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Chimera states on complex networks

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The model of nonlocally coupled identical phase oscillators on complex networks is investigated. We find the existence of chimera states in which identical oscillators evolve into distinct coherent and incoherent groups. We find that the coherent group of chimera states always contains the same oscillators no matter what the initial conditions are. The properties of chimera states and their dependence on parameters are investigated on both scale-free networks and Erdös-Rényi networks.

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I. INTRODUCTION

A chimera state refers to the spatial-temporal pattern in which identical oscillators evolve into distinct coherent and incoherent groups [1–6]. The state was first numerically observed by Kuramoto and his colleagues [1,2] in simulations of nonlocally coupled complex Ginzburg-Landau equations. Soon after, great interest on chimera states was attracted and a flurry of theoretical activities appeared. Abrams and Strogatz [3] found an exact solution for the state in a ring of phase oscillators with a cosine-kernel coupling. A clustered chimera state in a ring of oscillators [7,8] in which spatially distributed phase coherences are separated by incoherence and adjacent coherent groups are in antiphase was found in nonlocally coupled oscillators in the presence of time delay. In contrast, Zhu *et al.* found that a two-cluster chimera state can be realized in the absence of time-delayed coupling but with heterogeneous phase lags [9]. Abrams and Strogatz [10] considered a model consisting of two interacting subpopulations of oscillators and found a breathing chimera state. Pikovsky and Rosenblum [11] investigated an oscillator ensemble in which there are several subpopulations of identical units and a general heterogeneous coupling between them is assigned, and they acquired a quasiperiodic chimera state. Laing [12,13] studied chimera states using the Ott-Antonsen ansatz [14], which provides a low-dimensional description of the dynamics in globally coupled phase oscillators, and he pointed out that a breathing chimera state may exist in a one-dimensional system when the parameter heterogeneity is introduced to the system. Chimera states can manifest their presence in self-organized patterns. Kuramoto and his colleagues first observed spiral chimeras in a two-dimensional arrays of nonlocally coupled oscillators in which the core region of the spiral wave is occupied by incoherent oscillators [15]. Soon after, a theoretical analysis on spiral chimera states was presented by Martens *et al.* [16]. Interestingly, Gu *et al.* found that spiral chimeras may exist in complex oscillatory and chaotic systems [17]. Recently the experimental evidence on the chimera state has been presented either in optical or in chemical setups [18,19].

Up to now, the investigations on chimera states have focused on regular structures where oscillators sit on either a one-dimensional array or a two-dimensional lattice. The emergent chimera state is thought to be directly related to the phenomenon of unihemisphere sleep, the coexistence of synchronized and unsynchronized brain wave activities [20,21]. However, typical patterns in the connections among neurons in the brain are thought to be characterized by complex networks. Consequently, it will be interesting to ask whether chimera phenomena can be observed on complex networks, and that is our target in this work.

II. THE MODEL

For N identical phase oscillators nonlocally coupled together, the motion equation of the system can be described as

$$\frac{d\theta_i}{dt} = \omega - \frac{1}{N}\sum_{j=1}^{N} G_{ij}\sin(\theta_i - \theta_j + \alpha), \tag{1}$$

where θ_i represents the phase of oscillator i $(i = 1,\ldots,N)$, ω is the common natural frequency of phase oscillators, and we set $\omega = 0$ for convenience. The angle $\alpha(0 \leqslant \alpha \leqslant \frac{\pi}{2})$ is a tunable parameter that describes the phase lag between oscillators i and j. To account for the nonlocal coupling, we introduce the coupling function G_{ij} depending on the shortest length between oscillators i and j on the underlying complex network. Generally, $G_{ij}(d_{ij}) \geqslant 0$ and decreases with the shortest length d_{ij}. We assume that G_{ij} follows

$$G_{ij} = Ae^{-\kappa d_{ij}}, \tag{2}$$

where A is the global coupling strength and κ describes the strength of the nonlocal coupling. When $\kappa = 0$, the model becomes a globally coupled Kuramoto model. On the other hand, when κ increases, the model approaches a Kuramoto model on networks with locally coupling, and the coupling function G becomes the adjacent matrices of networks. Throughout this work, we set $N = 1024$, $\alpha = \pi/2 - 0.10$, $A = 1$, and $\kappa = 0.1$ unless otherwise specified.

