Quantum information-geometry of dissipative quantum phase transitions

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A general framework for analyzing the recently discovered phase transitions in the steady state of dissipation-driven open quantum systems is still lacking. To fill this gap, we extend the so-called fidelity approach to quantum phase transitions to open systems whose steady state is a Gaussian fermionic state. We endow the manifold of correlation matrices of steady states with a metric tensor g measuring the distinguishability distance between solutions corresponding to a different set of control parameters. The phase diagram can then be mapped out in terms of the scaling behavior of g and connections with the Liouvillean gap and the model correlation functions unveiled. We argue that the fidelity approach, thanks to its differential-geometric and information-theoretic nature, provides insights into dissipative quantum critical phenomena as well as a general and powerful strategy to explore them.

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I. INTRODUCTION

The occurrence of typical equilibrium phenomena in out-of-equilibrium driven condensed-matter systems (e.g., long-range order, topological order, and quantum phase transitions) has been recently discovered [1–4]. This poses new, fascinating, and challenging problems both at the theoretical and experimental levels. Indeed, it has been shown that dissipation processes can in principle be controlled and tailored in order to compete with system-free evolution and to realize fundamental protocols such as quantum state preparation [5], quantum simulation [6], and computation [7]. The natural question that arises is whether and how the methods typically used in the equilibrium realm can be adapted to characterize nonequilibrium problems. In particular, the occurrence of quantum phase transitions (QPTs) in nonequilibrium steady states (NESSs), which are the results of complex many-body dissipative evolutions, is far from being understood, and we still lack a comprehensive and systematic framework that can link equilibrium and nonequilibrium properties.

In this paper, we propose an information-geometric strategy for describing NESS-QPT based on the study of a quantity borrowed from quantum information theory, i.e., the fidelity $\mathcal{F}$ between quantum states. This general approach has been successfully applied to a large variety of ground-state QPTs (GS-QPTs) [8–11] and quantum chaos [12]. In the context of NESS-QPTs, the set of (control) parameters $\boldsymbol{\lambda} \in \mathcal{M}$ defines a Liouvillean superoperator $\mathcal{L}(\boldsymbol{\lambda})$ which drives the system, independently of the chosen initial state, to the corresponding (unique) NESS $\rho(\boldsymbol{\lambda})$. Depending on $\boldsymbol{\lambda}$, the NESS can exhibit quite different properties and the system can exhibit NESS-QPTs. The main idea behind the fidelity approach is the following: when dramatic structural changes occur in $\rho(\boldsymbol{\lambda})$, e.g., approaching a critical point, the geometric-statistical distance $d[\rho(\boldsymbol{\lambda}),\rho(\boldsymbol{\lambda}+\delta\boldsymbol{\lambda})]$ between two infinitesimally close states grows as they become more and more statistically distinguishable. Although there are several metrics in information geometry [13–15] for (mixed) density operators $\rho(\boldsymbol{\lambda})$ [16], here we concentrate on the Bures metric $ds_B^2 = 2[1-\mathcal{F}(\rho,\rho+d\rho)]$. The latter is written in terms of the Uhlmann fidelity [17] $\mathcal{F}$, and, in turn, represents the natural measure of *distinguishability*. The infinitesimal distance ds_B^2 is the fundamental tool of the fidelity approach: it has been shown that the study of its scaling behavior (extensive versus superextensive) allows a systematic study of GS-QPTs [10,18]. When expressed in terms of the physical $\boldsymbol{\lambda}$ of the model, the Bures distance provides a metric g onto the parameter manifold $\mathcal{M}$. The tensor elements of g allow one to study the responsiveness of the system to small variations of the physical parameters and eventually detect a phase transition. Although the metric tensor g is not coordinate-free, as long as the coordinate transformations are not singular themselves and are independent of the system size, the finite-size scaling of the metric is an intrinsic property of the family of quantum states that is parametrization-independent. This simple fact has been amply recognized in the corresponding fidelity approach for closed quantum systems [9,10] and applied here verbatim.

Dissipative QPTs are of a different nature from the standard QPTs at zero temperature. Accordingly, in spite of some obvious yet somewhat superficial similarity, their understanding calls for a different set of conceptual as well as mathematical tools. In the first place, stationary states are the result of an equilibration process: NESS-QPTs need a new equilibration time after the perturbation and, as such, they are not a result of an *adiabatic* reorganization of the (ground) state. From a mathematical point of view, a NESS is the zero eigenvalue *density matrix* of the *non-Hermitian* Liouvillean superoperator $\mathcal{L}$, as opposed to pure eigenvectors of a Hermitian Hamiltonian operator H. This implies that, on the one hand, one has to employ the more sophisticated information-geometry of mixed states and, on the other hand, that the whole wealth of powerful results stemming out of Hermiticity, e.g., the spectral theorem and perturbation theory, is simply not available in the dissipative case. The challenge here is to find a suitable way to parametrize the manifold of stationary states and pull back into the parameter manifold the state metric. This is in general a quite daunting task, but restricting to the physically relevant case of a quadratic Liouvillean can be achieved.

Specific models belonging to this class indeed display rich nonequilibrium features and NESS-QPTs, which have been characterized by studying long-range magnetic correlations and the Liouvillean spectral gap $\Delta_{\mathcal{L}}$ [1,19]. However, to the best of the authors' knowledge, a precise mathematical definition of a NESS-QPT is still lacking. This paper aims to be an attempt toward the use of the fidelity approach for *defining* the NESS-QPT.

We derive a general formula for the Bures distance over the set of Gaussian fermionic (GF) states and the metric tensor g over the parameter manifold. Then we discuss how the scaling of the metric implies both the closing of $\Delta_{\mathcal{L}}$ and the divergences of some two-point correlations. Finally, we apply our theoretical framework to exactly solvable models. Our analysis demonstrates that the NESS phase diagram can be accurately mapped by studying the (finite-size) scaling behavior of the metric tensor g; critical lines can be identified and the different phases distinguished.

II. BURES METRIC FOR GAUSSIAN FERMIONIC STATES

The calculation of the Bures distance is a notoriously hard task for large Hilbert spaces: standard methods [16] are computationally not applicable for many-body systems, and finding an efficient way to evaluate ds_B^2 is still a subject of active research [20]. Here we show a compact and efficient way to evaluate the Bures metric (for convenience, we use a rescaled metric $ds^2 = 8\,ds_B^2$) when the state space is restricted to the physically important case of Gaussian fermionic states. Consider a system of n fermion modes described by a set of $2n$ Majorana operators w_i. These operators are Hermitian, depend linearly on the fermionic creation and annihilation operators via $w_\ell = f_\ell + f_\ell^\dagger$, $w_{n+\ell} = i(f_\ell - f_\ell^\dagger)$, $\ell = 1,\ldots,n$, and satisfy the algebra $\{w_i,w_j\} = 2\delta_{ij}$. A GF state ρ, i.e., a Gaussian state in terms of the operators w_j, is completely specified by the two-point correlation functions $C_{ij} = \frac{1}{2}\langle[w_i,w_j]\rangle_\rho$, where the complex $2n\times 2n$ matrix C is imaginary and antisymmetric.

With this natural parametrization, the metric can be pulled back from the many-body Liouville space to the manifold of the two-point correlation functions. Indeed, in Appendix A we have shown that the fidelity metric around the GF state ρ specified by the correlation function C is given by

$$ds^2 = \mathrm{Tr}[dC(\mathbb{1} - \mathrm{Ad}_C)^{-1} dC] \;=:\; \big\|(\mathbb{1} - \mathrm{Ad}_C)^{-\frac{1}{2}}\, dC\big\|_2^2, \tag{1}$$

where $\mathrm{Ad}_C X := C X C^\dagger = C X C$ is the adjoint action, and $^{-1}$ refers to the pseudo-inverse. In particular, when ρ is pure, $\mathrm{Sp}(C) = \{\pm 1\}$ and the above equation reduces to $ds^2_{\mathrm{pure}} = \|dC\|_2^2/2$.

