Experimental observation of extreme multistability in an electronic system of two coupled Rössler oscillators

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We report the first experimental observation of extreme multistability in a controlled laboratory investigation. Extreme multistability arises when infinitely many attractors coexist for the same set of system parameters. The behavior was predicted earlier on theoretical grounds, supported by numerical studies of models of two coupled identical or nearly identical systems. We construct and couple two analog circuits based on a modified coupled Rössler system and demonstrate the occurrence of extreme multistability through a controlled switching to different attractor states purely through a change in initial conditions for a fixed set of system parameters. Numerical studies of the coupled model equations are in agreement with our experimental findings.

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I. INTRODUCTION

Multistability is a common occurrence in many nonlinear dynamical systems, corresponding to the coexistence of more than one stable attractor for the same set of system parameters [1]. A large number of theoretical and experimental studies have explored this phenomenon in a variety of physical [2–5], chemical [6,7], and biological [8,9] systems. A curious and novel manifestation of this phenomenon arises when a system can have an infinite number of coexisting attractors, where each attractor is associated with a particular set of initial conditions [10,11]. Extreme multistability was first demonstrated in a system of two coupled identical Lorenz oscillators by Sun *et al.* [10] and was subsequently investigated in the three-variable autocatalator model by Ngonghala *et al.* [11]. More recently, Hens *et al.* [12] have demonstrated the existence of extreme multistability in a system of two coupled Rössler oscillators and in a chemical autocatalator model [13].

In all these studies, a special coupling was applied between two three-variable chaotic systems to form six-variable coupled systems. Numerical simulations of the synchronization of the coupled systems were carried out for a fixed set of system parameters and only the initial conditions were changed. The synchronized systems were found to evolve to different attractor states (fixed points, limit cycles, chaotic states) purely through changes in initial conditions, typically with a change in the initial condition of just one of the state variables. The origin of this behavior lies in the synchronization dynamics of the two coupled subsystems. Two properties are essential for the appearance of extreme multistability in two coupled n-dimensional nonlinear systems: (i) the complete synchronization of $n-1$ state variables of the two systems and (ii) the synchronization of the remaining state variable in each subsystem according to a conserved quantity K in the long-term limit $t \to \infty$. The conserved quantity has a profound effect on the synchronization dynamics; it characterizes the synchronization manifold. It also leads to a neutrally stable direction in the steady states and orbits which gives rise to a dependence on the initial conditions of the asymptotic state at $t \to \infty$. In addition, perturbations give rise to new dynamical states, as the system is shifted from one synchronization manifold to another.

Extreme multistability might have important consequences in the reproducibility of certain experimental systems. For example, some chemical reactions, such as the chlorite-thiosulfate reaction [14] and the chlorite-iodide reaction [15], consistently exhibit irreproducibility: despite great care to ensure reproducibility, these reactions show a random long-term behavior for the same set of experimental conditions. The cause of the irreproducibility is not known; extreme multistability offers a possible mechanism for the behavior. However, to the best of our knowledge, there is no direct experimental verification of this new type of dynamical behavior in a controlled laboratory investigation. In this paper, we report experimental observations of a coupled electronic circuit system that displays extreme multistability.

II. MODEL SYSTEM

Our experiments are carried out on an analog circuit system closely based on the model investigated by Hens *et al.* [12], consisting of a set of coupled Rössler equations, namely,

$$\dot{x}_1 = -y_1 - z_1, \tag{1a}$$

$$\dot{y}_1 = x_2 + ay_2 + \alpha(x_2 - x_1), \tag{1b}$$

$$\dot{z}_1 = b + 2x_2 z_2 - cz_1, \tag{1c}$$

$$\dot{x}_2 = x_1 - x_2 - y_1 - z_1, \tag{1d}$$

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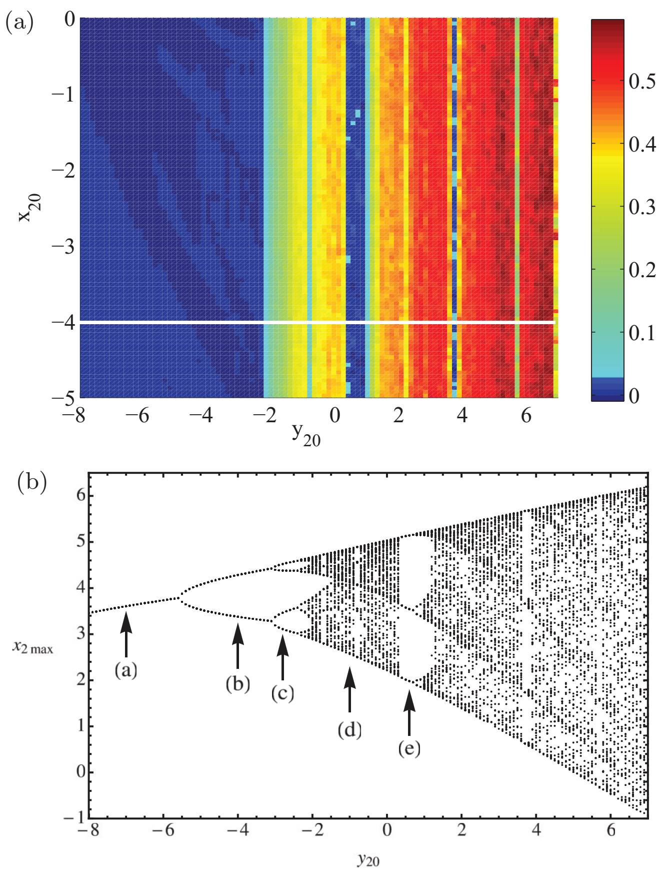

