^2(\kappa\xi,\nu)}{\rho_3 + 5.6133\,\mathrm{sn}^2(\kappa\xi,\nu)}, \tag{28}$$

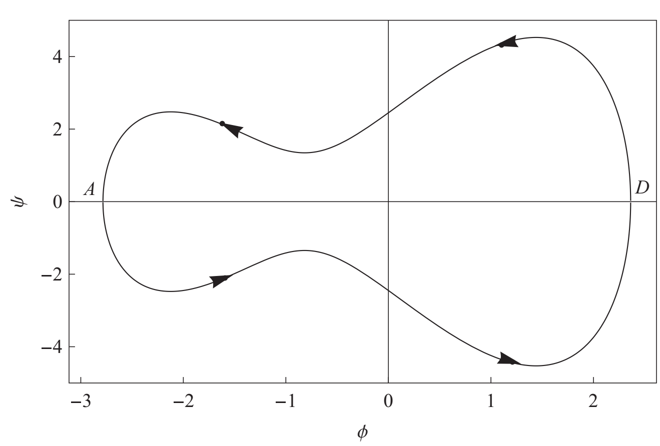

FIG. 2. Phase diagram from (21) for $H = 3$ or $c_2 = 6$ showing the closed curve with the roots $\alpha_1(A)$ and $\alpha_4(D)$ at its boundary.

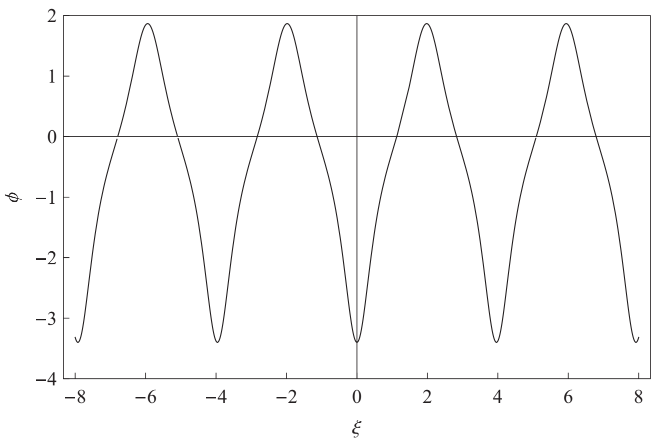

FIG. 3. Solution $\phi(\xi)$ from (28) as a function of ξ.

with

$$\rho_1 = 7.6112 + 2.7492i, \quad \rho_2 = 1.3117 + 4.2589i, \quad \rho_3 = -2.2389 - 0.8087i \tag{29a}$$

and

$$\kappa = -0.3867 + 1.2916i, \quad \nu = 0.6040 + 0.9183i. \tag{29b}$$

In Fig. 3 we display ϕ of (28) as a function of ξ. Clearly, the traveling wave solution closely resembles the so-called triangular wave. The Fourier series representation of a triangular wave is given by [14]

$$U = \sum_{k=0}^{\infty} (-1)^k \frac{\sin(2k+1)\xi}{(2k+1)^2}. \tag{30}$$

The plot of U as a function of ξ is symmetric about the ξ axis. However, $\phi(\xi)$ in Fig. 3 is not symmetrical about the ξ axis. Rather it behaves more like an internal wave.

Case 2. For $H = 2$ the plot of χ as a function of ϕ is shown in Fig. 4. Looking closely at this figure we see that the curve $\chi = f(\phi)$ has undergone a lower vertical displacement with respect to that in Fig. 1 such that the ϕ axis is now a tangent