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III. RESULTS AND ANALYSIS

To show evidence for the existence of a chimera state, we first monitor the snapshot of phases of oscillators when the system has reached a steady state. Then we consider the effective angular velocities of oscillators. The effective angular velocity of oscillator i is defined as $\langle \omega_i \rangle = \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} \dot{\theta}_i \, dt$. In simulations, $T = 1000$ and t_0 is sufficiently large such that the system has evolved into a steady state. We also consider the fluctuation of the instantaneous angular velocity σ_i of oscillator i around its effective velocity, which is defined as $\sigma_i^2 = \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} (\dot{\theta}_i - \langle \omega_i \rangle)^2 \, dt$. A zero σ_i indicates that the oscillator i rotates at a constant angular velocity, which helps to distinguish a stationary chimera state from a breathing one.

Furthermore, singling out a chimera state on complex networks is not trivial. On a regular network such as a one-dimensional array or a two-dimensional lattice, a chimera state is easily visualized since both coherent and incoherent oscillators form compact clusters in space and the boundary between them can be easily found. However, the concept of the space is lost on complex networks, which makes it not straightforward to figure out a chimera state. To have a clear view of chimera states on complex networks, we rearrange the order of oscillators as follows. First, we ascend oscillators according to their effective angular velocities such that $i \geqslant j$ if $|\langle \omega_i \rangle| \geqslant |\langle \omega_j \rangle|$. Second, if there exists a plateau in the graph of $|\langle \omega_i \rangle|$ on which oscillators have the same effective angular velocity, we will further rearrange the order of oscillators on the plateau. We label the oscillator with the highest degree on the plateau with 1. Then the other oscillators on the plateau are ordered according to their distance away from the first one. For those oscillators with the same distance from the first one, $i < j$ if their degrees satisfy $k_i > k_j$.

We consider two types of complex networks: Erdös-Rényi networks (ERN) and scale-free networks (SFNs). The number of neighbors of an oscillator follows Poission distributions on ERNs, while, for SFNs, the number of neighbors of an oscillator follows power law distributions. In this work, we adopt the Barabasi-Albert model (BAM) for SFNs and, for both ERNs and SFNs, we first let their mean degrees be $\langle k \rangle = 4$. The numerical results are presented in Fig. 1, in which the top panels are for ERNs and the bottom panels are for SFNs. The snapshots of the phases of oscillators in Fig. 1(a) show the formation of two groups, a typical pattern of chimera states. As shown in Fig. 1(a), the left group is a coherent one in which oscillators have nearly the same phases and the right one is an incoherent one in which oscillators randomly distribute their phases in the range of $[-\pi,\pi]$. The relative phase differences between oscillators in the coherent group are unchanged with time, while those in the incoherent group change irregularly. The effective angular velocity presented in Fig. 1(b) shows that the oscillators in the coherent group have the same $\langle \omega \rangle$, which is around $\langle \omega \rangle = -0.53$. On the other hand, the oscillators in the incoherent group advance their phases at a relatively low $|\langle \omega \rangle|$. The observed chimera state is a stationary one as evidenced by $\sigma = 0$ in the coherent group in Fig. 1(c).

Clearly, there do exist chimera states on complex networks where parts of the oscillators are synchronized while others are out of synchronization. Interestingly, though ERNs and