FIG. 1. (Color) (a) Plot of the maximum Lyapunov exponent (color coded) in the (x_{20},y_{20}) space with parameter values fixed at $\alpha=0.02, a=0.2, b=0.2, c=5.7$ and the other initial conditions fixed at $x_{10}=y_{10}=z_{10}=z_{20}=0$. (b) Maxima of x_2 plotted against initial values of y_2 along the line $x_{20}=-4.0$.

$$\dot{y}_2 = x_2 + a y_2, \tag{1e}$$

$$\dot{z}_2 = b + 2x_2 z_2 - c z_2, \tag{1f}$$

where α, a, b, and c are constants (with $c>0$) and (x_1,y_1,z_1) and (x_2,y_2,z_2) are the state variables of the two subsystems. If we set $x_1=x_2$, $y_1=y_2$, and $z_1=z_2$, then the two subsystems become decoupled, and each individual subsystem represents a Rössler oscillator. The factor of 2 multiplying the nonlinear terms in (1c) and (1f) arises from a scaling down of the original Rössler system variables by a factor of 2. This is done to restrict the output signal voltage range, in the circuit implementation of the equations, to within ±15 V in order to avoid saturation of the circuit. The coupled system (1) is a variant of the set analyzed in [12] in that (1b) of our system is different from the corresponding equation in [12]. Our system becomes identical to that of [12] for $\alpha=-1$. However, the basic property of extreme multistability is still preserved in the modified system, as can be seen from an analysis of a reduced set of equations that govern the differences ("errors") of the corresponding state variables of the subsystems, namely,

$$\dot{e}_1 = -e_1, \tag{2a}$$

$$\dot{e}_2 = -\alpha e_1, \tag{2b}$$

$$\dot{e}_3 = -c e_3, \tag{2c}$$

where $e_1=x_1-x_2$, $e_2=y_1-y_2$, and $e_3=z_1-z_2$. Upon complete synchronization, the error dynamics evolves to a stationary state that defines the relationship between the state variables. From (2) we see that e_1 and e_3 both go to zero asymptotically, while e_2 tends to a constant value, which corresponds to the previously mentioned conserved quantity K. It should be mentioned here that a dynamical system made up of two coupled subsystems does not have independent variables upon complete synchronization and hence is overdetermined since the variables of one subsystem are equal to the corresponding variables of the other subsystem. This is true for all synchronization systems that attain complete synchronization in the long-term limit $t\to\infty$ [16,17]. In the synchronization dynamics of systems that exhibit extreme multistability, the error dynamics also evolves to a stationary state, but now two of the corresponding variables of the subsystems are related by a conserved quantity; that is, the variables are related by a constant (or a more complex relation) that depends on the initial conditions. This conserved quantity can be introduced into one of the subsystems as a bifurcation parameter that, while providing insights into the asymptotic behavior, can be misinterpreted as a description of the overall synchronization dynamics. As discussed in [12], system (1) possesses an infinite number of attractors corresponding to different values of K. Further, the system admits a Lyapunov function $V=e_1^2+e_3^2$ such that $dV/dt<0$, ensuring stability of the attractor states. As a numerical demonstration of the multiple attractor states, we solve (1) for a fixed set of system parameters ($\alpha=0.02, a=0.2, b=0.2, c=5.7$) and vary the initial conditions y_{20} from −8.0 to 7.0 and x_{20} from −5.0 to 0.0 while keeping all the other initial conditions fixed at $x_{10}=y_{10}=z_{10}=z_{20}=0$. Figure 1(a) shows a plot of the maximum Lyapunov exponent (color coded) in the (x_{20},y_{20}) space. Figure 1(b) shows the attractor states that exist along the line $x_{20}=-4.0$.

III. EXPERIMENTAL SYSTEM AND RESULTS

We next turn to the experimental implementation of Eq. (1). Figure 2 shows a picture of the experimental setup that was constructed to study the dynamics of the coupled Rössler system. A detailed circuit diagram of the system is available

FIG. 2. (Color online) Experimental setup of the two coupled Rössler oscillators system with relay and timer circuits to change the initial conditions.

