Manifold structures of unstable periodic orbits and the appearance of periodic windows in chaotic systems

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Manifold structures of the Lorenz system, the Hénon map, and the Kuramoto-Sivashinsky system are investigated in terms of unstable periodic orbits embedded in the attractors. Especially, changes of manifold structures are focused on when some parameters are varied. The angle between a stable manifold and an unstable manifold (manifold angle) at every sample point along an unstable periodic orbit is measured using the covariant Lyapunov vectors. It is found that the angle characterizes the parameter at which the periodic window corresponding to the unstable periodic orbit finishes, that is, a saddle-node bifurcation point. In particular, when the minimum value of the manifold angle along an unstable periodic orbit at a parameter is small (large), the corresponding periodic window exists near (away from) the parameter. It is concluded that the window sequence in a parameter space can be predicted from the manifold angles of unstable periodic orbits at some parameter. The fact is important because the local information in a parameter space characterizes the global information in it. This approach helps us find periodic windows including very small ones.

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I. INTRODUCTION

Chaotic dynamical systems are interesting research subjects which are studied not only in mathematics, physics, and engineering but also in biology and economics [1]. In a parameter space, there are periodic regions, regarded as periodic windows, inside chaotic regions of chaotic dynamical systems. The appearance of a periodic window is often related to a saddle-node bifurcation. A stable periodic orbit in a periodic window collides with an unstable periodic orbit (UPO) at an edge of the window and the two periodic orbits disappear. At the same time, the periodic window vanishes. It is known that a great many periodic windows usually exist in a parameter space, so that they are very common in nonlinear systems. There are some works on a mechanism constructing periodic windows and on the fat fractal structure of chaos and periodic windows in a parameter space [2]. Moreover, it is known that periodic windows are useful for controlling chaos [3]. However, there are still many open problems concerning periodic windows (e.g., the sequence of periodic windows). In particular, there is a difficulty in finding small periodic windows. We clarify the relation between a sequence of periodic windows in a parameter space and manifold structures of UPOs, from which we can identify a sequence of periodic windows in a parameter space. Although the Lyapunov exponents of UPOs are useful to study chaotic systems [4,5], manifold structures of UPOs have not been discussed in detail so far.

We use the covariant Lyapunov vectors (CLVs) [6] to investigate manifold structures, especially the angle between a stable and an unstable manifold of each UPO. CLVs span the Oseledec subspaces corresponding to the Lyapunov exponents [7]. By using CLVs, the degree of hyperbolicity and the effective dimension of the chaotic system can be measured [6,8]. CLVs are vectors which are given by iterating generic vectors backward in time within a subspace spanned by Gram-Schmidt vectors. Although the calculation of CLVs of chaotic sets is usually time consuming, our calculation in this paper mainly focuses on CLVs of UPOs with relatively small calculation cost.

This paper is organized as follows. In Sec. II, we analyze manifold structures of UPOs in the Lorenz system when we change some parameters. We characterize the appearance of periodic windows in terms of the manifold structures of UPOs. In Secs. III and IV, we perform similar analyses for the Hénon map and the Kuramoto-Sivashinsky system. Finally, in Sec. V, we summarize our results.

II. LORENZ SYSTEM

We first discuss manifold structures of the Lorenz system [9]:

$$\dot{x} = -\sigma x + \sigma y, \quad \dot{y} = -xz + rx - y, \quad \dot{z} = xy - bz.$$

Unless otherwise stated, we choose $\sigma = 10$ and $b = 8/3$, with r being the control parameter. It was found that the Lorenz system is (singular) hyperbolic at $r = 28$ [10], and as r increases further the system comes to possess tangencies between a stable manifold and an unstable manifold [9,11]. As in the case of general chaotic dynamical systems, there are infinitely many periodic windows in the Lorenz system. However, since the sizes of most periodic windows are very small, it is difficult to detect them. A periodic window of the Lorenz system starts via an inverse period doubling bifurcation and finishes by a saddle-node bifurcation as r increases from 28 [9].