This is an interesting result, but it is just the first step of our analysis. In fact, the crucial physical information is contained in the external parameters $\{\lambda_\mu\} \in \mathcal{M}$ of the model. As $dC = \sum_\mu d\lambda_\mu \partial_\mu C$, we obtain

$$ds^2 = \sum_{\mu,\nu} g_{\mu\nu} d\lambda_\mu d\lambda_\nu, \quad g_{\mu\nu} = \sum_{rs}{}' \frac{(\partial_\mu C)_{rs}(\partial_\nu C)_{sr}}{1 - c_r c_s}, \tag{2}$$

where $C = \sum_r c_r |r\rangle\langle r|$, with $c_r \in \mathbf{R}$ and $(\partial C)_{rs} = \langle r|\partial C|s\rangle$, i.e., the sum in the above equation is performed in the basis in which C is diagonal and it is restricted over the elements such that $c_r c_s \neq 1$. The infinitesimal distance ds^2 encodes the statistical distinguishability between two infinitesimally close Gaussian fermionic states; this result is completely general and it can be used to study the geometrical properties of manifolds of GF states. Equations (1) and (2) provide the basic tool for studying the phase transitions occurring when the NESS are GF states. In this respect, a first qualitative indication that the scaling behavior of the metric can spot QPTs is suggested by the following inequality (see Appendix D): $ds^2 \leqslant 2n\, P_C\, \|dC\|_\infty^2$, where $P_C = \|(\mathbb{1} + C^{\otimes 2})^{-1}\|_\infty$ and $\|A\|_\infty$ refers to the maximum singular value of A. If $P_C = O(1)$, a superextensive behavior of ds^2 implies some sort of singularity in the correlation functions that may reflect the occurrence of a phase transition.

III. DISSIPATIVE SOLVABLE MODEL

We consider a Markovian dissipative open quantum system evolution [21] governed by the Lindblad master equation,

$$\frac{d\rho}{dt} = \mathcal{L}\rho := -i[\mathcal{H},\rho] + \sum_\mu (2L_\mu\, \rho\, L_\mu^\dagger - \{L_\mu^\dagger\, L_\mu,\rho\}), \tag{3}$$

with a quadratic Hamiltonian $\mathcal{H} = \sum_{ij} H_{ij}\, w_i w_j$ and linear Lindblad operators $L_\mu = \sum_i \ell_{\mu i}\, w_i$, where the matrices H and ℓ depend on the parameters $\boldsymbol{\lambda} \in \mathcal{M}$ defining the specific model. In the following, we obtain the steady state Ω, namely the state for which $d\Omega/dt = \mathcal{L}\Omega = 0$, and pull back the set of admissible NESS to the parameter manifold. The Liouvillean can be written as a quadratic form in terms of the following set of $2n$ creation and annihilation superoperators:

$$a_j^\dagger \cdot = -\frac{i}{2}\, W\, \{w_j,\cdot\}, \quad a_j \cdot = -\frac{i}{2}\, W\, [w_j,\cdot], \tag{4}$$

where $W = i^n \prod_{j=1}^{2n} w_j$ is a Hermitian idempotent operator which anticommutes with all the w_j. A direct calculation proves that the operators defined in Eq. (4) satisfy the canonical anticommutation relations (CAR), $\{a_j^\dagger,a_k\} = \delta_{jk}$, and that $\mathcal{L} = -\sum_{ij}(X_{ij}\, a_i^\dagger a_j + Y_{ij}\, a_i^\dagger a_j^\dagger/2)$, where $X = 4(iH + \mathrm{Re}M) \equiv X^*$, $Y = -8i\,\mathrm{Im}M \equiv -Y^* \equiv -Y^T$, with $M_{ij} = \sum_\mu \ell_{\mu i}\ell_{\mu j}^* \equiv M^\dagger$. This result was derived in [22], but thanks to our definition (4), complex transformations [23] for unifying the different parity sectors are avoided. The *two-point* correlation functions in the steady state, $C_{ij} = \langle[w_i,w_j]\rangle$, are obtained from the solution of the following Sylvester equation [23]:

$$X\,C + C\,X^T = Y. \tag{5}$$

As shown in Appendix B, the matrix C also plays a central role in the diagonalization of the Liouvillean. To simplify our analysis, we assume the real matrix X to be diagonalizable, i.e., $X = UxU^{-1}$ for $x = \mathrm{diag}(\{x_i\})$, $x_i \in \mathbf{C}$, as this condition is always satisfied in our numerical simulations; the general (nondiagonalizable) case is discussed in Appendix C. The transformation $\mathbf{d} = U^{-1}(\mathbf{a} + C\mathbf{a}^\dagger)$, $\mathbf{d}^\times = U^T\mathbf{a}^\dagger$, realizes a nonunitary Bogoliubov transformation and brings $\mathcal{L}$ to the diagonal form $\mathcal{L} = -\sum_k x_k\, d_k^\times d_k$. The (unnormalized) steady state Ω is then obtained as the $\mathbf{d}$-vacuum, ($d_i\Omega = 0$, $\forall\; j =$

$1,\ldots,2n$), i.e.,

$$\Omega = e^{-\frac{1}{2}\mathbf{a}^\dagger \cdot C\mathbf{a}^\dagger}(\mathbb{1}), \tag{6}$$

where the identity operator is the **a**-vacuum. The physical conditions for the existence and uniqueness of the steady state are given in [24]: if $\Delta := 2\min_i \mathrm{Re}(x_i) > 0$, then the solution of (5) is unique and every initial state converges for $t\to\infty$ to the unique steady state (6). The gap Δ represents both the inverse of the time scale for reaching the steady state and the gap of the Liouvillean: $\min\{|\sum_j x_j n_j| : n_j \in \{0,1\}\} \equiv \Delta$.

If $\Delta > 0$, the steady state $\Omega(\boldsymbol{\lambda})$ is unique, and since $\mathcal{L}$ smoothly depends on the parameters $\boldsymbol{\lambda} \in \mathcal{M}$, it is a smooth function of $\boldsymbol{\lambda}$ [25]. If the gap $\Delta(n)\to 0$ for $n\to\infty$, the steady state $\Omega(\boldsymbol{\lambda})$ may become a *nondifferentiable* function of $\boldsymbol{\lambda}$. However, NESS-QPT are *not* defined by the closing of the Liouvillean gap. Nevertheless, the scaling properties of $\Delta(n)$ have been used as indicators of NESS criticality [19,26–28]. Motivated by this, in Appendix D we derive the following upper bound, which relates the behavior of $\Delta(n)$ and ds^2:

$$\frac{ds^2}{n} \leqslant 2\frac{P_C}{\Delta^2}(\|dY\|_\infty + 2\|dX\|_\infty)^2. \tag{7}$$

The latter is the dissipative analog of the GS-QPT one given in [10], where it was shown that superextensivity of ds^2 implies the closing of the Hamiltonian gap [10] and the occurrence of criticality. Here the bound intriguingly links the geometric quantity ds^2 to the dynamical property Δ, and it provides the following information: if the numerator of the right-hand side in (7) is $O(1)$, then any superextensive behavior of $ds^2 = O(n^{\alpha+1})$, $\alpha > 0$, implies that the Liouvillean gap Δ closes at least as $O(n^{-\alpha/2})$. Therefore, the geometric properties of the NESS manifold set the minimal time scales for the reaching of the steady state.