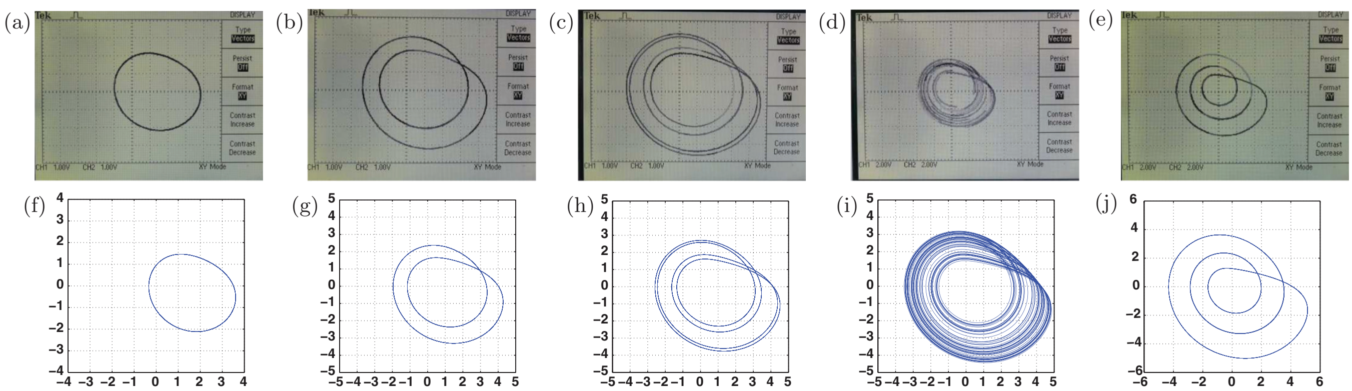

FIG. 3. (Color online) Oscilloscope images of the phase plots y_1 vs x_2 of various attractor states with parameters $\alpha = 0.02, a = 0.2, b = 0.2, c = 5.7$, initial conditions $x_{10} = y_{10} = z_{10} = z_{20} = 0, x_{20} = -4.0$, and (a) $y_{20} = -7.0$, (b) $y_{20} = -4.0$, (c) $y_{20} = -2.0$, (d) $y_{20} = -1.0$, and (e) $y_{20} = 0.6$. The plots shown in (f), (g), (h), (i), and (j) are phase plots from numerical solutions of (1) corresponding to the initial conditions of (a), (b), (c), (d), and (e), respectively.

as Supplemental Material [18]. A regulated power supply of ± 15 V energizes the circuit, and the system parameters α, a, b, and c are controlled with circuit resistors. As a benchmark exercise, each individual Rössler oscillator was separately tested by varying the system parameters to obtain its various attractor states, ranging from periodic states to chaotic dynamics, and the behavior was then compared to numerical simulations of the Rössler equations. Extreme care was taken to ensure that the two oscillator systems were as nearly "identical" as possible within practical limits. This entailed careful weaning of all the component elements (resistors, capacitors) to match their values as closely as possible and the removal of any intrinsic drifts or biases within the operational amplifiers and multipliers [19]. The two oscillators were then coupled to each other as shown in Fig. 2, and their parameters were fixed at the values mentioned earlier. It should be mentioned that for the parameters chosen in our experiment the individual Rössler oscillators (when decoupled) were in the chaotic state.

To change the initial conditions of the dynamics of the circuit, we have employed a strategy of imposing external voltages on selective nodes of the operational amplifiers as well as shorting relevant capacitors of the circuit initially to set the values of some of the state variables to zero. To implement this combined strategy in a controlled manner we have developed and attached an additional system of relay circuits to the coupled Rössler circuits (indicated by a label in Fig. 2). For details of the circuit diagram of the relay circuit, see [18]. When this circuit is energized, the timer portion of the circuit produces a high output for 6 s, and the relay is turned on through the relay driver circuit. Due to this, the capacitor responsible for generating the x_1 signal gets shorted in the coupled circuit, and hence x_1 is set to 0 V initially. Similarly, y_1, z_1, and z_2 are also set to 0 V. After 6 s the output of the timer circuit falls to a low value, and the relay gets switched off. Due to this, the capacitors no longer remain shorted, and the circuit runs with the applied initial conditions for x_1, y_1, z_1, and z_2. The initial conditions of x_2 and y_2 are changed using independent external voltage sources [18]. Using the above strategy, we have run the coupled circuit for a number of initial conditions without changing the circuit parameters. Some typical results in the form of oscilloscope images of phase plots of y_1 vs x_2 are shown in the top row of Fig. 3, indicating the various periodic attractor states as well as a chaotic state.