The knowledge of CLVs allows calculating the angle between a stable and an unstable manifold at some point

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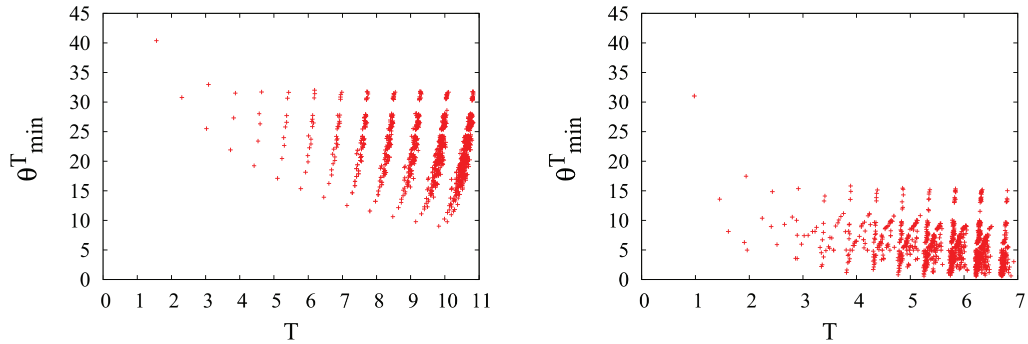

FIG. 1. (Color online) Minimum manifold angle $\theta^T_{\rm min}$ of each UPO with period T at $r = 28$ (left) and $r = 60$ (right).

by determining the angle between subspace E^s spanned by contracting CLVs and subspace E^u spanned by expanding CLVs. The angle $[0^\circ \leqslant \angle(E^s,E^u) < 90^\circ]$ is defined as follows [12]:

$$\angle(E^s,E^u) = \cos^{-1}\left[\max_{\substack{|\mathbf{u}^s|=|\mathbf{u}^u|=1\\ \mathbf{u}^s\in E^s,\mathbf{u}^u\in E^u}} |(\mathbf{u}^s,\mathbf{u}^u)|\right]\frac{180}{\pi}.$$

We calculate the angle at every sample point along an orbit. We will sometimes call the angle between a stable and an unstable manifold the manifold angle.

Statistical properties of chaotic systems have been characterized in terms of UPOs embedded in the attractor [4,5]. From now on we estimate the relative positions of periodic windows in a parameter space through the manifold structures of UPOs. Here, the manifold structure of a chaotic attractor is captured by the angle between a stable and an unstable manifold of a UPO. The minimum angle $\theta^T_{\rm min}$ between a stable and an unstable manifold (minimum manifold angle) of each UPO with period T is defined as

$$\theta^T_{\rm min} \equiv \min_{0\leqslant t<T} \angle(E^s,E^u)_t,$$

and is considered as a quantity representing the manifold structure of each UPO.

Figure 1 shows the minimum manifold angles of hundreds of numerically detected UPOs with period T at $r = 28$ and $r = 60$. It is found that UPOs in the case of $r = 28$ tend to have large minimum manifold angles, whereas some UPOs at $r = 60$ have very small ones. This result suggests that the tangency of a chaotic attractor at $r = 60$ [11] can be characterized by UPOs which have small minimum manifold angles, that is, UPOs which pass near the tangency [13].

To see the changes of manifold structures of UPOs, the parameter dependence of the minimum manifold angles for five UPOs with periods $T = 4.7986$, 1.9094, 1.4514, 3.4090, and 2.9155 detected at $r = 60$ [Fig. 2 (left)] is investigated. Figure 2 (left) shows that the minimum manifold angles decrease monotonically as r increases. The minimum manifold angle of each UPO takes the minimum value at one of the edges, a saddle-node bifurcation point, of the corresponding periodic window. Note that the value of the minimum manifold angle is not zero but a small positive value at the saddle-node bifurcation point [14]. Figure 2 (left) also shows that the order of the minimum manifold angles of five UPOs holds for any parameter r. This indicates that if there is a UPO which has a smaller minimum manifold angle at a certain parameter, the corresponding periodic window exists closer to the parameter. Conversely, if the minimum manifold angle of a UPO at a certain parameter is larger, the corresponding periodic window exists farther from the parameter. Table I shows more examples of the relation between the minimum manifold angles of UPOs at $r = 60$ and the parameter values $r_{\rm SN}$ of the edges of the corresponding periodic windows, where