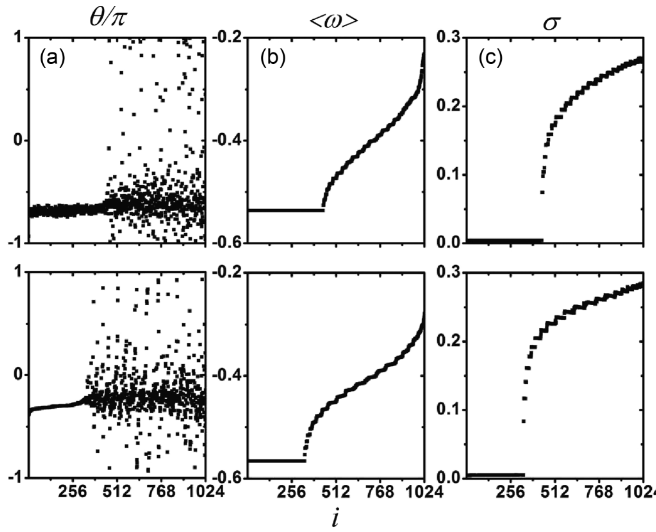

FIG. 1. Column (a) shows the the snapshots of the phase profile of oscillators. Column (b) shows the effective angular velocities of oscillators $\langle \omega \rangle$ averaged over 1000 time units. Column (c) shows the fluctuation of the instantaneous angular velocity of oscillators σ. The top panels are for ERNs and the bottom panels for SFNs. The mean degree of networks $\langle k \rangle = 4$. $N = 1024$, $A = 1$, $\kappa = 0.1$, and $\alpha = \pi/2 - 0.1$.

SFNs are quite different in their topological properties, the chimera states realized on these two types of networks in Fig. 1 share great similarity except for the higher effective angular velocities in value and smaller coherent group for the chimera state on SFNs. Actually, even the buildup of chimera states on these two types of networks looks like each other. In Fig. 2 we present the time evolutions of the model (1) with random initial conditions. The transients for both networks are short, which evolve into chimera states only after a few cycles. The time evolutions of model 1 in Fig. 2 also show that the dynamics of oscillators in the incoherent group is characterized by the

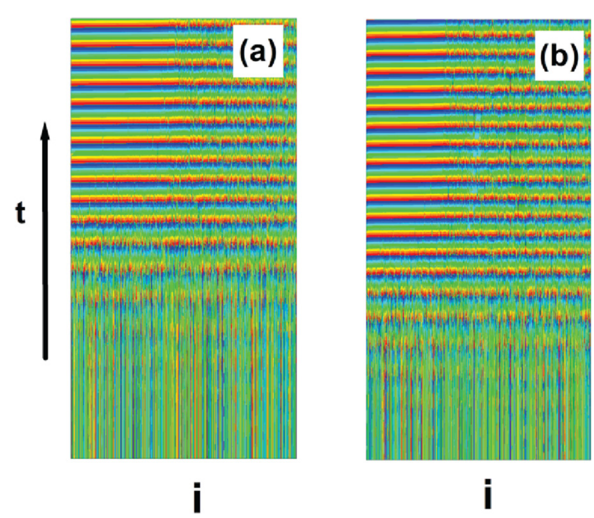

FIG. 2. (Color online) The time evolutions of oscillators show the buildup of a chimera state on an ERN (a) and an SFN (b) from random initial conditions. The parameters are $N = 1024$, $A = 1$, $\kappa = 0.1$, and $\alpha = \pi/2 - 0.1$.

existence of a large amount of phase slips, which interrupt their synchronization with those in the coherent group.

We will make several remarks on chimera states on ERNs and SFNs. First, the coherent group is composed of oscillators which are connected on networks, and the incoherent group is fragmented. Simulations show that there is only one connected cluster for the synchronized oscillators. Differently from this, the incoherent group consists of a large number of desynchronized clusters which are separated by the coherent group on networks. In particular, most of desynchronized clusters consist of only one or two oscillators. Second, chimera states on complex networks can always be formed no matter what the initial conditions are. The generation of a chimera state on regular networks in previous studies always has required special initial conditions due to its coexistence with a trivial fully synchronized state and its narrow attraction basin [6]. In contrast, chimera states on complex networks can be evolved from arbitrary initial conditions in which phases of oscillators are randomly chosen from $[-\pi,\pi]$; one example has been shown in Fig. 2. Furthermore, chimera states on complex networks are always unique in the sense that, under the same dynamical parameters and network parameters, the coherent group always contains the same oscillators no matter what initial conditions are, which can be examined by numerical simulations. By reshuffling the phases of oscillators when a chimera state has been built and then letting the model evolve again, we find that the same set of coherent oscillators is obtained on SFNs, and most of oscillators in the coherent group on ERNs are always there. Generally, on a one-dimensional array (or a two-dimensional lattice) with periodic boundary conditions, the coherent group in chimera states may appear in any locations around the array (or lattice) since oscillators are indistinguishable due to the translation symmetry in the model. However, oscillators on complex networks are distinguishable according to their different neighborhoods, and, as a result, the coherent group in a chimera state is always attracted to the same locations on complex networks for different initial conditions.