These states correspond to those identified in the diagram of Fig. 1 by the labels of the images in Fig. 3. The plots in the bottom row of Fig. 3 show corresponding numerical solution results of Eq. (1) using the same initial conditions.

One of the remarkable aspects of the synchronization dynamics that gives rise to extreme multistability, like in the coupled Rössler circuit studied here, is that the coupled system has both dissipative dynamics and conservative dynamics. The dissipative dynamics is manifested in the nature of the dynamical state, which is a true attractor, with an infinite number of initial conditions that take the system to that attractor. The conservative dynamics is a consequence of the conserved quantity, which gives rise to a neutrally stable direction and, consequently, a dependence on the initial conditions. Thus the circuit is characterized by infinitely many attractors, each associated with a particular value of the conserved quantity $K = y_1 - y_2$ (y_1 and y_2 are asymptotic values taken at very large times), where the basin of attraction is made up of all sets of initial conditions that evolve asymptotically to the particular value of K associated with the attractor. One can arbitrarily introduce a dependence on the initial conditions into any dynamical system described by a set of differential equations; however, in coupled systems undergoing synchronization, the conserved quantity arises from the synchronization dynamics.

The conserved quantity gives rise to a direction of neutral stability for the stationary states as well as the periodic and chaotic orbits (in addition to that associated with the direction along the orbit). Hence, if the system is in a particular periodic orbit, say period 2, a perturbation that does not satisfy the condition of the conserved quantity will give rise to the evolution of the system to a new attractor. The new attractor may differ only quantitatively; for example, a small perturbation might shift the period-2 dynamics to a new period-2 dynamics that differs in amplitude. However, larger perturbations give rise to the evolution of the system to qualitatively new attractors, such as a period-4 or period-8 attractor. This characterization of the effects of perturbations also applies to the effects of different initial conditions. Because the coupled system exhibits period-doubling bifurcations, perturbations or different initial conditions permit the sampling of any of an infinite number of qualitatively different attractors. Even if the coupled system did not display chaotic dynamics, the same mechanism would give rise to an infinite number of quantitatively different attractors. In principle it should be possible to visit each of these asymptotic attractors in a continuous fashion by altering the initial conditions in an infinitesimal manner, an extremely challenging task in an experimental setup. However, evidence of this continuous transition can be observed in a transient manner by deliberately introducing a slight mismatch in the two circuits. We have carried out such an exercise (by slightly changing the value of one of the resistors, see details in [18]) such that the term $x_2 - x_1$ on the right-hand side of Eq. (1b) is changed to $x_2 - x_1 + \delta x_2$, where δ is a small quantity. Then integrating (2b), one gets $e_2 = K - \alpha\delta \int dt x_2(t)$. Thus the value of the constant K that e_2 acquires changes with time, and the rate of change is controlled by the constants α and the mismatch value δ. We have observed such a continuous drift of the coupled system through the various states represented by Fig. 1 when the two oscillators are slightly detuned by bringing about such a deliberate change of a small amount. The results can also be reproduced exactly by a numerical solution of the mismatched system. The temporal evolution through various states can be viewed in the video clip provided as Supplemental Material for this paper [18]. This demonstration provides additional support for the existence of extreme multistability in the coupled Rössler system (1).

IV. CONCLUSIONS

To conclude, we have described an experimental demonstration of extreme multistability using an electronic circuit implementation of two coupled Rössler attractors. The theoretical model on which the experiment is based has a simple set of equations. In particular, the error dynamics equations [(2a)–(2c)] of this model involve linear terms and are easily solvable. It is possible in principle to obtain extreme multistability from more complicated error dynamics that includes time dependent and nonlinear terms. However, for the sake of simplicity and in the interest of minimizing the technical complexity of the experimental system we have avoided dealing with complicated error dynamics.

A restrictive feature of this type of dynamical system is the requirement that the coupled subsystems be identical or nearly identical. This restriction and the need to devise a method to change the initial conditions in a controlled manner pose serious technical challenges. We have successfully overcome these challenges in our electronic circuit system and have demonstrated that two chaotic circuits can be sufficiently matched to give rise to extreme multistability when appropriately coupled. It is likely that extreme multistability will be a rarity in most physical, chemical, and biological systems; however, the combined conservative and dissipative features give rise to dynamics that might find technological uses, such as the ability to easily select qualitatively different dynamical states from an infinite number of possibilities. In addition, slightly mismatched systems that might occur in natural settings may display a temporal evolution through various dynamical states, as observed in our experimental electronic system.

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[18] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevE.89.022918 for detailed circuit diagrams of the experimental setup and a video clip showing the temporal transition of the system through various attractor states when there is a slight mismatch in the system parameters of the two oscillators.
[19] The multiplier used in the circuit is an AD734, and the operational amplifiers are all OP400.