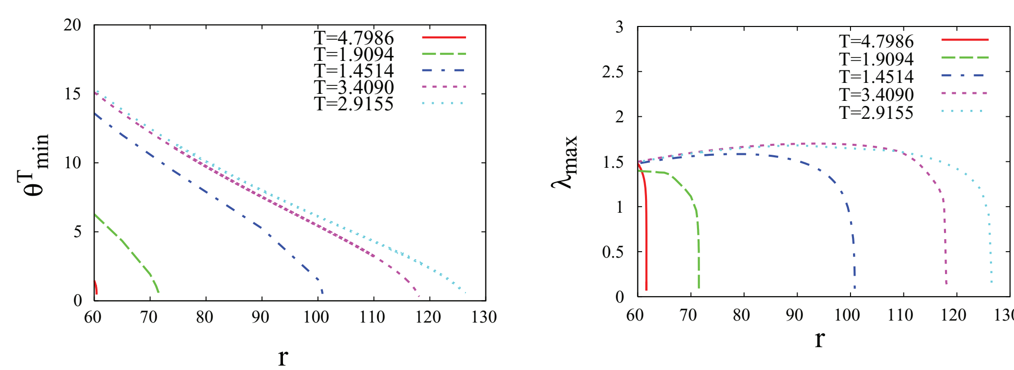

FIG. 2. (Color online) Dependence of minimum manifold angles $\theta^T_{\rm min}$ on parameter r for five UPOs with period T at $r = 60$ (left). Dependence of the maximum Lyapunov exponents $\lambda_{\rm max}$ on parameter r for five UPOs with period T at $r = 60$ (right).

TABLE I. Relation between minimum manifold angles $\theta^T_{\rm min}$ of stable and unstable manifolds at points on each UPO with period T at $r = 60$ and the saddle-node bifurcation point ($r^T_{\rm SN}$) of the corresponding periodic window. S and N represent the origin of UPOs at the corresponding saddle-node bifurcation points, the saddle orbit (S) and the node orbit (N). Periodic windows corresponding to UPOs which have small (large) manifold angles exist near (away from) the parameter.

$\theta^T_{\rm min}$	T	S/N	$r^T_{\rm SN}$
1.00	5.2497	S	60.42
1.24	4.7767	N	60.25
1.47	5.2558	N	60.42
1.56	5.7184	S	60.51
2.32	4.7986	N	61.63
5.97	2.9737	N	76.82
6.27	1.9094	S	71.53
7.42	3.0216	N	92.51
7.97	3.3820	S	86.40
8.13	1.6127	N	100.79
13.59	1.4514	S	100.79
14.12	3.4090	S	118.13
15.37	2.9155	S	126.52
31.02	0.9773	S	312.96

the saddle-node bifurcation occurs. We select five UPOs with small minimum manifold angles and nine UPOs with relatively large minimum manifold angles at $r = 60$. The result gives us the idea that the manifold angles of UPOs are useful to estimate the sequence of periodic windows and the relative position of the edges of the periodic windows in the parameter space.

