

Core solutions of rigidly rotating spiral waves in highly excitable media

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Analytical spiral wave solutions for reaction-diffusion equations play an important role in studying spiral wave dynamics. In this paper, we focus on such analytical solutions in the case of highly excitable media. We present numerical evidence that, for rigidly rotating spiral waves in highly excitable media, the species values in the spiral core region do harmonic oscillations but not relaxation ones, and their amplitudes grow linearly with the distance from the rotation center. An analytical solution is proposed to describe such spiral wave dynamics, and the quantitative comparisons between the numerical results and the analytical solutions show that the proposed spiral core solution works well in highly excitable media.

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I. INTRODUCTION

Rotating spiral waves and spiral formations are ubiquitous in diverse physical, chemical, and biological systems. They have been found in the Belousov-Zhabotinsky (BZ) reaction [1–3], in the catalytic oxidation of CO on a Pt(110) surface [4], during the aggregation of slime mold amoebae [5] and in the cardiac tissue [6]. Such self-sustained spiral wave activity also plays an essential role in cardiac arrhythmia and fibrillation [7–11].

Theoretical investigations of spiral waves have been performed predominantly in reaction-diffusion (RD) systems. A general RD system in two dimensions can be described by the following partial differential equations:

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{F}(\mathbf{u}, \lambda) + \mathbf{D} \nabla^2 \mathbf{u}, \quad (1)$$

where $\mathbf{u}(\mathbf{r}, t)$ is a space- and time-dependent vector describing the various chemical or biological species, $\mathbf{F}$ is a kind of nonlinear vectorial function representing the chemical or biological kinetics, $\mathbf{D}$ generally is a diagonalized diffusion matrix, and λ is a control parameter.

When the parameter choice yields oscillatory motion, and is close to the supercritical Hopf bifurcation ($\lambda = \lambda_0$), the spiral wave solution for the RD system is of the form [12]

$$u_i = u_{0i} + \sqrt{\lambda - \lambda_0} R(T, \mathbf{r}) \eta_i \cos[\Theta(T, \mathbf{r}) + \omega t - \xi_i] + O(\lambda - \lambda_0). \quad (2)$$

Here, u_{0i} , ω , η_i , and ξ_i are coefficients and $T = (\lambda - \lambda_0)t$. The evolution equations for $R(T, \mathbf{r})$ and $\Theta(T, \mathbf{r})$, including their solutions, have been extensively studied in Ref. [12].

It is shown that, in polar coordinates (r, θ) , there are $R = c_1 r$ and $\Theta = c_2 T - \sigma \theta + \psi_0$ as $r \rightarrow 0$ [12], where c_1 , c_2 , and ψ_0 are coefficients, and σ denotes the chirality of the spiral wave ($\sigma = +1$ for an anticlockwise spiral wave, and $\sigma = -1$ for a clockwise one). Then the spiral core solution in oscillatory systems close to the supercritical bifurcation has the form

$$u_i = u_{0i} + \rho_i r \cos(-\sigma \theta + \omega_0 t - \alpha_i) + O(r^2) + O(\lambda - \lambda_0). \quad (3)$$

Here, $\rho_i = c_1 \eta_i \sqrt{\lambda - \lambda_0}$, $\omega_0 = \omega + c_2(\lambda - \lambda_0)$, and $\alpha_i = \xi_i - \psi_0$.

As we know, exploring specific forms of the spiral wave solutions in excitable systems, especially analytical solutions around the spiral core, plays an important role in studying spiral wave dynamics. However, in excitable media, although the theories describing spiral waves have well been developed [13–20], the analytical spiral wave solutions for RD equations have little been investigated.

In this paper, we investigate the core properties of the rigidly rotating spiral waves in highly excitable media. We provide the numerical evidence that the values of species u_i in the spiral core region act like some harmonic oscillations with time, and their amplitudes grow linearly with the distance from the rotation center. Therefore, the behaviors of species u_i near the rotation center are similar to the ones described by Eq. (3) for spiral core in oscillatory system which is close to the supercritical Hopf bifurcation. An analytical spiral core solution analogous to Eq. (3) is proposed to describe the dynamics of the core of the rigidly rotating spiral wave in highly excitable media. To examine the validity of the core solution, quantitative comparisons between the numerical results and the analytical solution are also performed, after the coefficients of the core solution are fixed with a high degree of precision.