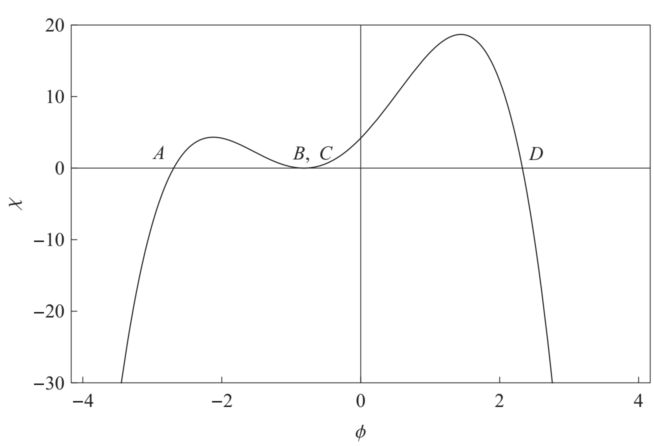

FIG. 4. Hamiltonian $H = 2$. Here χ is a function of ϕ showing that the equation $\chi = 0$ has two real roots indicated by A and D and only one real root in between them (point B or C).

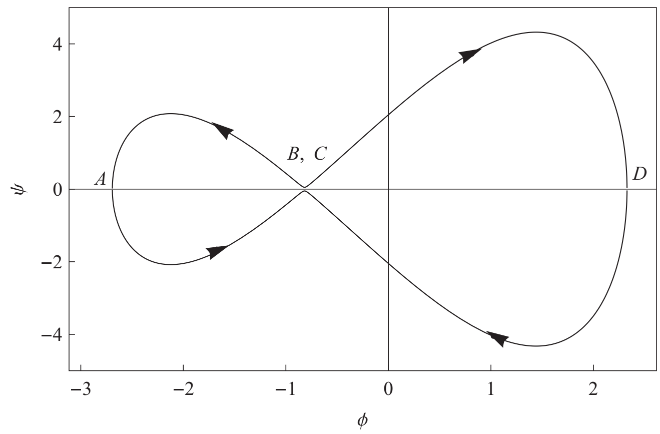

FIG. 5. Phase diagram from (21) for $H = 2$ showing a homoclinic phase path that enters into or emerge from the saddle or hyperbolic equilibrium point $(\alpha_2,0)$ denoted by B or C in the figure.

to the minimum of the curve. We have labeled the point of contact by B or C. In this case the roots of the equation $\chi = 0$ are given by

$$\alpha_1 = -2.692\,23, \quad \alpha_2 (= \alpha_3) = -0.8175, \quad \alpha_4 = 2.3272. \tag{31}$$

The corresponding phase diagram is presented in Fig. 5. The phase trajectories join the equilibrium point $(\alpha_2,0)$ to itself and thereby complete the homoclinic path. With α_2 as a repeated root we will now have to integrate (25) in order to find $\phi(\xi)$. Unfortunately, the integral (26) becomes undefined in this case and cannot be used to get the solution. We therefore make use of another integral given by [15]

$$\int \frac{dx}{\sqrt{(x-\alpha_1)(x-\alpha_2)^2(x-\alpha_4)}} = \ln\left(\frac{a_+(x)[c(x) + \alpha_1 a_-(x) - b_-(x)]}{a_-(x)[c(x) + \alpha_1 a_+(x) - b_+(x)]}\right), \tag{32}$$

where

$$a_\pm(x) = \sqrt{\alpha_2 - \alpha_1} \pm \sqrt{x - \alpha_1}, \tag{33a}$$

$$b_\pm(x) = \alpha_4 \sqrt{\alpha_2 - \alpha_1} \pm \alpha_2 \sqrt{x - \alpha_1}, \tag{33b}$$

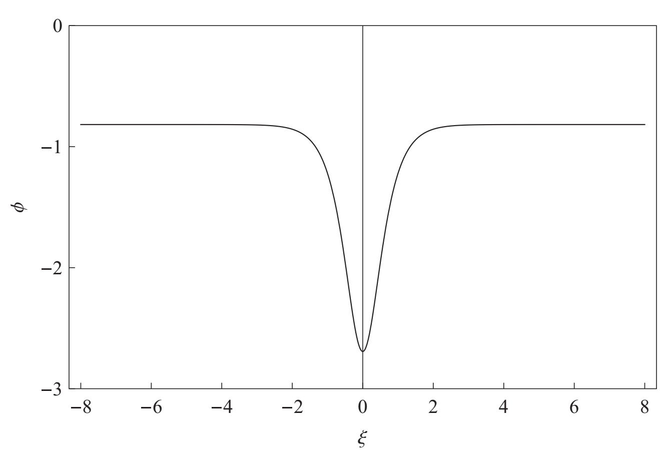

FIG. 6. Solution $\phi(\xi)$ of (35) as a function of ξ.