It should also be noted that periodic windows are not constructed completely in ascending order of the minimum manifold angles of UPOs (e.g., the first two UPOs in Table I), although periodic windows can be constructed in ascending order at a rough estimate. This is because there are two UPOs (e.g., the first and the third UPOs in Table I) which have the same symbol sequences [9] but are different orbits; one is a saddle orbit and the other is a node orbit at the edge of the periodic window. The order of the minimum manifold angles of these two UPOs can change, because UPOs due to node orbits and those due to saddle orbits exist on different branches (see Fig. 3). However, from Table I the order of minimum manifold angles of UPOs in a set of UPOs (S) originating from saddle orbits [UPOs (N) originating from node orbits] cannot change, because UPOs (S/N) due to saddle/node orbits exist on the branch b_s/b_n (see Fig. 3). In general, a UPO denoted by S/N on the branch b_s/b_n cannot cross another UPO (S/N) when the parameter r changes, because the dimension of the attractor on the Poincaré surface is almost unity. On the other hand, in the Lorenz system at (around) $r = 28$, there are no UPOs with the same symbolic sequences. Thus saddle-node bifurcation points of periodic windows deriving from the UPOs are made completely in ascending order of minimum manifold angles [15].

Figure 4 shows the relation between the minimum manifold angle $\theta^T_{\rm min}$ and the distance Δ_T $(:= r^T_{\rm SN} - 13.926)$ from the saddle-node bifurcation point $r^T_{\rm SN}$ of each UPO with period T to the homoclinic bifurcation point ($\sim$13.926) [9]. The minimum manifold angles and the distances can be

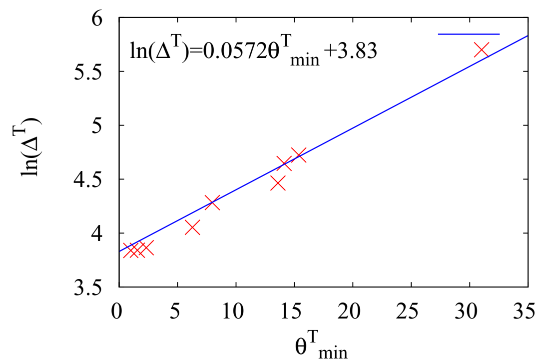

FIG. 4. (Color online) Relation between the minimum manifold angle $\theta^T_{\rm min}$ and $\ln \Delta^T$ for each UPO with period T at $r = 60$. The solid line shows $\ln(\Delta^T) = 0.0572\theta^T_{\rm min} + 3.83$.

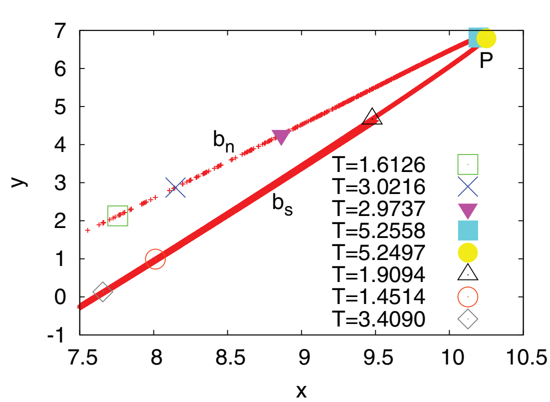

FIG. 3. (Color online) Chaotic attractor (small +) and some UPOs with period T on a Poincaré section (x,y) of $z = r - 1$ at $r = 60$. Poincaré plots of UPOs originating from saddle orbits at the saddle-node bifurcation points exist only on the branch b_s, whereas those originating from node orbits exist on both branches b_s and b_n. A saddle orbit on b_s (e.g., $T = 5.2497$ at $r = 60$) and a node orbit on b_n (e.g., $T = 5.2558$ at $r = 60$) will collide near the point P.

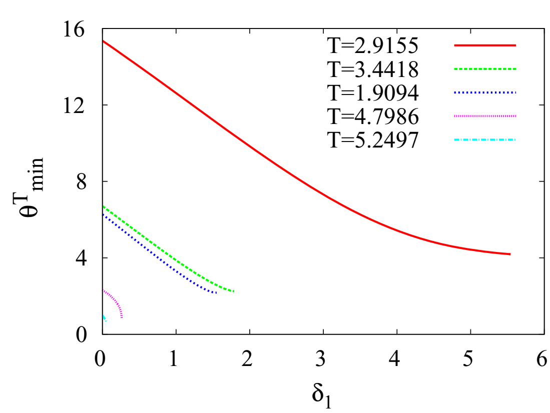

FIG. 5. (Color online) Dependence of the minimum manifold angles $\theta^T_{\rm min}$ on parameter δ_1 for five UPOs with period T at $r = 60$. Parameters r and σ are changed as $r = r_0 - \delta_1$ and $\sigma = \sigma_0 - \delta_1$, where $r_0 = 60$ and $\sigma_0 = 10$.