In the next sections, we specialize our results to particular solvable instances of (3) and we perform numerical and analytical analyses aimed at validating the importance and usefulness of the fidelity approach to NESS-QPT and at comparing the scaling properties of Δ and ds^2. In particular, we will focus on the scaling of metric tensor $g_{\mu\nu}$ and of its largest eigenvalue $|g|$, which depends on the control parameters of the model. However, the scaling of the metric tensor *is* a coordinate-free concept, as long as the coordinate transformations over the control manifold do not depend on n. This latter is always going to be the case on physical grounds, as the control manifold's dimension is unrelated to the Hilbert space one and therefore does not scale with the number of particles.

IV. BOUNDARY-DRIVEN *XY* SPIN CHAIN

We now concentrate on a solvable spin-$\frac{1}{2}$ model exhibiting a NESS-QPT [1]. Coherent interactions are described by the XY Hamiltonian

$$H = \sum_{i=1}^{n-1}\left(\frac{1+\gamma}{2}\sigma_i^x\sigma_{i+1}^x + \frac{1-\gamma}{2}\sigma_i^y\sigma_{i+1}^y\right) + h\sum_{i=1}^{n}\sigma_i^z, \tag{8}$$

where σ_j^α are the Pauli operators acting on the jth spin. The two boundary spins of the chain are coupled to two (thermal) reservoirs via the Lindblad operators $L_L^\pm = \sqrt{\Gamma_L^\pm}\sigma_1^\pm$, $L_R^\pm = \sqrt{\Gamma_R^\pm}\sigma_n^\pm$, where $\sigma_j^\pm = (\sigma_j^x + i\sigma_j^y)/2$, and the strengths $\Gamma_{L,R}^\pm$ depend on the reservoir parameters as well as on their temperature [23]. Due to the Jordan-Wigner transformation, such a model can be exactly described by a quadratic Majorana master equation (3). The steady state of the resulting dissipative Markovian evolution is therefore Gaussian, and different phases can be identified depending on the parameters (h,γ) of the Hamiltonian (8). Along the lines $h=0$, $\gamma=0$, and for $h > h_c = |1-\gamma^2|$, magnetic correlations are short-ranged, i.e., the correlation function $C_{ij}^{zz} = \langle\sigma_i^z\sigma_j^z\rangle$ exhibits an exponential decay, $C_{ij}^{zz} \approx e^{-|i-j|/\xi}$, with a localization length $\xi \approx \sqrt{2h_c/(h-h_c)}/8$. On the other hand, for $h < h_c$ a phase with long-range magnetic correlations emerges which is characterized by nondecaying structures in C_{ij}^{zz} and a strong sensitivity to small changes of the parameters. Around the critical point h_c, one finds a power-law behavior $C_{ij}^{zz} \approx |i-j|^{-4}$.

TABLE I. Scaling analysis of the gap Δ and of the maximum eigenvalue of the fidelity metric $g_{\mu\nu}$. These laws do not depend on the particularly chosen rate $\Gamma_{L,R}^\pm$. An asterisk denotes that the lines $h=0$ and $\gamma=0$ consist of a short-range region embedded in the long-range phase; one finds (see discussion in the text) $|g| \approx g_{hh}$ for $h=0$ and $|g| \approx g_{\gamma\gamma}$ for $\gamma=0$.

Phase	Parameters	Δ	$\lvert\mathbf{g}\rvert$
Critical *	$h=0$	n^{-3}	n^6
Long-range	$0<\lvert h\rvert<h_c$	n^{-3}	n^3
Critical	$\lvert h\rvert \approx h_c$	n^{-5}	n^6
Short-range	$\lvert h\rvert > h_c$	n^{-3}	n
Critical *	$\gamma=0, \lvert h\rvert<h_c$	n^{-3}	n^2

In Table I we summarize the scaling analysis performed. Our results show that the Liouvillean gap and the metric encode different information. Indeed, unlike the Hamiltonian gap ruling ground state QPT, the Liouvillean gap Δ closes for $n\to\infty$ both at the critical point and for $h \neq h_c$, both in the long-range and short-range phase. As the reservoirs act only at the boundaries of the spin chain, the eigenvalues x_k of the matrix X for $n \gg 1$ are a small perturbation of the $n\to\infty$ case, where $x_k = \pm 4i\omega_k$, where $\omega_k = \sqrt{(\cos k - h)^2 + \gamma^2\sin^2 k}$ is the quasiparticle dispersion relation of the Hamiltonian (8). In particular, x_k gains a small real part and one finds a gap $\Delta = O(n^{-3})$ for $h \neq h_c$ and $\Delta = O(n^{-5})$ for $h = h_c$. Therefore, the scaling of the Liouvillean gap allows one to identify the transition from the short-range phase to the long-range phase only along the critical line $h = h_c$, while the transition occurring at the $h=0$ (or $\gamma=0$) line can only be appreciated by evaluating the long-rangedness of the magnetic correlations. The question that naturally arises is how the different phases and transitions can be precisely characterized in a way similar to what happens for GS-QPTs. This question becomes more compelling if one compares the above results with the scaling of the geometric tensor $g_{\mu\nu}$, and in particular of its largest eigenvalue $|g|$; see Table I and Fig. 1 for specific values of the parameters.

A first important result is that the tensor g is able to identify the transitions between short-range and long-range

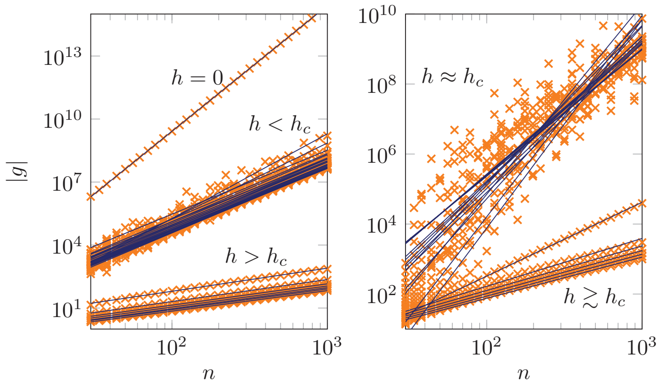

FIG. 1. (Color online) Scaling of $|g|$ for $\gamma = 0.6$ and $h \in [0,0.8]$ (left) and for $\gamma = 0.5$ and $h \in [0.735,0.755]$ (right). In both cases, $\Gamma_L^+ = 0.3$, $\Gamma_L^- = 0.5$, $\Gamma_R^+ = 0.1$, $\Gamma_R^- = 0.5$. Points represent the numerical data, while lines are linear fits, whose results are summarized in Table I. $|g|$ fluctuates slightly as a function of n in the long-range phase, and the relative amplitude of the fluctuations increases close to the critical field h_c. Due to finite-size effects and to the differential nature of the geometric tensor, the value where $|g|$ takes its maximum is slightly smaller than h_c, and this difference depends on n.