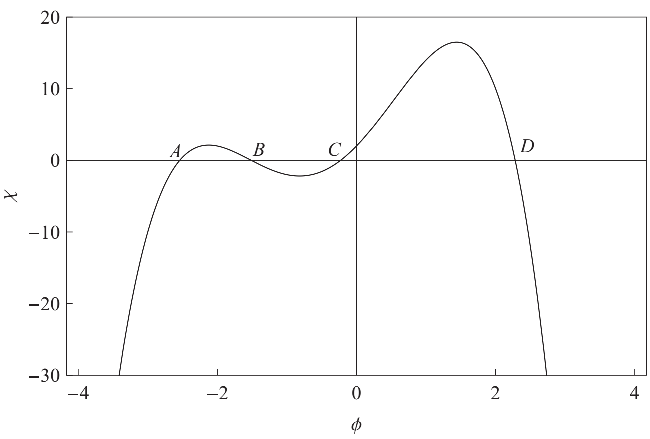

FIG. 7. Hamiltonian $H = 1$. Here χ is a function of ϕ showing that the equation $\chi = 0$ has four distinct real roots indicated by points A, B, C, and D.

and

$$c(x) = \sqrt{(\alpha_2 - \alpha_1)(\alpha_2 - \alpha_4)(x - \alpha_4)}. \tag{34}$$

From (25), (31), and (32) we obtain

$$\phi = -\frac{2.1953 + 1.0757\sinh^2(1.1965\xi)}{0.8154 + 1.3159\sinh^2(1.1965\xi)}. \tag{35}$$

We plot in Fig. 6 $\phi(\xi)$ of (35) as a function of ξ. Here the internal traveling wave corresponds to a dark soliton. Dark solitons have been experimentally observed in a number of systems ranging from optics to plasmas [16], but not in water. Only recently, Chabchoub *et al.* [17] found these solitons on the surface of water.

Case 3. For $H = 1$ the plot of χ as a function of ϕ as displayed in Fig. 7 shows that the curve $\chi = f(\phi)$ intersects the ϕ axis at four different points denoted by A, B, C, and D.

Note that in Fig. 4 for $H = 2$ points B and C were coincident. In the present case, however, these two points are separated by a distance BC. The separation resulting from the vertical displacement of the curve $\chi = f(\phi)$ leads to two different closed curves in the phase space (Fig. 8).

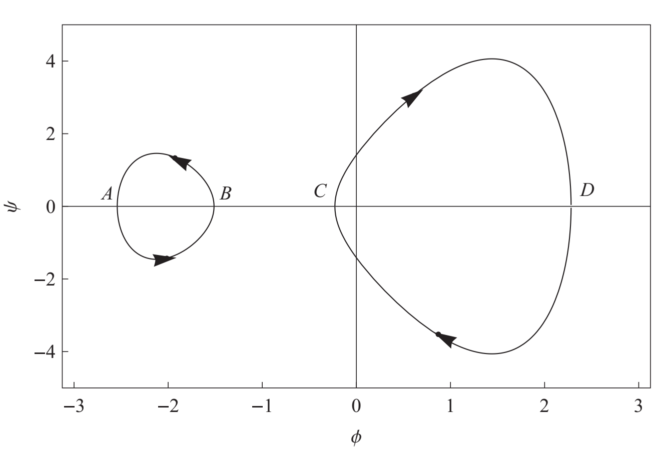

FIG. 8. Phase diagram from (21) for $H = 1$ or $c_2 = 2$ showing two different closed curves separated by a distance BC.