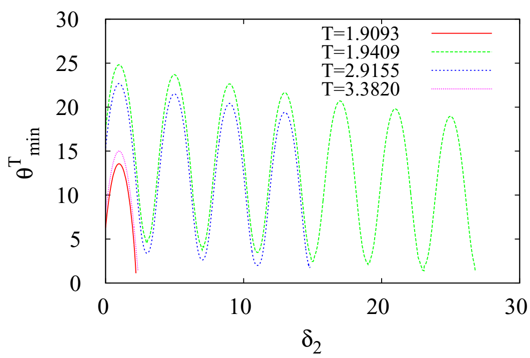

FIG. 6. (Color online) Dependence of the minimum manifold angles $\theta^T_{\rm min}$ on parameter δ_2 for four UPOs with period T at $r = 60$. Parameters r, σ, and b are changed as $r = r_0 + \delta_2$, $\sigma = \sigma_0 + 5\sin(\delta_2\pi/2)$, and $b = b_0 + \sin(\delta_2\pi/2)$, where $r_0 = 60$, $\sigma_0 = 10$, and $b_0 = 8/3$.

fitted by using the least-squares method, and we find that $\Delta^T = \exp(\alpha\theta^T_{\rm min} + 3.83)$ with $\alpha = 5.72 \times 10^{-2}$ (rms error: 2.2×10^{-3}). 3.83 is chosen such that when the minimum manifold angles ($\theta^T_{\rm min}$) of UPOs at $r = 60$ are almost 0, the value of $r^T_{\rm SN}$ is 60. We can predict a parameter value of the corresponding window from the minimum manifold angle $\theta^T_{\rm min}$ of some UPOs. We cannot obtain the results by employing the maximum Lyapunov exponent, which cannot reflect local structures, because the order of the Lyapunov exponents can change [see Fig. 2 (right)]. Thus the minimum manifold angle is an important quantity in analyzing chaotic dynamical systems which will open the door to solving unexplored problems.

In the above results, we have calculated the minimum manifold angles of UPOs by changing only the parameter value r. Here, we discuss the minimum manifold angles by varying various parameters simultaneously. Figure 5 shows the dependence of the minimum manifold angles on the parameter δ_1, where the parameters r and σ are changed as $r = r_0 - \delta_1$ and $\sigma = \sigma_0 - \delta_1$ ($r_0 = 60$ and $\sigma_0 = 10$). The angles in this case also change monotonically and the order of minimum manifold angles does not change as in the case of changing the only parameter r.

These results imply the robustness of the order of the minimum manifold angles. This property is consistent with a result about saddle-node bifurcation lines in a parameter plane of the Lorenz system [16]. On the other hand, it is imagined that the angles do not change monotonically in some cases. In fact, we can find such a case by changing parameters in a complex way, i.e., $r = r_0 + \delta_2$, $\sigma = \sigma_0 + 5\sin(\delta_2\pi/2)$, and $b = b_0 + \sin(\delta_2\pi/2)$, where δ_2 is a control parameter, and $r_0 = 60$, $\sigma_0 = 10$, and $b_0 = 8/3$ (see Fig. 6). However, even in this case, the minimum manifold angles still keep the order.