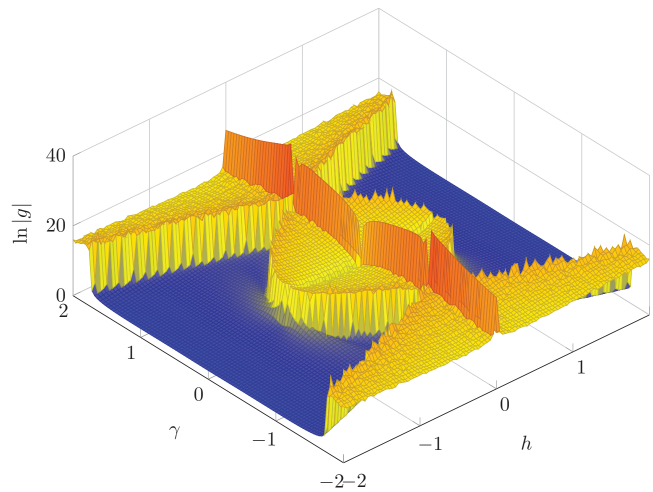

FIG. 2. (Color online) Maximum eigenvalue $|g|$ of the fidelity metric (2) for $n = 250$. The Lindblad parameters are the same as in Fig. 1. The larger value of $|g|$ close to the phase transition line $h = h_c(\gamma)$ is not evident in Fig. 2 because of the numerical mesh, and because the actual values of $|g|$ for $h \approx h_c$ can be comparable to those of the long-range phase, depending on n (e.g., see Fig. 1). The qualitative form of Fig. 2 is not affected by different values of the Lindblad parameters $\Gamma_{L,R}^{\pm}$ and by the dimension n.

phases. On the "transition lines" $h = 0$ and $h = h_c$ one has that $|g| = O(n^6)$, while in the rest of the phase diagram $|g| < O(n^6)$. Furthermore, a closer inspection of the elements of g shows that while $g_{hh}(h = 0,\gamma) = O(n^6)$, one has that $g_{\gamma\gamma}(h = 0,\gamma) = O(n)$: the scaling is superextensive only if one moves away from the line $h = 0$ (g_{hh}) and enters the long-range phase, while if one moves along the $h = 0$ line ($g_{\gamma\gamma}$), i.e., if one remains in the short-range phase, the scaling is simply extensive and it matches the scaling displayed in the other short-range phase $h > h_c$. On the other hand, the transition occurring at $\gamma = 0$ has a different scaling: $g_{\gamma\gamma} = O(n^2)$ while $g_{hh} \approx 0$. These findings can be further confirmed by a detailed study [29] based on the analytical results available for $\gamma \ll 1$ [23]. The understanding of the transitions occurring at $h = 0$ or $\gamma = 0$ is complicated by the fact that each of these lines corresponds both to the critical line separating two phases, but also to one of those phases. Therefore, in these lines there are mixed features: there is both a phase with short-range correlations and a critical line separating two phases, which is detected by the superextensive scaling of some metric tensor elements. Moreover, it turns out that the introduction of the magnetic field or the anisotropy drives different transitions whose specificity is accounted for by the different superextensive scalings of different components of the metric tensor.

Another important result shown in Table I is that the metric tensor is able to signal the presence of long-range correlations: within the long-range phase, ds^2 scales superextensivity as $|g| = O(n^3)$, and this superextensive behavior is different from that displayed at the transition lines. One is therefore led to conjecture that all long-range phases have a critical character, due to the presence of long range correlations.

The findings discussed above demonstrate that the metric tensor g, being directly linked to the correlation properties of the Gaussian NESS, encodes all the relevant information about the dissipative phase transition featured by the model (8). In particular, the specificity of the different phases (short-range versus long-range) and the information about the physically relevant parameters (whether it is the magnetic field or the anisotropy) that drive the different transitions are properly accounted for. As shown in Fig. 2, the complete phase diagram can indeed be reconstructed with the study of the single function g. While these results are specific to the model examined, the connection established in (2) roots the behavior of g in the correlation properties of the general class of GF states. Accordingly, one expects the fidelity approach to have a broader scope of application. We would like to stress that there are compelling questions that are still unanswered. In the first place, there is the relation between g and other relevant quantities that have been used so far to characterize NESS-QPT. For the model (8), these are the ranges of correlations and the finite-size scaling of the Liouvillean gap Δ. The latter does not entirely capture the criticality phenomenon, and further investigation of the relation between criticality in NESS-QPT and geometrical and dynamical aspects is in order [30]. Notice also that, in the XY model, different types of symmetries (discrete versus continuous) are broken moving away from the $h = 0$ or $\gamma = 0$ line. It would be interesting to understand whether the scaling exponents of ds^2 at different lines can be related to different nonequilibrium universality classes. Extending the present results to non-Gaussian states [31] and transitions [32] is also an important future direction.

V. TRANSLATIONALLY INVARIANT CASE

To support the generality of the geometric approach in understanding dissipative phase transitions, we apply our theoretical framework to a different dissipative model, first introduced in [19]. We consider an XY spin chain on a ring where each site is coupled to the environment via $L_i^- = \epsilon\mu f_i$, $L_i^+ = \epsilon\nu f_i^\dagger$: these Lindblad operators describe a competition between particle-loss and particle-gain processes. The *closed boundary conditions* and the *uniform* interaction with the environment make the phase diagram very different from the previous one.

In this case, the quadratic Liouvillean is translationally invariant and can be diagonalized with a Fourier transformation together with a Bogoliubov transformation. In the Fourier basis, the two-point correlation function matrix takes the following form [19] in the weak-coupling limit $\epsilon \to 0$:

$$C = i\frac{\Lambda}{2}\bigoplus_k \begin{pmatrix} 0 & 1+e^{iq_k} \\ -1-e^{-iq_k} & 0 \end{pmatrix}, \qquad (9)$$

where $q_k = -2\arctan(\frac{\gamma\sin\phi_k}{h-\cos\phi_k})$, and where $\phi_k = 2\pi k/n$ is the quasimomentum, n is the length of the chain, and $\Lambda = \frac{\nu^2-\mu^2}{\nu^2+\mu^2}$. The eigenvalues of the above matrix are $c_{k,\pm} = \pm\Lambda\cos\frac{q_k}{2}$. The metric (2) can be obtained by writing the matrix dC in the basis in which C is diagonal. A straightforward calculation yields

$$ds^2 = \frac{\Lambda^2}{2}\sum_k \frac{1-\Lambda^2\cos^2\frac{q_k}{2}\cos q_k}{1-\Lambda^4\cos^4\frac{q_k}{2}}(dq_k)^2, \qquad (10)$$

$$dq_k = 2\gamma\frac{\sin\phi_k}{\omega_k^2}dh - 2\frac{(h-\cos\phi_k)\sin\phi_k}{\omega_k^2}d\gamma, \qquad (11)$$

where $\omega_k = \sqrt{(\cos\phi_k - h)^2 + \gamma^2\sin^2\phi_k}$ is the dispersion relation of the XY model. An extensive behavior of (10) is given by the continuous limit $\sum_k \to \frac{n}{2\pi}\int_0^{2\pi} d\phi$: if the resulting integral is convergent, no superextensive behavior can occur. However, from (11) it is clear that a possible (the only?) source of a divergent behavior of dq_k^2 is the vanishing of the gap $\min_k \omega_k$. It is known that in the XY model this condition occurs only for $h = 1$, where one finds for $\phi \simeq O(n^{-1})$ that $\min_k \omega \approx O(n^{-1})$. Hence

$$\max_k dq_k \approx O(n)\, dh + O(n^{-1})\, d\gamma, \qquad (12)$$

from which

$$|g| \approx g_{hh} = O(n^2), \quad \text{for} \quad h = 1. \qquad (13)$$

On the other hand, for $\gamma \to 0$, $\omega \simeq |h - \cos\phi|$, so if $h = \cos\phi + O(n^{-1})$ we obtain

$$dq_k|_{\gamma\to 0} = -2\frac{\phi_k}{(h-\cos\phi_k)}d\gamma \simeq O(n)d\gamma, \qquad (14)$$

again recovering the scaling $|g| = O(n^2)$.

In this particular translationally invariant case, the critical points match the known values for GS-QPT: for $\gamma \neq 0$ there is a critical field $h = 1$, while in the XX case the whole segment $|h| < 1$ is critical. The information-geometric content of this dissipative phase transition is not as rich as the one in Table I, and again the scaling of the metric tensor allows one a precise mapping of the phase diagram.