Though chimera states on both ERNs and SFNs seem to be similar in the sense of their macroscopic properties, a difference exists on the organizations of chimera states on ERNs and on SFNs. To see it, we consider what kind of oscillators are prone to being synchronized. Since all oscillators are identical in the sense of their dynamics when isolated, the difference between them results from their locations on networks. Therefore, we focus on the number of neighbors (the degree) of oscillators and monitor the degree distribution of oscillators in the coherent group. The results for different mean degrees $\langle k\rangle$ are presented in Fig. 3 in which the degree distributions of the underlying networks are presented as a comparison. On an ERN, the degree distributions of oscillators in the network and in the coherent group are almost the same. On the other hand, though the degree distribution of oscillators in the coherent group on an SFN still follows a power law, the exponent gets increased in value. The difference of the degree distributions between oscillators in the coherent groups and those on networks suggests that all oscillators have the same probability to be synchronized on ERN networks while oscillators with high degrees are more prone to be synchronized than those with low degrees on SFNs.

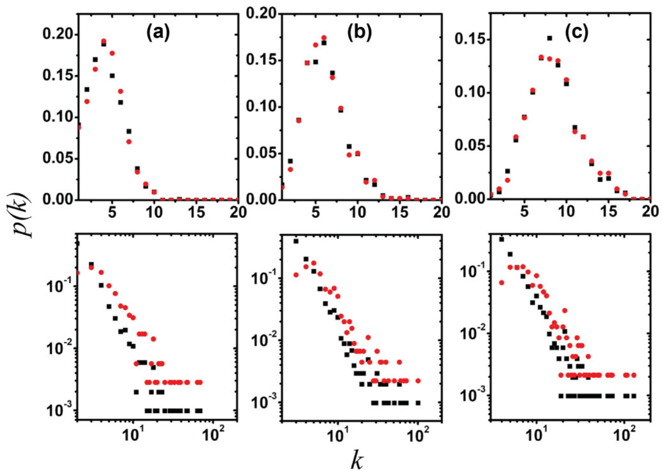

FIG. 3. (Color online) The degree distributions of oscillators in the coherent group (red circle symbols) and the degree distributions of the underlying networks (black square symbols) for different mean degrees. The top panels are for ERNs, and the bottom panels are for SFNs. The mean degree $\langle k\rangle = 4$ in column (a), $\langle k\rangle = 6$ in column (b), and $\langle k\rangle = 8$ in column (c), respectively. The parameters are $N = 1024$, $A = 1$, $\kappa = 0.1$, and $\alpha = \pi/2 - 0.1$.

Generally, in coupled oscillators, the strength of interaction of an oscillator with its environment measures its ability to be synchronized. The difference in the organizations of chimera states on ERNs and on SFNs can be further evidenced by different behaviors of the interaction of oscillators with their environments on ERNs and on SFNs. In the model of nonlocally coupled oscillators on complex networks, the interaction strength of oscillator i with its environment can be defined as $S_i = \sum_{j=1}^{N} Ae^{-\kappa d_{ij}}$. For an oscillator with high degree, its interaction strength S_i with the environment will be high since the oscillator always has the shortest lengths to

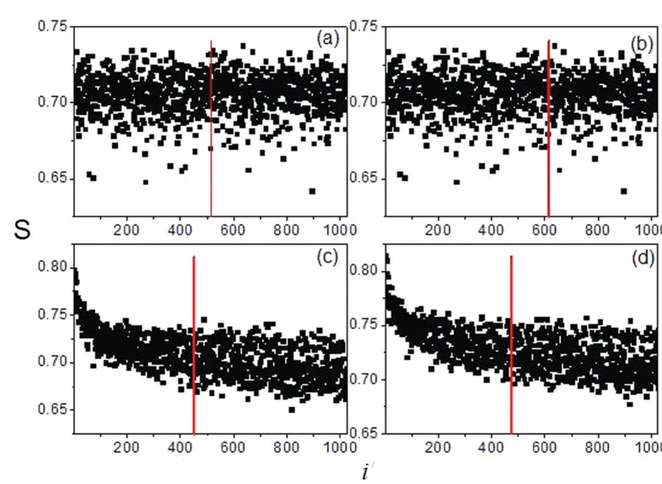

FIG. 4. (Color online) The profiles of S on ERNs with mean degree $\langle k\rangle = 6$ (a) and $\langle k\rangle = 8$ (b), respectively. The profiles of S on SFNs with mean degree $\langle k\rangle = 6$ (c) and $\langle k\rangle = 8$ (d), respectively. The oscillators on the left side of the red line are in the coherent group. The parameters are $N = 1024$, $A = 1$, $\kappa = 0.1$, and $\alpha = \pi/2 - 0.1$.