III. HÉNON MAP

Here, we consider the Hénon map of real variables:

$$x_{n+1} = a - x_n^2 + 0.3y_n, \quad y_{n+1} = x_n.$$

We show minimum manifold angles at points on UPOs at $a = 1.4$ in Fig. 7 (left). We choose four UPOs (period 13) with various minimum manifold angles at the parameter and calculate the parameter dependence of them [see Fig. 7 (right)]. Figure 7 (right) shows that the minimum manifold angle along each UPO decreases monotonically as a decreases until the corresponding branch vanishes via the saddle-node bifurcation, and that the order of minimum manifold angles for the four UPOs holds for any parameter a. This indicates that if there is a UPO which has a small minimum manifold angle at a certain parameter, the corresponding periodic window exists near the parameter. The parameter at which the minimum manifold angle takes the minimum value corresponds to the saddle-node bifurcation point. We have confirmed that these properties hold for many other branches of UPOs. Notice that these results for the Hénon map are almost the same as those obtained in the Lorenz system.

IV. KURAMOTO-SIVASHINSKY SYSTEM

Next, we consider the Kuramoto-Sivashinsky (KS) system [17]. The KS system is one of the simplest partial differential equations that can exhibit spatiotemporal chaos. It has the form

$$u_t + 2uu_x + u_{xx} + \nu u_{xxxx} = 0,$$

where $u = u(x,t)$ ($x \in [0,2\pi]$) is a real variable and ν is the control parameter. We impose the 2π periodic boundary condition $u(x,t) = u(x + 2\pi,t)$. To calculate the equation, we adopt the spectral method with Fourier decomposition $u(x,t) = \sum_{k=-\infty}^{\infty} b_k(t)e^{ikx}$. To simplify the system we restrict

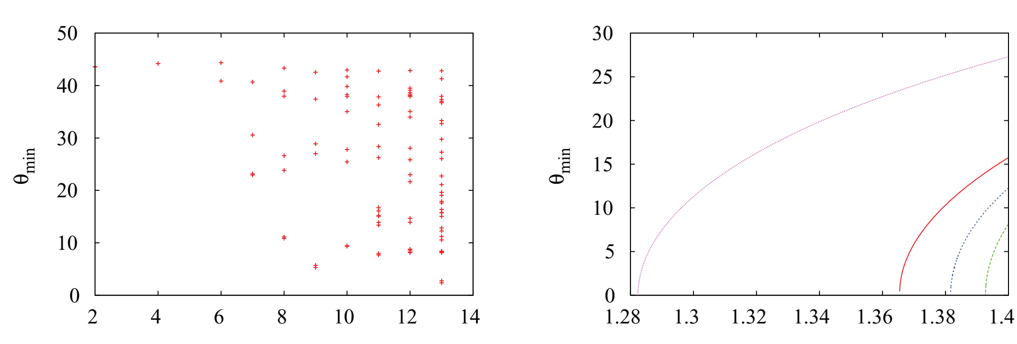

FIG. 7. (Color online) Minimum manifold angle $\theta_{\rm min}$ of each UPO with period p at $a = 1.4$ (left). Dependence of minimum manifold angles $\theta_{\rm min}$ on parameter a for four UPOs with period 13 at $a = 1.4$ (right).

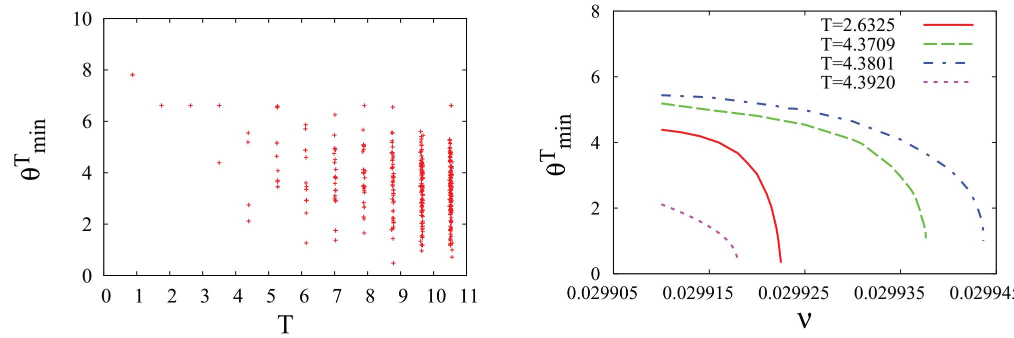

FIG. 8. (Color online) Minimum manifold angle $\theta^T_{\rm min}$ of each UPO with period T at $\nu = 0.02991$ (left). Dependence of minimum manifold angles $\theta^T_{\rm min}$ on parameter ν for four UPOs with period T at $\nu = 0.02991$ (right).