VI. CONCLUSIONS

In this paper, we developed an information-geometric framework for studying dissipative critical phenomena exhibited by the nonequilibrium steady states of Markovian evolutions described by a quadratic Fermionic Liouvillean. The presented results represent a step toward a full mathematical understanding of dissipative quantum critical phenomena via information-geometric concepts. Indeed, to the best of our knowledge, a precise definition and characterization of NESS-QPT is still lacking in the literature. We first derived a general formula for the infinitesimal Bures distance between Gaussian fermionic (mixed) states. This in turn allows one to define a metric tensor g on the manifold of steady states corresponding to different sets of control parameters. The intuitive underlying idea is that a transition between two structurally different phases should be reflected by the statistical distinguishability of pairs of infinitesimally close steady states. The method does not require knowledge about or the existence of any order parameters, as the tensor g is directly connected to the two-point correlation functions which define the Gaussian fermionic steady states. We have shown that the superextensive behavior of the tensor g implies some singularity for $n \to \infty$ in the derivative of the correlation functions. We have applied the method to specific (XY) models and shown that the scaling of the geometric tensor enables one to identify the critical lines and to distinguish between different phases characterized by short- or long-ranged correlations. The metric tensor encodes also for the direction of maximal distinguishability in the parameter manifold, thus allowing a detailed study of the sensitivity of the steady state to small variations of some control parameters. This is a crucial point for experimental applications of dissipative evolution. The scope of the information-geometric approach extends well beyond the important quadratic case analyzed in this paper, and it may pave the way to the systematic study of general nonequilibrium critical phenomena. This in turn would allow the investigation of a broad class of systems and processes which are natural candidates for the preparation of desired quantum states and the realization of quantum protocols.

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APPENDIX A: PROOF OF EQ. (1)

We consider a Gaussian fermionic state written in the following form:

$$\rho = e^{-\frac{i}{4}\sum_{ij} G_{ij}\, w_i w_j}/Z, \qquad (A1)$$

where the matrix G has to be real and antisymmetric. Accordingly, G can be cast in the canonical form by an orthogonal matrix Q, i.e.,

$$G = Q^T \bigoplus_{k=1}^{n} \begin{pmatrix} 0 & g_k \\ -g_k & 0 \end{pmatrix} Q, \quad Q^T = Q^{-1}, \qquad (A2)$$

and it has eigenvalues $\pm i g_k$. Moreover, let $z_i = \sum_j Q_{ij} w_j$ be the new Majorana operators. Hence

$$\rho = \frac{1}{Z}\prod_k \left[\cosh\left(\frac{g_k}{2}\right) - i\sinh\left(\frac{g_k}{2}\right) z_{2k-1}z_{2k}\right], \tag{A3}$$

$$Z = \prod_k 2\cosh\left(\frac{g_k}{2}\right) = \sqrt{\det\left[2\cosh\left(i\frac{G}{2}\right)\right]}, \tag{A4}$$

where we used the fact that the eigenvalues of iG are $\pm g_k$. As $C_{ij} = \frac{1}{2}\langle[w_i,w_j]\rangle = \frac{2i}{Z}\frac{\partial Z}{\partial G_{ij}}$, one can show that

$$C = \tanh\left(i\frac{G}{2}\right). \tag{A5}$$

The correlation matrix $C = C^\dagger = -C^T$ is diagonal in the same basis of G, and its eigenvalues read $c_k = \tanh(g_k/2)$. Hence

$$\rho = \prod_k \frac{1 - i c_k\, z_{2k-1}z_{2k}}{2}, \tag{A6}$$

where $|c_k| \leqslant 1$. Note that for $c_k = \pm 1$, one has $g_k = \pm\infty$, making the ansatz (A1) not well-defined, unlike Eq. (A6). The latter possibility occurs, for instance, for pure states, as is clear from the following explicit expression for the purity of the states (A1) and the states (A1) and (A6):

$$\mathrm{Tr}[\rho^2] = \frac{\det[2\cosh(i\,G)]^{\frac{1}{2}}}{\det\left[2\cosh\left(i\frac{G}{2}\right)\right]} = \sqrt{\det\left(\frac{1+C^2}{2}\right)}. \tag{A7}$$

We now derive the proof of Eqs. (1) and (2), dividing the different steps into three lemmas. At first we assume $c_k \neq \pm 1$ and then we extend the result for including pure states.

Lemma 1. Let ρ,ρ' be two GF states (A1) parametrized by G,G', respectively. Then

$$\mathcal{F}(\rho,\rho') = \mathrm{Tr}\sqrt{\sqrt{\rho}\rho'\sqrt{\rho}} \tag{A8}$$

$$= \frac{\det[\mathbb{1} + \sqrt{e^{iG/2}e^{iG'}e^{iG/2}}]^{\frac{1}{2}}}{\det[\mathbb{1}+e^{iG}]^{\frac{1}{4}}\ \det[\mathbb{1}+e^{iG'}]^{\frac{1}{4}}}. \tag{A9}$$

Proof. This lemma is a direct consequence of the fact that the quadratic Majorana operators form a Lie algebra:

$$\left[\frac{\mathbf{w}\cdot A\mathbf{w}}{4},\frac{\mathbf{w}\cdot B\mathbf{w}}{4}\right] = \frac{\mathbf{w}\cdot[A,B]\mathbf{w}}{4}, \tag{A10}$$

and accordingly

$$e^{\frac{i}{4}\mathbf{w}\cdot A\mathbf{w}}e^{\frac{i}{4}\mathbf{w}\cdot B\mathbf{w}} = e^{\frac{i}{4}\mathbf{w}\cdot D\mathbf{w}}, \quad e^A e^B = e^D. \tag{A11}$$

Thanks to the above identity,

$$\sqrt{\sqrt{\rho}\rho'\sqrt{\rho}} \propto \exp\left(\frac{1}{4}\sum_{ij}\left(\frac{\log[e^{-iG/2}e^{-iG'}e^{-iG/2}]}{2}\right)_{ij} w_i w_j\right), \tag{A12}$$

and using (A4) we find

$$\mathcal{F}(\rho,\rho') = \frac{\det\left[\cosh\left(\frac{1}{4}\log e^{-iG/2}e^{-iG'}e^{-iG/2}\right)\right]^{\frac{1}{2}}}{\sqrt{\det\left[\cosh\left(i\frac{G}{2}\right)\right]^{\frac{1}{2}}\ \det\left[\cosh\left(i\frac{G'}{2}\right)\right]^{\frac{1}{2}}}}, \tag{A13}$$

which is equivalent to (A9).

A convenient parametrization of Eq. (A9) is obtained in terms of the correlation function by defining the new matrix $T = e^{iG}$. Then

$$C = \frac{T-\mathbb{1}}{T+\mathbb{1}}, \quad T^T = T^{-1}, \quad T^\dagger = T, \tag{A14}$$

$$\mathcal{F}(\rho,\rho') =: \mathcal{F}(T,T') = \frac{\det[\mathbb{1} + \sqrt{\sqrt{T}\,T'\sqrt{T}}]^{\frac{1}{2}}}{\det[\mathbb{1}+T]^{\frac{1}{4}}\ \det[\mathbb{1}+T']^{\frac{1}{4}}}. \tag{A15}$$

The following lemma conveys the metric pull back within the manifold of states parametrized by T:

Lemma 2. Let $ds^2 = 8\,ds_B^2 = 16[1 - \mathcal{F}(T,T+dT)]$ be the fidelity metric around the state (A1) pulled back in the space of the matrices T, and let $dT = \partial_\mu T\, d\lambda_\mu$, where $\lambda_\mu \in \mathcal{M}$ are the parameters of the model. Then the fidelity metric can be cast in the form $ds^2 = \sum_{\mu\nu} g_{\mu\nu}\, d\lambda_\mu\, d\lambda_\nu$, where the geometric tensor is

$$g_{\mu\nu} = 2\sum_{ij}\frac{(\partial_\mu T)_{ij}\,(\partial_\nu T)_{ji}}{(1+t_i)(1+t_j)(t_i+t_j)}. \tag{A16}$$

In (A16), the sum is performed in the basis in which T is diagonal, i.e., we set $T = \sum_i t_i |i\rangle\langle i|$ and $(\partial_\mu T)_{ij} = \langle i|\partial_\mu T|j\rangle$.