The numerical findings around the spiral core are presented in Sec. II, including the oscillation forms of the species and the relations between the amplitudes and the distance from the rotation center. The proposed analytical core solution is given in Sec. III, and the other unknown coefficients of the solution are fixed. Besides, in Sec. III, the quantitative comparisons between the numerical results and the analytical solution, and the parameter region where the core solution is valid, are also discussed.

II. NUMERICAL CORE SOLUTION

In this work, a classic two-variable FitzHugh-Nagumo (FHN) model is considered [21,22],

$$\begin{aligned} \frac{\partial u}{\partial t} &= f(u, v)/\varepsilon + D_u \nabla^2 u, \\ \frac{\partial v}{\partial t} &= \varepsilon g(u, v), \end{aligned} \quad (4)$$

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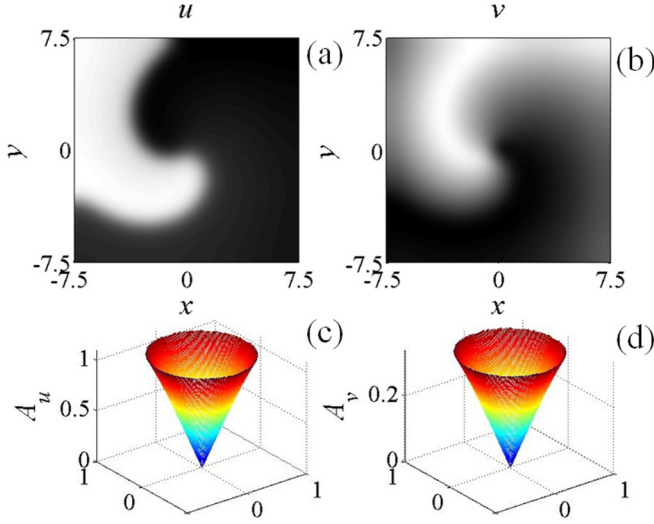


FIG. 1. (Color online) (a), (b) Snapshots of the spatial distributions of the fast variable u and the slow variable v . (c), (d) Amplitude spatial distributions for u and v , respectively, around the rotation center, where A_u and A_v denote the corresponding oscillation amplitudes for u and v .

where species u and v are the fast activator and the slow inhibitor, respectively, whose time ratio is ε , and the diffusion coefficient D_u is set to 1 for this single-diffusive system. The kinetics of this model are given by $f(u, v) = (u - u^3/3 - v)$ and $g(u, v) = (u + \beta - \gamma v)$. Here parameters β and γ control the properties of the medium.

In the simulation, the chosen parameters for excitable medium are $\varepsilon = 0.2$, $\beta = 0.5$, and $\gamma = 0.8$. The same form of the FHN model as Eq. (4) with those chosen parameters is also discussed in Ref. [23], which shows that, this way can form a rigidly rotating spiral wave with the radius of the spiral tip trajectory being zero. Equation (4) is integrated on a 1201×1201 uniform grid with no-flux boundary conditions via Euler algorithm. In order to study the fine structures around the spiral core, small space step and time step are used ($\Delta x = \Delta y = 0.0125$ and $\Delta t = 2.5 \times 10^{-5}$), and a nine-point finite difference scheme is also applied to compute the Laplacian term $\nabla^2 u$.

Spatial distributions of the fast variable u and the slow variable v are shown in Figs. 1(a) and 1(b). For all of the grid points in the discretized medium, both values of u and v keep oscillating with time, and the oscillation situation around the rotation center can be known from the amplitude spatial distributions of u and v . From Figs. 1(c) and 1(d) we know that near the rotation center, there is a grid point whose amplitudes of both u and v always remain in a minimal level, and they increase linearly with the distance from the rotation center. For convenience, Cartesian coordinates (x, y) are established in the medium plane with the minimum oscillation grid point being the origin $(0, 0)$.