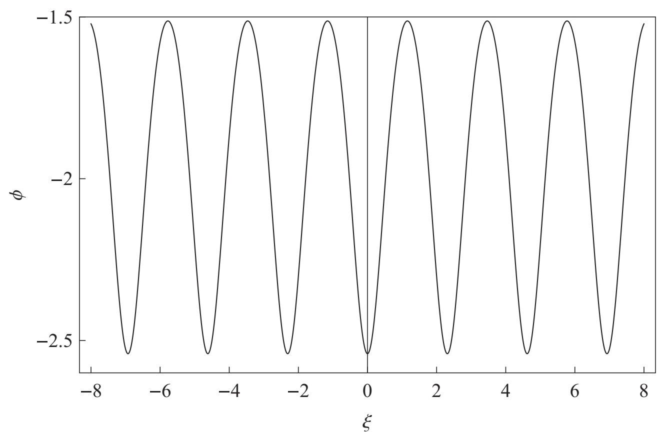

FIG. 9. Solution $\phi(\xi)$ of (35) as a function of ξ.

The roots of $\chi = 0$ are now given by

$$\begin{aligned} \alpha_1 &= -2.5408, \quad \alpha_2 = -1.5125, \\ \alpha_3 &= -0.2281, \quad \alpha_4 = 2.2814. \end{aligned} \tag{36}$$

From (25), (26), and (36) we get the trigonometric solution

$$\phi = \frac{1.5297 - 1.6397\sin^2\theta}{-0.6020 + 1.0841\sin^2\theta}, \tag{37}$$

with

$$\theta = \mathrm{am}(1.2343i\xi, 1.417). \tag{38}$$

Here am$(\cdot)$ stands for the Jacobi amplitude function. We display in Fig. 9 the trigonometric solution (37) as a function of ξ. As expected, the curve represents an oscillatory solution of the Gardner equation. We now examine what happens if we choose to work with $H = 0$ and negative values of the Hamiltonian.

Case 4. For $H = 0$ we plot χ as a function of ϕ in Fig. 10. By comparing this plot with that in Fig. 7 we see that points A and B have now merged into one point A or B. Consequently, the equation $\chi = 0$ will have one repeated root. The values of the roots are

$$\alpha_1(\alpha_2) = -2.1150, \quad \alpha_3 = 0.0099, \quad \alpha_4 = 2.2337. \tag{39}$$

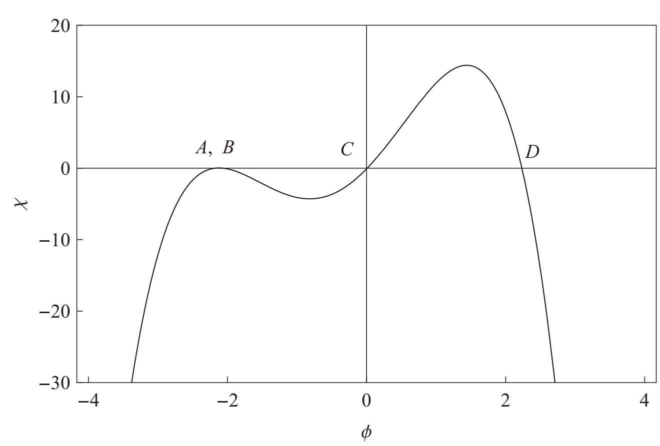

FIG. 10. Hamiltonian $H = 0$. Here χ is a function of ϕ showing that the points A and B of Fig. 7 have merged at A or B.