Proof. Proceeding along the same lines as Section 3 of [33], we obtain for $T' = T + dT$

$$\sqrt{\sqrt{T}T'\sqrt{T}} = T + \sum_{ij}|i\rangle\langle j|\frac{\sqrt{t_i t_j}}{t_i+t_j}dT_{ij} - \sum_{ijk}|i\rangle\langle k|\,dT_{ij}dT_{jk} \times \frac{\sqrt{t_i t_j^2 t_k}}{(t_i+t_j)(t_j+t_k)(t_i+t_k)} + O(dT)^3. \tag{A17}$$

Due to the above expression and to Eq. (A15), the fidelity $\mathcal{F}(T,T+dT)$ can be written in terms of some infinitesimal operators δ,∂,

$$\mathcal{F}(T,T+dT) \simeq \frac{\det[(\mathbb{1}+T)(\mathbb{1}+\partial)]^{\frac{1}{2}}}{\det[\mathbb{1}+T]^{\frac{1}{4}}\ \det[(\mathbb{1}+T)(\mathbb{1}+\delta)]^{\frac{1}{4}}} = \frac{\det[\mathbb{1}+\partial]^{\frac{1}{2}}}{\det[\mathbb{1}+\delta]^{\frac{1}{4}}} = e^{\frac{1}{2}\mathrm{Tr}\log(1+\partial)-\frac{1}{4}\mathrm{Tr}\log(1+\delta)} \simeq e^{\frac{1}{2}\mathrm{Tr}(\partial-\delta/2)-\frac{1}{4}\mathrm{Tr}(\partial^2-\delta^2/2)}, \tag{A18}$$

where

$$\delta = (1+T)^{-1}\,dT = \sum_{ij}|i\rangle\langle j|\frac{1}{1+t_i}dT_{ij}, \tag{A19}$$

$$\partial = (1+T)^{-1}(\sqrt{\sqrt{T}T'\sqrt{T}} - T) = \sum_{ij}|i\rangle\langle j|\frac{\sqrt{t_i t_j}}{t_i+t_j}\frac{1}{1+t_i}dT_{ij} - \sum_{ijk}|i\rangle\langle k|\,dT_{ij}dT_{jk} \times \frac{\sqrt{t_i t_j^2 t_k}}{(t_i+t_j)(t_j+t_k)(t_i+t_k)}\frac{1}{1+t_i}. \tag{A20}$$

The elements of Eq. (A18) become

$$\mathrm{Tr}(\partial-\delta/2)=-\frac{1}{4}\sum_{ij}|dT_{ij}|^2\frac{1}{(t_i+t_j)^2}\left(\frac{t_j}{1+t_i}+\frac{t_i}{1+t_j}\right), \tag{A21}$$

$$\mathrm{Tr}\,\delta^2=\sum_{ij}|dT_{ij}|^2\frac{1}{(1+t_i)(1+t_j)}, \tag{A22}$$

$$\mathrm{Tr}\,\partial^2\simeq\sum_{ij}|dT_{ij}|^2\frac{t_it_j}{(t_i+t_j)^2}\frac{1}{(1+t_i)(1+t_j)}, \tag{A23}$$

so that

$$\mathcal{F}(T,T+dT)\simeq 1-\frac{1}{8}\sum_{ij}\frac{|dT_{ij}|^2}{(1+t_i)(1+t_j)(t_i+t_j)}, \tag{A24}$$

which completes the proof.

Before proving Eq. (1) we introduce the following lemma, which will be used for analytical continuations to the pure state manifold:

Lemma 3. Let $f(x,y):=(x-y)^2(1-xy)^{-1}$ be a function defined in $[-1,1]^2-\{z^+,z^-\}$, $z^\pm:=(\pm1,\pm1)$. Then $f(x,y)\leqslant 4$ and $\lim_{(x,y)\to z^\pm}f(x,y)=0$.

Proof. The upper bound is found thanks to $1-xy=1-[(x+y)^2-(x-y)^2]/4\geqslant(x-y)^2/4$. To show that $\lim_{(x,y)\to z^\pm}f(x,y)=0$, let us restrict f to the $x\geqslant0$, $y\geqslant0$ part of the domain to analyze the limit to z^+. The limit z^- follows because of the $(x,y)\to(-x,-y)$ symmetry of f. One can write $y=1+m\,(x-1)$ or $x=1+m\,(y-1)$ with with $m\in[0,1]$. Because of the $(x,y)\to(y,x)$ symmetry of f, we can consider just the first case. One obtains $f(x,y)=(1-x)\frac{(1-m)^2}{1+mx}\leqslant 1-x$, which is in a disk of radius δ centered on z^+ and is upper bounded by δ. This shows that $\forall\epsilon>0,\exists\delta=\delta(\epsilon)$ s.t $\|(x,y)-z^+\|\leqslant\delta\Rightarrow f(x,y)\leqslant\epsilon$ [with $\delta(\epsilon)=\epsilon$], i.e., the claim.

Proof of Eq. (1). Equation (2) is obtained directly from lemma A. Indeed, from Eq. (A14),

$$dC=dT\frac{1}{1+T}-\frac{T-1}{T+1}dT\frac{1}{T+1}=2\frac{1}{T+1}dT\frac{1}{T+1}. \tag{A25}$$

Inserting the above equation in (A16), and noting that C and T are diagonal in the same basis, $c_i=\frac{t_i-1}{t_i+1}$, one obtains

$$g_{\mu\nu}=\sum_{ij}\frac{(\partial_\mu C)_{ij}\,(\partial_\nu C)_{ji}}{1-c_ic_j}. \tag{A26}$$

The singular behavior of (A26) for $c_i=\pm1$ is just apparent. Indeed, let $iG|j\rangle=g_j|j\rangle$ $(j=1,\ldots,2n)$, $\mathrm{Sp}(iG)=\{g_j\}\subset\mathbf{R}$, and then $C=\sum_j c_j|j\rangle\langle j|$, $c_j:=\tanh(g_j/2)$. By differentiation, $dC=\sum_j((1-c_j^2)\frac{dg_j}{2}|j\rangle\langle j|+c_j(|dj\rangle\langle j|+|j\rangle\langle dj|))$. One has, therefore, the matrix elements $(dC)_{jj}=(1-c_j^2)dg_j$ and $(dC)_{ij}=(c_i-c_j)\langle di|j\rangle$ $(i\neq j)$. Plugging these into (A26), we obtain

$$ds^2=\frac{1}{4}\sum_j\left(1-c_j^2\right)dg_j^2+\sum_{i\neq j}f(c_i,c_j)\,|\langle di|j\rangle|^2. \tag{A27}$$

Now one can see easily that for $c_j\to\pm1$, the first (diagonal) contribution in (A27) vanishes while the second, thanks to lemma A, is upper bounded by $4\sum_{i\neq j}|\langle di|j\rangle|^2$ for *all* $c_i,c_j\in(-1,1)$ and vanishes for $(c_i,c_j)\to z^\pm$: even if (A26) has been derived for C such that $c_i\neq\pm1$, we can perform the limit $|c_i|\to1,(\forall\,i)$, and in this way *extend* the metric to the pure state manifold just by setting c_ic_j to -1 (e.g., the case $c_ic_j=1$ gives a vanishing contribution).