Although the values u and v on all of the medium grid points keep oscillating, their oscillating forms are quite different for different parts of the medium. Relaxation oscillation, a typical characteristic for excitable systems, dominates most regions of the medium, especially in the region far away from the

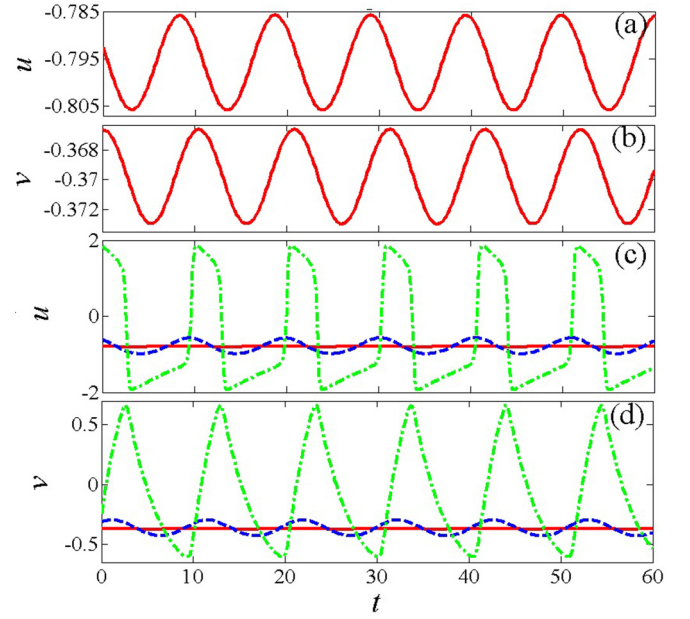


FIG. 2. (Color online) Plots of u - t and v - t for the grid points in the medium. (a), (b) Plots for the minimum oscillation grid point $(0, 0)$. (c), (d) Cases for the grid point $(10\Delta x, 10\Delta y)$ (dashed lines) and $(1200\Delta x, 1200\Delta y)$ (dot-dashed lines). And for comparison, solid lines are still for grid point $(0, 0)$, which is the same as (a) and (b).

rotation center [see dot-dashed lines in Figs. 2(c) and 2(d)]. However, in the region around the rotation center, harmonic oscillation, which is thought to be a typical property for oscillatory media close to the supercritical Hopf bifurcation, is found [see Figs. 2(a), 2(b) and dashed lines in Figs. 2(c) and 2(d)]. Due to the monotone relations between the u , v amplitudes (A_u , A_v) and the distance r , the oscillations for the minimum oscillation grid point are extremely small and almost negligible compared to those for the grid points which are far away from the rotation center [solid lines in Figs. 2(c) and 2(d) are almost straight].

The above two different kinds of oscillation forms can also be reflected on the u - v phase plane (Fig. 3). For typical relaxation oscillation, the u - v trajectories for grid points on the edge of the medium approximately look like parallelograms [see Fig. 3(a)], while, as for harmonic oscillation, the u - v trajectories appear to be elliptical [see Fig. 3(b)]. On the other hand, the intersection of the two nullclines [$f(u, v) = 0$ and $g(u, v) = 0$] is on the left side of the point C which is the minimal value point for the nullcline $f(u, v) = 0$, and the intersection is close to the point C [see Fig. 3(a)]. Therefore, one can see that the medium we study is highly excitable, which is quite susceptible to perturbations which could cause an excitation.

III. ANALYTICAL CORE SOLUTION

From the core solution (3) for spiral waves in oscillatory systems close to the supercritical Hopf bifurcation, we know that the system species u_i do harmonic oscillations with different initial phases and their amplitudes are proportional to the distance from the rotation center ($r = 0$). In the study of the rigidly rotating spiral waves in highly excitable media, our