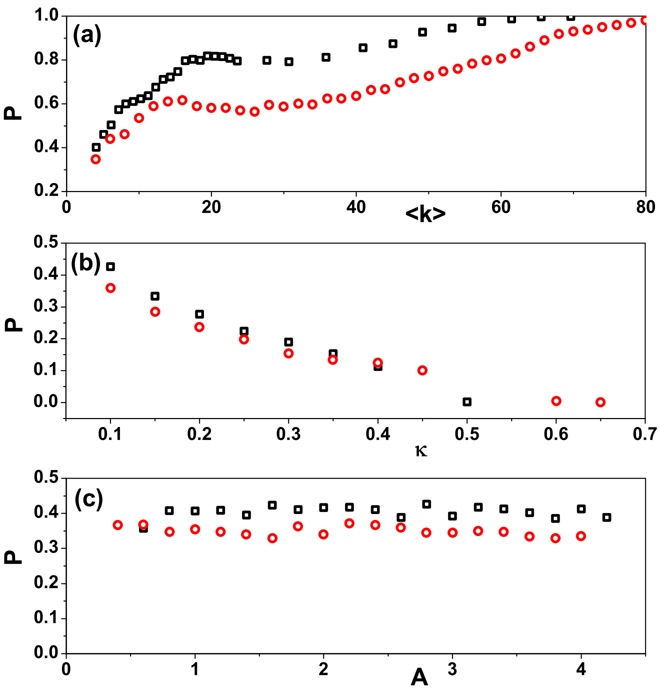

FIG. 5. (Color online) The dependence of the size of the coherent group P on the mean degree $\langle k\rangle$ (a), the strength of the nonlocally coupling κ (b), and the global coupling strength A (c) for ERNs (the black square) and SFNs (the red circles), respectively. The parameters are $N = 1024$, $A = 1$, $\kappa = 0.1$, and $\alpha = \pi/2 - 0.1$.

others. Under the same arrangement of oscillators in Fig. 1, we present S_i for different mean degrees $\langle k\rangle$ in Fig. 4. For ERNs, there is no difference in S_i between oscillators in the coherent group and oscillators in the incoherent group, which means that the formation of the coherent cluster in chimera states on ERNs has no preference for oscillators. In contrast, for SFNs, oscillators with the highest S are always in the coherent group. As a result, the coherent group in chimera states on SFNs always condensates onto those oscillators with the highest degrees.

Now we study the dependence of chimera states on the parameters of the model. We focus on the size of the coherent group. For this aim, we consider the quantity $P = N_s/N$ with N_s the number of oscillators in the coherent group. As shown in Fig. 5, the dependence of P on the mean degree $\langle k\rangle$ of the underlying network, the strength of the nonlocal coupling κ, and the global coupling strength A is independent of the type of networks. Furthermore, Fig. 5(a) shows that P undergoes a fast growth at small $\langle k\rangle$ and, then, grows towards $P = 1$ very slowly at large $\langle k\rangle$. On the other hand, Fig. 5(a) shows that increasing κ disfavors the coherent group in chimera states. At sufficiently strong κ where the model is reduced to the one similar to the locally coupled oscillators, chimera states yield to an incoherent state. Though A is the global coupling, Fig. 5(c) shows no clear dependence of chimera states on A provided that A is not sufficiently weak.

IV. CONCLUSION

In summary, we have studied the dynamics of nonlocally coupled identical phase oscillators on scale-free networks and Erdös-Rényi networks. We found that chimera states can spontaneously emerge out of arbitrary initial conditions for both types of complex networks. We found that oscillators in the coherent group tend to be those with high degrees on SFNs, while there is no preference for degree for oscillators on ERNs. We also found that the coherent group always contains the same oscillators regardless of initial conditions. We investigated the dependence of chimera states on the system parameters and found that the size of coherent group always increases with the mean degree of the underlying networks and decreases monotonically with the strength of the nonlocal coupling.

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