The basis-independent expression Eq. (1) follows from (A26),

$$ds^2=\sum_{\mu\nu}g_{\mu\nu}d\lambda_\mu d\lambda_\nu=\langle(\mathbf{1}-\mathrm{Ad}C)^{-1}(dC),dC\rangle \tag{A28}$$

where $dC=\sum_\mu d\lambda_\mu\partial_\mu C$, and $\mathrm{Ad}C(X):=CXC^\dagger=CXC=(L_C\circ R_C)(X)$ is the adjoint action. To see this, let us first write $dC=\sum_{ij}(dC)_{ij}|i\rangle\langle j|$, where $C|i\rangle=c_i|i\rangle$. Then $(1-\mathrm{Ad}C)^{-1}(dC)=\sum_{ij}(dC)_{ij}(1-c_ic_j)^{-1}\,|i\rangle\langle j|$ and $\langle(1-\mathrm{Ad}C)^{-1}(dC),dC\rangle=\sum_{ij}(dC)_{ij}{}^*(1-c_ic_j)^{-1}\langle|i\rangle\langle j|,dC\rangle=\sum_{ij}(dC)_{ij}{}^*(dC)_{ij}(1-c_ic_j)^{-1}$. The zero contribution to the sum (A26) for $c_ic_j=1$ is considered thanks to the pseudo-inverse.

One can show that Eq. (1) reduces to the known expressions when ρ is a thermal state [11] and when ρ is a pure state [34], provided that the appropriate matrices T or C are used. In Appendix B, this theorem is applied to NESS-QPT where C is given by the solution of the Sylvester equation (5).

APPENDIX B: LIOUVILLEAN STEADY STATE

We call $\mathcal{R}$ the 4^n-dimensional operator spaces generated by $\prod_j w_j^{s_j}$, $(s_j\in\{0,1\})$, and we use the notation $|\mathbf{s})$ to refer to the elements of $\mathcal{R}$, normalized with respect to the Hilbert-Schmidt inner product, i.e., $(\mathbf{s}|\mathbf{s})\equiv\mathrm{Tr}[s^\dagger s]=1$ for $|\mathbf{s})\in\mathcal{R}$.

Following the notation introduced in Sec. III, the Liouvillean $\mathcal{L}:\mathcal{R}\to\mathcal{R}$ introduced in (3) can be written as

$$\mathcal{L}=-\frac{1}{2}(\mathbf{a}^\dagger\quad\mathbf{a})\begin{pmatrix}X & Y\\ 0 & -X^T\end{pmatrix}\begin{pmatrix}\mathbf{a}\\ \mathbf{a}^\dagger\end{pmatrix}-\frac{1}{2}\mathrm{Tr}\,X. \tag{B1}$$

The superoperator $a_j^\dagger$ is the Hermitian conjugate of a_j in $\mathcal{R}$.

If C is the matrix solution of (5), then

$$\begin{pmatrix}X & Y\\ 0 & -X^T\end{pmatrix}=\begin{pmatrix}U & -C\,U^{-T}\\ 0 & U^{-T}\end{pmatrix}\begin{pmatrix}x & 0\\ 0 & -x\end{pmatrix}\times\begin{pmatrix}U^{-1} & U^{-1}C\\ 0 & U^T\end{pmatrix}. \tag{B2}$$

We show now that the latter transformation is a nonunitary Bogoliubov transformation [35] and that everything is consistent. It is known that nonunitary Bogoliubov transformations are isomorphic to the group of orthogonal complex matrices $O(4n,\mathbb{C})$. This condition can be expressed in a simple way thanks to Eq. (2.6) of [35], i.e.,

$$\hat V\,\Sigma^x\,\hat V^T=\Sigma^x,\quad \Sigma^x=\sigma^x\otimes\mathbb{1}_{2n}. \tag{B3}$$

It is simple to show that the transformation $\hat V$,

$$\hat V=\begin{pmatrix}U^{-1} & U^{-1}C\\ 0 & U^T,\end{pmatrix}, \tag{B4}$$

satisfies that condition. We define new diagonal creation and annihilation operators as

$$\begin{pmatrix} \mathbf{d} \\ \mathbf{d}^\times \end{pmatrix} = \hat{V} \begin{pmatrix} \mathbf{a} \\ \mathbf{a}^\dagger \end{pmatrix}. \tag{B5}$$

Since $\mathcal{V}$ is a nonunitary Bogoliubov transformation, the operators d_i and $d_j^\times$ satisfy the CAR algebra, but $d_j^\times \neq d_j^\dagger$. Moreover, using $(\mathbf{a}^\dagger \quad \mathbf{a}) = \binom{\mathbf{a}}{\mathbf{a}^\dagger}^T \Sigma^x$, it is simple to show that

$$\mathcal{L} = -\frac{1}{2}(\mathbf{d}^\times \quad \mathbf{d}) \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \begin{pmatrix} \mathbf{d} \\ \mathbf{d}^\times \end{pmatrix} - \frac{1}{2} \operatorname{Tr} X, \tag{B6}$$

i.e.,

$$\mathcal{L} = -\sum_j x_j\, d_j^\times\, d_j. \tag{B7}$$

Note also that the transformation (B5) can be written thanks to Eq. (2.16) of [35] into the form

$$d_j = \mathcal{V} a_j \mathcal{V}^{-1}, \quad d_j^\times = \mathcal{V} a_j^\dagger \mathcal{V}^{-1}, \tag{B8}$$

where

$$\mathcal{V} =: \exp\left(-\tfrac{1}{2}\mathbf{a}^\dagger\, C\, \mathbf{a}^\dagger + \mathbf{a}^\dagger\,(U-1)\,\mathbf{a}\right) : , \tag{B9}$$

and $:\exp(\cdot):$ refers to the normal ordering of the exponential.

It is now possible to express the stationary state of the Liouvillean, i.e., the state Ω such that $\mathcal{L}\Omega = 0$, as the **d**-vacuum, i.e., $d_j|\Omega) = 0$. The identity operator, i.e., the element $|\mathbf{0}) \in \mathcal{R}$, is the **a**-vacuum, i.e., $a_i|\mathbf{0}) = 0, \forall\, j = 1,\ldots,2n$, and in particular $(\mathbf{0}|\mathcal{L} = 0$. The **d**-vacuum can be readily obtained from the Bogoliubov transformation: $|\Omega) = \mathcal{V}|\mathbf{0})$. Indeed, as $a_j|\mathbf{0}) = 0$, one has $d_j|\Omega) = \mathcal{V} a_j \mathcal{V}^{-1}\mathcal{V}|\mathbf{0}) = 0$. Hence,

$$|\Omega) = \mathcal{V}|\mathbf{0}) = e^{-\frac{1}{2}\mathbf{a}^\dagger C\, \mathbf{a}^\dagger}|\mathbf{0}). \tag{B10}$$

We now show that the state (B10) is exactly (A6). Thanks to the transformation Q defined in (A2) and the direct relation (A5), one can write the imaginary antisymmetric matrix $C = Q^T \bigoplus_k \begin{pmatrix} 0 & ic_k \\ -ic_k & 0 \end{pmatrix} Q$. Then, using the definition (B10),

$$\begin{aligned} \frac{1}{2}\mathbf{a}^\dagger\, C\, \mathbf{a}^\dagger \rho &= \frac{1}{8}(\mathbf{w}\cdot C\mathbf{w}\rho + 2\mathbf{w}\cdot C\rho\mathbf{w} + \rho\mathbf{w}\cdot C\mathbf{w}) \\ &= \frac{i}{4}\sum_k c_k[z_{2k-1}z_{2k}\rho + z_{2k-1}\rho z_{2k} \\ &\quad - z_{2k}\rho z_{2k-1} + \rho z_{2k-1}z_{2k}] \\ &=: \sum_k \mathcal{G}_k(\rho). \end{aligned} \tag{B11}$$

As

$$\mathcal{G}_k(\mathbb{1}) = i\, c_k\, z_{2k-1}z_{2k}, \quad \mathcal{G}_k(z_{2k-1}z_{2k}) = 0, \tag{B12}$$

it is clear that

$$\Omega \propto e^{-\frac{1}{2}c^\dagger C c^\dagger}|0) \propto \prod_k e^{-\mathcal{G}_k}\mathbb{1} = \prod_k (1 - i\, c_k\, z_{2k-1}z_{2k}), \tag{B13}$$

thus recovering Eq. (A1).