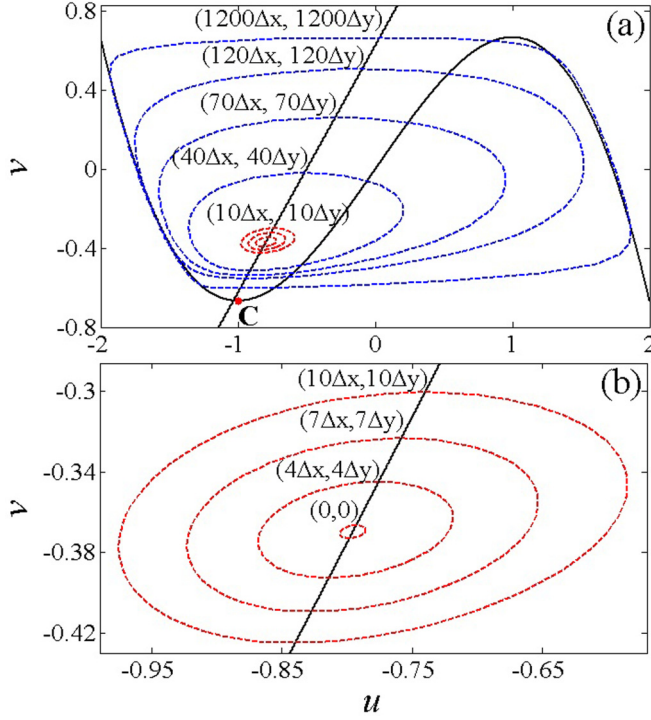


FIG. 3. (Color online) (a) u - v phase plane for excitable system. Solid lines are two nullclines [$f(u,v)=0$ and $g(u,v)=0$]. Dashed lines are trajectories parametrized by time for different grid points in the medium. Point C is the minimal value of nullcline $f(u,v)=0$. (b) Detail information of the central part in (a), and the innermost circle is for the origin $(0,0)$.

numerical simulations show that the behaviors of the species u and v are analogous to the ones described by Eq. (3) in the oscillatory systems close to the supercritical Hopf bifurcation, including the harmonic oscillations and the linear relations between the amplitudes and the distance from the rotation center (see Figs. 1 and 2). Here, one can see the spiral core solutions in both cases should have a similar form. Therefore, we assume that, for a rigidly rotating spiral wave in highly excitable media, the spiral core solution has the form

$$\begin{aligned} u &= u_0 + \rho_1 r \cos(-\sigma\theta + \omega_0 t - \alpha_1) + O(r^2), \\ v &= v_0 + \rho_2 r \cos(-\sigma\theta + \omega_0 t - \alpha_2) + O(r^2), \end{aligned} \quad (5)$$

where $r = \sqrt{(x - x_0)^2 + (y - y_0)^2}$, $\theta = \arctan[(y - y_0)/(x - x_0)]$, and (x_0, y_0) is the rotation center in the defined coordinates. ω_0 is the angular velocity of the rotating spiral, u_0, v_0, ρ_1 , and ρ_2 are four coefficients, and α_1 and α_2 are two initial rotation phases.

Now we can perform quantitative comparisons between the results obtained numerically and the ones calculated from the spiral core solution (5). First, from numerical simulation, the angular velocity ω_0 of the anticlockwise spiral is measured to be 0.6059 and σ is +1. According to the u and v oscillation situation around the rotation center, the other unknown coefficients of the core solution (5) can be fixed as $u_0 = -0.7956$, $v_0 = -0.3697$, $\rho_1 = 1.616$, $\rho_2 = 0.5153$, $\alpha_1 = 3.138$, $\alpha_2 = 4.450$, $x_0 = -0.00473$, and $y_0 = 0.00391$. Here, we should note that these coefficients are measured with