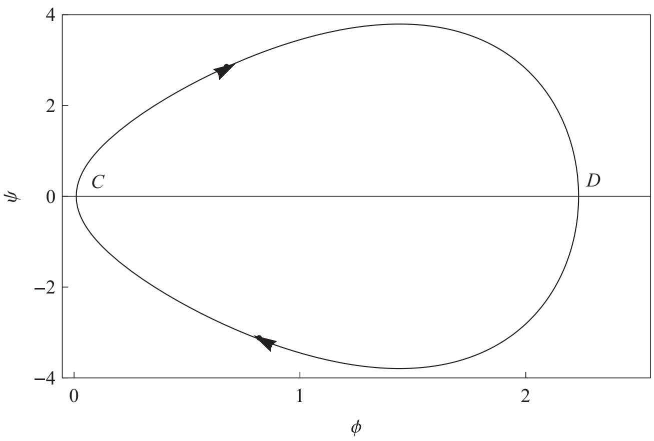

FIG. 11. Phase diagram from (21) for $H = 0$ showing one closed curve.

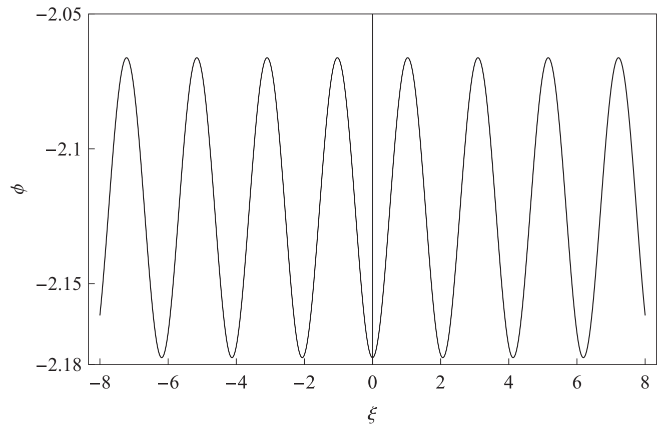

FIG. 12. Solution $\phi(\xi)$ of (40) as a function of ξ.

We present in Fig. 11 the phase plot obtained from (21). The phase trajectory consists of a single closed curve. Comparing the curves in Figs. 8 and 11, it appears that one of the closed loops (small one on the left) in Fig. 8 has disappeared in the phase diagram for $H = 0$. Using the values of the roots in (39), the solution of the Gardner equation can be found as

$$\phi = \frac{4.9148 - 4.7844 \sin\theta'}{-2.2572 + 2.3155 \sin\theta'}, \tag{40}$$

with

$$\theta' = \mathrm{am}(1.5334 i\xi, 0.9737). \tag{41}$$

Figure 12 gives ϕ of (40) as a function of ξ. From Figs. 9 and 12 it is clear that curves in both figures look alike; however, there is a striking difference. The amplitude of the wave represented by (35) is about ten times larger than that represented by (40). This implies that the solution found for $H = 0$ represents a really feeble wave. Thus it remains an interesting curiosity to examine if the Gardner equation for our chosen values of the parameters could support solutions for $H < 0$. It is easy to verify that for $H < 0$, ψ in (21) is complex for all values of ϕ such that we do not have any phase trajectory in this case. Consequently, there exists no physical solution for negative values of H.

IV. CONCLUSION

The Gardner equation can model a wide variety of physical phenomena that appear in plasma physics [6], stratified fluid flows [8], and quantum fluid dynamics [7]. In view of this, many special methods have been used to obtain solutions of the Gardner equation. Inverse spectral method has also been discovered by Grimshaw *et al.* [11] with a view to generate solitons and breathers in the equation. Based on some elementary concepts in the dynamical systems theory, we present in this work a direct method to construct different types of solutions supported by the equation. Using the so-called traveling coordinate we convert (1) to an ordinary differential equation. The stable points of the converted equation provide, in a rather natural way, a useful constraint for the physically admissible parameters of the Gardner equation. The parameters used in [11] are found to obey the constraint relation. For these parameters we reduced the Gardner equation to the Hamiltonian form and subsequently presented its traveling wave solutions for different values of the Hamiltonian H.