our attention to the subspace of odd functions $u(x,t) = -u(-x,t)$, and assume $b_k(t) = -i a_k(t)/2$ [18,19]. The set of equations is given by

$$\dot{a}_k(t) = (k^2 - \nu k^4) a_k(t) + k \sum_m a_m(t) a_{k-m}(t),$$

where $a_0 = 0$, $1 \leqslant (k,m) \leqslant N$, and N is the truncation order. We choose $N = 16$. Here, we again investigate the relation between positions of periodic windows and manifold structures of UPOs in the KS system around the parameter $\nu = 0.02991$.

We show minimum manifold angles at points on each of numerically detected hundreds of UPOs at $\nu = 0.02991$ in Fig. 8 (left). This figure indicates that there are some UPOs which pass near the tangencies. That is, the tangencies can be characterized by the UPOs as in the case of the Lorenz system. Next, we calculate the parameter dependence of minimum manifold angles for four UPOs with periods $T = 2.6325$, 4.3709, 4.3801, and 4.3920 at $\nu = 0.02991$. Figure 8 (right) shows that the minimum manifold angle along each UPO decreases monotonically as ν increases until the corresponding branch vanishes via the saddle-node bifurcation. This indicates that if there is a UPO which has a small minimum manifold angle at a certain parameter, the corresponding periodic window exists near the parameter. Furthermore, Fig. 8 (right) indicates that the order of minimum manifold angles for four UPOs holds for any parameter ν. The parameter at which the minimum manifold angle takes the minimum value corresponds to the saddle-node bifurcation point. It should be noted that the minimum manifold angle at the edge is around 0 but bounded from 0 [14]. We have confirmed that these properties hold for many other branches of UPOs. Notice that these results for the KS system are almost the same as those given in the Lorenz system and the Hénon map. We conjecture that the reason for this is the low dimensionality of these attractors. In fact, the Kaplan-Yorke dimension of the KS system is $D = 2.2$.

V. SUMMARY

In this paper, we have obtained results on the relation between the manifold structures of unstable periodic orbits (UPOs) and the appearance of periodic windows corresponding to the UPOs for the Lorenz system, the Hénon map, and the Kuramoto-Sivashinsky system. In these systems, the minimum manifold angle between a stable and an unstable manifold at each point on a UPO decreases as a parameter approaches the edge of the corresponding periodic window, although the minimum manifold angle does not necessarily change monotonically in a parameter space. The most important point is that the order of the minimum manifold angles of UPOs that originated from the only saddle branches (only node branches) of the saddle-node bifurcation is preserved. This property cannot be found when we employ Lyapunov exponents.

Furthermore, the manifold angle becomes very small but is positive at the edge of a periodic window where the saddle-node bifurcation occurs, suggesting that the system at the edge of each periodic window is nonhyperbolic without tangencies. It has been found from a different study [15] that the three systems studied here possess tangencies at the Feigenbaum points, where period doubling bifurcations occur infinitely many times. A saddle-node bifurcation point locates near the Feigenbaum point in a parameter space. Thus, the minimum manifold angle of a UPO originating from a saddle-node bifurcation takes the minimum value around the corresponding bifurcation point. Furthermore, when the minimum manifold angle of a UPO at a certain parameter is small (large), the corresponding periodic window exists near (away from) the parameter. This result indicates that by calculating manifold angles of UPOs we can predict the window sequence in a parameter space even if the corresponding periodic windows considered are quite small. In future work, it should be clarified whether similar results can be obtained in higher dimensional systems.

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