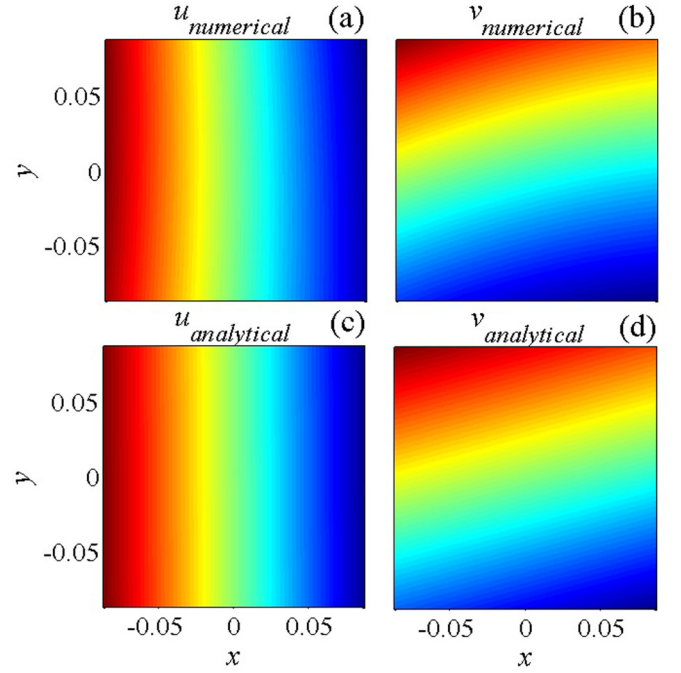


FIG. 4. (Color online) (a), (b) u and v patterns at $t = 0$ are obtained by linear interpolation with the numerical data of 15×15 grid points around the rotation center, which are gotten from the integration of Eq. (4). (c), (d) The corresponding u and v patterns calculated from the core solution (5) at $t = 0$.

high precision, and the maximum relative standard deviation of them is less than 1% (see the Appendix for details).

Then, as assumed in Ref. [15], the size of the spiral core region is ε , and in our simulation, the number of the involved grid points is about $N = \varepsilon/\Delta x = 16$. Figures 4(a) and 4(b) are u and v patterns obtained by linear interpolation with numerical data of 15×15 grid points around the rotation center, which are gotten from the integration of Eq. (4), and Figs. 4(c) and 4(d) are the corresponding ones calculated from the core solution (5) at the same moment. One can see that the u and v patterns gotten from solution (5) are in good agreement with the numerical results. Quantitative differences between these two sets of patterns are relatively small; even the maximum difference is less than 0.005 [see Figs. 5(a) and 5(b)] and the relative difference is less than 2% [see Figs. 5(c) and 5(d)]. It thus indicates that solution (5) works well in the neighborhood of the rotation center.

In our further study, harmonic oscillations near the rotation center of the spiral wave are also found in the Barkley model [24] for highly excitable media. And a similar phenomenon is also reported in the Luo-Rudy model for cardiac tissue [25]. Therefore, the core solution (5) may be model independent in describing rigidly rotating spiral waves in highly excitable media. Note that solutions similar to Eq. (5) had been applied to analyze the spiral dynamics near the rotation center in excitable systems [26–28]. In this paper, we further discuss the reason for using this solution and its properties. After appropriate fitness, coefficients of the solution (5) can be fixed with a high degree of precision, and the validity of the solution is further supported by the good

but not relaxation ones, and their amplitudes grow linearly with the distance from the rotation center. Analytical solution (5) is proposed to describe the spiral dynamics around the core. After the coefficients of solution (5) are measured with a high degree of precision, quantitative comparisons are performed between the simulation results and the analytical solutions. These two sets of results turn out to be in good agreement with each other, which shows that the core solution (5) can work well in describing the dynamics of rigidly rotating spirals in highly excitable media. The same conclusion can also be obtained in the study of the Barkley model. Therefore, the spiral core solution (5) may be model independent in describing the dynamics of rigidly rotating spiral waves in highly excitable media.

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APPENDIX: MEASUREMENT OF SOME PARAMETERS OF SOLUTION (5)

1. Coefficients u_0 and v_0

From the spiral core solution (5), Figs. 2 and 3, we know for the considered small area around the rotation center that, u_0 and v_0 are the mean values of the u and v harmonic oscillations, respectively. Table I lists the u_0 and v_0 values measured from the grid points around the rotation center. $\bar{\mu}$, S , and S_r denote the average value, the standard deviation [$S = \sqrt{\sum_{i=1}^N (\mu_i - \bar{\mu})^2 / (N - 1)}$], and the relative standard deviation ($S_r = S / |\bar{\mu}|$), respectively. In Table I, the average values of u_0 and v_0 are -0.7956 and -0.3697 , respectively, and the standard deviations and the relative ones are very small ($S_r < 1\%$), which shows the high precision of the measurements. Similar error analysis can be applied to the other tables below.