The results presented by us for $H = 3$, 2, and 1 are found in terms of Jacobi elliptic, hyperbolic, and trigonometric functions, respectively. The choice $H = 0$ led to an extremely weak trigonometric solution and there exists no physically acceptable solution for $H < 0$. The Gardner equation considered here can provide us with only three types of internal waves as found by us. The Jacobi elliptic function solutions can be obtained for all values of H satisfying $3 \geqslant H > 2$. The solution for $H > 3$ can also result in terms of the Jacobi elliptic function. Hyperbolic function solutions in the form of dark solitons can be found for $2 \geqslant H > 1$. A trigonometric traveling wave solution of appreciable amplitude can appear for $1 \geqslant H > 0$. The dark solitons reported here have not been observed as internal waves, although recently these have been generated on the surface of water [17]. The numerical values of the coefficients a, b, and μ in (1) depend on the stratification feature and water depth [18]. The theoretical model presented in this work is quite straightforward such that it can be judiciously used to generate internal waves corresponding to any form of stratification and any depth of water.

[1] R. M. Muira, C. S. Gardner, and M. D. Kruskal, J. Math. Phys. **9**, 1204 (1968).

[2] N. A. Kudryashov and D. I. Sinelshchikov, Acta Appl. Math. **113**, 41 (2011).

[3] M. Wadati, J. Phys. Soc. Jpn. **32**, 1681 (1972).

[4] M. Lakshmanan and K. M. Tamizhmani, J. Math. Phys. **26**, 1189 (1985).

[5] W. Mafliet and W. Hereman, Phys. Scr. **54**, 563 (1996); S. Liu, Z. Fu, S. Liu, and Q. Zhaoa, Phys. Lett. A **289**, 69 (2001); Z. Fu, S. Liu, S. Liu, and Q. Zhaoa, *ibid.* **290**, 72 (2001);

Z. Fu, S. Liu, and S. Liu, Chaos, Solitons Fractals **20**, 301 (2004); A. M. Wazwaz, Appl. Math. Comput. **187**, 1131 (2007); A. Bakir, Commun. Nonlinear Sci. Numer. Simulat. **14**, 1038 (2009).

[6] S. Watanabe, J. Phys. Soc. Jpn. **53**, 950 (1984); M. S. Ruderman, T. Talipova, and E. Pelinovsky, J. Plasma Phys. **74**, 639 (2008).

[7] E. Demler and A. Maltsev, Ann. Phys. (NY) **326**, 1775 (2011).

[8] R. Grinshaw, *Internal Solitary Waves in Environmental Stratified Flows*, edited by R. Grimshaw (Kluwer, Dordrecht, 2002), p. 1.

[9] K. R. Helfrich and W. K. Melville, Annu. Rev. Fluid Mech. **38**, 395 (2006).

[10] R. Grimshaw, D. Pelinovsky, E. Pelinovsky, and A. Slunyaev, Chaos **12**, 1070 (2002); R. Grimshaw, E. Pelinovsky, T. Talipova, M. Ruderman, and R. Erdélyi, Stud. Appl. Math. **114**, 189 (2005); R. Grimshaw, D. Pelinovsky, E. Pelinovsky, and T. Talipova, Physica D **159**, 35 (2001).

[11] R. Grimshaw, A. Slunyaev, and E. Pelinovsky, Chaos **20**, 013102 (2010).

[12] D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations*, 4th ed. (Oxford University Press, Oxford, 2007).

[13] A. Das, *Integrable Model*, World Scientific Lecture Notes in Physics Vol. 30 (World Scientific, Singapore, 1989).

[14] B. Talukdar, S. K. Ghosh, A. Saha, and D. Pal, Phys. Rev. E **88**, 015201 (2013).

[15] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, 1927).

[16] Y. S. Kivshar and B. Luther-Davies, Phys. Rep. **298**, 81 (1998).

[17] A. Chabchoub, O. Kimmoun, H. Branger, N. Hoffmann, D. Proment, M. Onorato, and N. Akhmediev, Phys. Rev. Lett. **110**, 124101 (2013).

[18] E. Pelinovsky, A. V. Slunyaev, O. E. Polukhina, and T. G. Talipova, in *Solitary Waves in Fluid*, edited by R. Grimshaw (WIT Press, Southampton, 2007), p. 85.