TABLE I. Measurement of u_0 and v_0 .

	u_0	v_0
(0,0)	-0.7958	-0.3698
($\Delta x, 0$)	-0.7955	-0.3697
($-\Delta x, 0$)	-0.7958	-0.3698
(0, Δy)	-0.7957	-0.3698
($\Delta x, \Delta y$)	-0.7954	-0.3696
($-\Delta x, \Delta y$)	-0.7957	-0.3697
(0, $-\Delta y$)	-0.7955	-0.3697
($\Delta x, -\Delta y$)	-0.7952	-0.3695
($-\Delta x, -\Delta y$)	-0.7955	-0.3696
$\bar{\mu}$	-0.7956	-0.3697
S	1.9×10^{-4}	1.0×10^{-4}
S_r	0.024%	0.027%

TABLE II. Measurement of x_0 and y_0 .

	x_0	y_0
$A_{u,00}, A_{u,01}, A_{u,-10}$	-0.00473	0.00390
$A_{v,00}, A_{v,01}, A_{v,-10}$	-0.00473	0.00390
$A_{u,00}, A_{u,01}, A_{u,0-1}$	-0.00476	0.00389
$A_{v,00}, A_{v,01}, A_{v,0-1}$	-0.00475	0.00389
$A_{u,00}, A_{u,10}, A_{u,-10}$	-0.00473	0.00394
$A_{v,00}, A_{v,10}, A_{v,-10}$	-0.00473	0.00393
$A_{u,00}, A_{u,0-1}, A_{u,10}$	-0.00474	0.00393
$A_{v,00}, A_{v,0-1}, A_{v,10}$	-0.00474	0.00392
$A_{u,00}, A_{u,0-1}, A_{u,-10}$	-0.00472	0.00399
$A_{v,00}, A_{v,0-1}, A_{v,-10}$	-0.00472	0.00397
$A_{u,00}, A_{u,01}, A_{u,10}$	-0.00481	0.00387
$A_{v,00}, A_{v,01}, A_{v,10}$	-0.00479	0.00388
$A_{u,10}, A_{u,0-1}, A_{u,-10}$	-0.00472	0.00391
$A_{v,10}, A_{v,0-1}, A_{v,-10}$	-0.00472	0.00390
$A_{u,10}, A_{u,0-1}, A_{u,01}$	-0.00469	0.00388
$A_{v,10}, A_{v,0-1}, A_{v,01}$	-0.00470	0.00388
$A_{u,01}, A_{u,10}, A_{u,-10}$	-0.00471	0.00388
$A_{v,01}, A_{v,10}, A_{v,-10}$	-0.00472	0.00388
$\bar{\mu}$	-0.00473	0.00391
S	2.9×10^{-5}	3.3×10^{-5}
S_r	0.62%	0.85%

2. Position for rotation center (x_0, y_0)

According to solution (5), in the core region, the u and v amplitudes near the rotation center are proportional to the distance from it [see Figs. 1(c) and 1(d)], that is, $A_u = \rho_1 r$, $A_v = \rho_2 r$. Taking the first line data in Table II, for example, there is

$$\frac{A_{u,00}}{\sqrt{(x_0 - 0)^2 + (y_0 - 0)^2}} = \frac{A_{u,01}}{\sqrt{(x_0 - 0)^2 + (y_0 - \Delta y)^2}} = \frac{A_{u,-10}}{\sqrt{(x_0 + \Delta x)^2 + (y_0 - 0)^2}},$$

where $A_{u,00}$ and $A_{v,00}$ are the amplitudes of u and v at (0, 0), and the ones at (0, Δx) are $A_{u,01}$ and $A_{v,01}$, and so on. $A_{u,00}$, $A_{u,01}$, and $A_{u,-10}$ can be measured numerically. Then x_0 and y_0 can be fixed by solving the above equations. We can see that the amplitude data on three different grid points

TABLE III. Measurement of ρ_1 and ρ_2 .

	ρ_1	ρ_2
r_{00}	1.618	0.5157
r_{10}	1.614	0.5150
r_{-10}	1.618	0.5159
r_{01}	1.619	0.5163
r_{11}	1.613	0.5146
r_{-11}	1.615	0.5153
r_{0-1}	1.615	0.5150
r_{1-1}	1.614	0.5148
r_{-1-1}	1.614	0.5148
$\bar{\mu}$	1.616	0.5153
S	2.3×10^{-3}	5.8×10^{-4}
S_r	0.13%	0.12%

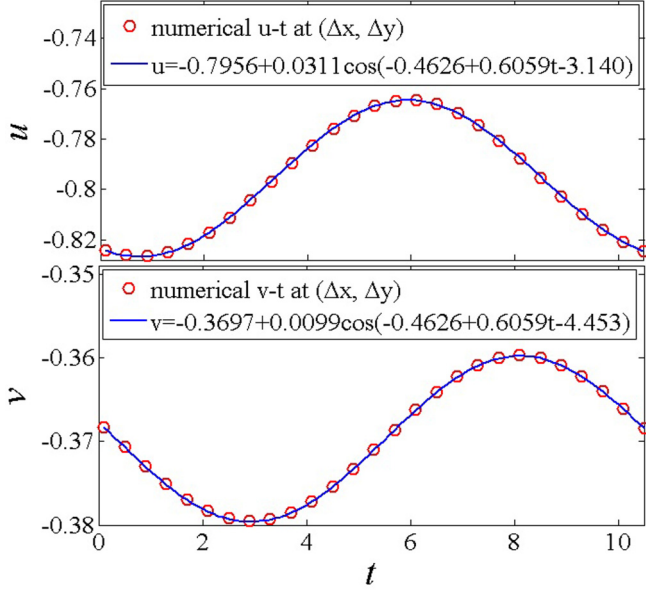


FIG. 7. (Color online) Solid points and lines show the numerical and fitting results of u - t and v - t , respectively, over a period of time.

can fix a set of x_0 and y_0 , and Table II lists the results gotten from the combinations of the grid points around the rotation center. Thus the location of the rotation center is measured to be $(-0.00473, 0.00391)$.

3. Coefficients ρ_1 and ρ_2

Around the rotation center, according to $A_u = \rho_1 r$, $A_v = \rho_2 r$, after knowing the values of A_u , A_v , and r , we can get the coefficients ρ_1 and ρ_2 . The distance from the neighboring grid points $(0, 0)$ and $(0, \Delta x)$ to the rotation center (x_0, y_0) can be

TABLE IV. Measurement of α_1 and α_2 .

	α_1	α_2
$(0, 0)$	3.139	4.452
$(\Delta x, 0)$	3.139	4.452
$(-\Delta x, 0)$	3.131	4.444
$(0, \Delta y)$	3.143	4.455
$(\Delta x, \Delta y)$	3.140	4.453
$(-\Delta x, \Delta y)$	3.137	4.450
$(0, -\Delta y)$	3.136	4.449
$(\Delta x, -\Delta y)$	3.138	4.450
$(-\Delta x, -\Delta y)$	3.135	4.447
$\bar{\mu}$	3.138	4.450
S	3.4×10^{-3}	3.3×10^{-3}
S_r	0.11%	0.07%

marked as r_{00} and r_{01} , respectively, and so on. From Table III we can get that $\bar{\rho}_1 = 1.616$, $\bar{\rho}_2 = 0.5153$.

4. Coefficients α_1 and α_2

All the unknown coefficients of solution (5) have been fixed precisely except α_1 and α_2 , which are initial rotation phases for u and v , respectively. Now we can use the least-squares method to get them. For a grid point in the medium, keep changing α_1 and α_2 gradually until solution (5) gives the best fit with the numerical results (the sum of the squares of their differences is minimum), and then take the α_1 and α_2 as the desired values. Taking grid point $(\Delta x, \Delta y)$, for example, solution (5) with $\alpha_1 = 3.140$ and $\alpha_2 = 4.453$ agrees best with the numerical results (see Fig. 7). In the same way, we can get the initial rotation phases α_1 and α_2 from the eight other grid points, and the average values are $\bar{\alpha}_1 = 3.138$ and $\bar{\alpha}_2 = 4.450$ (see Table IV).

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