The conditions for the existence and uniqueness of (B13) are given in [24]. We now study those conditions and express them in terms of the spectral gap. The correlation matrix $C \in M_{2n}(\mathbf{C})$ is the matrix solution of Eq. (5). To study the solution of that equation, it is useful to consider the (noncanonical) "vectorizing" isomorphism $\phi : M_{2n}(\mathbf{C}) \to (\mathbf{C}^{2n})^{\otimes 2} \,/\, |i\rangle\langle j| \to |i\rangle \otimes |j\rangle$. This is also a Hilbert-space isomorphism, namely $\langle \phi(A), \phi(B)\rangle = \langle A, B\rangle = \operatorname{Tr}(A^\dagger B)$. One can directly check that if $R_X(C) := CX$ and $L_X(C) := XC$, then $\phi(R_X(C)) = (\phi \circ R_X \circ \phi^{-1} \circ \phi)(C) = (\mathbf{1} \otimes X^T)\phi(C)$, and $\phi(L_X(C)) = (\phi \circ L_X \circ \phi^{-1} \circ \phi)(C) = (X \otimes \mathbf{1})\phi(C)$. Applying ϕ to both sides of (5), one then obtains $(\tilde{C} := \phi(C),\, \tilde{Y} := \phi(Y))$

$$(X \otimes \mathbf{1} + \mathbf{1} \otimes X)\tilde{C} =: \hat{X}\tilde{C} = \tilde{Y}, \tag{B14}$$

where $\tilde{C}, \tilde{Y} \in (\mathbf{C}^{2n})^{\otimes 2}$, $\hat{X} \in \operatorname{End}(\mathbf{C}^{2n})^{\otimes 2} \cong M_{4n^2}(\mathbf{C})$. There are three different key operators in the formalism for obtaining the steady state:

(i) The Liouvillean $\mathcal{L} : \operatorname{End}((\mathbf{C}^2)^{\otimes n}) \to \operatorname{End}((\mathbf{C}^2)^{\otimes n})$, a $2^{2n} \times 2^{2n}$ matrix. Its complex spectrum, from (B7), is given by

$$\operatorname{Sp}(\mathcal{L}) = -\left\{ x_{\mathbf{n}} := \sum_{j=1}^{2n} x_j n_j \,/\, n_j = 0,1,\, x_j \in \operatorname{Sp}(X) \right\}. \tag{B15}$$

Notice that $0 \in \operatorname{Sp}(\mathcal{L})$, i.e., $\mathcal{L}$ is always noninvertible, and that the steady state (e.g., our Gaussian one $\mathbf{n} = \mathbf{0}$) is in the kernel of $\mathcal{L}$. If this latter is one-dimensional (unique steady state), the gap of $\mathcal{L}$ can be defined as $\Delta_{\mathcal{L}} := \min_{\mathbf{n}\neq\mathbf{0}} |x_{\mathbf{n}}|$.

(ii) The map $X : \mathbf{C}^{2n} \to \mathbf{C}^{2n}$, a $2n \times 2n$ *real* diagonalizable matrix. Its spectrum is $\{x_j\}_{j=1}^{2n} \subset \mathbf{C}$ and (because of reality) is invariant under complex conjugation. On physical grounds (stability) we *must* have $\operatorname{Re} x_j \geqslant 0, \forall\, j$. Indeed, the time scale for convergence $\rho(t) \to \rho(\infty)$ is dictated by $\tilde{\Delta}^{-1}$, where $\tilde{\Delta} = \min_{\mathbf{n}\neq\mathbf{0}} \operatorname{Re} x_{\mathbf{n}}$.

(iii) The map $\hat{X} = X \otimes \mathbf{1} + \mathbf{1} \otimes X : \mathbf{C}^{2n} \otimes \mathbf{C}^{2n} \to \mathbf{C}^{2n} \otimes \mathbf{C}^{2n}$, a $4n^2 \times 4n^2$ matrix. It spectrum is $\{x_i + x_j\}_{i,j=1}^{2n} \subset \mathbf{C}$ and the minimum (in modulus) is given by $\Delta_{\hat{X}} := \min_{i,j} |x_i + x_j|$. Note also that

$$\Delta_{\hat{X}}^{-1} = \|\hat{X}^{-1}\|_\infty. \tag{B16}$$

For the uniqueness of the steady state, we must have $\hat{X}$ invertible, i.e., $\Delta_{\hat{X}} > 0$.

Proposition 1. If $\Delta = \min_j 2\operatorname{Re}(x_j) > 0$, then

$$\Delta = \Delta_{\mathcal{L}} = \Delta_{\hat{X}}. \tag{B17}$$

Proof. $|x_{\mathbf{n}}| = |\sum_{j=1}^{2n} x_j n_j| \geqslant |\operatorname{Re}(\sum_{j=1}^{2n} n_j x_j)|$. The first bound can be saturated by choosing the n_j's in such a way that only a set P of complex-conjugate pairs $x_p^\pm$ of eigenvalues are present. In this case, $|\operatorname{Re}(\sum_{j=1}^{2n} n_j x_j)| = 2\sum_{p\in P} \operatorname{Re} x_p$, where we used the assumption $(\forall\, p) \operatorname{Re} x_p \geqslant 0$. Using again positivity of all the terms, this sum can be made as small as possible by choosing $|P| = 1$ and minimizing over $p = 1,\ldots,n$. This shows that $\Delta_{\mathcal{L}} = \min_{\mathbf{n}} |x_{\mathbf{n}}| = 2\min\{\operatorname{Re} x_p\}_{p=1}^n$. It is clear now that a similar argument shows that $\Delta_{\hat{X}} = \min\{|x_i + x_j|\}_{i,j=1}^{2n}$ is given by the same expression, i.e., $\Delta_{\mathcal{L}} = \Delta_{\hat{X}}$. Finally, $\Delta = 2\min_{\mathbf{n}} \operatorname{Re} x_{\mathbf{n}} \equiv 2\tilde{\Delta} = 2\min_p \operatorname{Re} x_p = \Delta_{\mathcal{L}}$.

APPENDIX C: NONDIAGONALIZABLE CASE

The nondiagonalizable case has been extensively handled in [24]. In Appendix B, we assumed X to be diagonalizable for simplicity, and also because the matrices X encountered in our numerical simulations were diagonalizable. Here we briefly discuss the general case. The matrix X can always be put in the Jordan canonical form, i.e., $X = U\, x^J\, U^{-1}$ with $x^J = \oplus_b J_{\ell_b}(x_b)$,

$$J_{\ell_b}(x_b) = \begin{pmatrix} x_b & 1 & & & \\ & x_b & 1 & & \\ & & x_b & 1 & \\ & & & \ddots & \ddots \end{pmatrix} : . \tag{C1}$$

x_b are (possibly equal) eigenvalues of X and ℓ_b is the dimension of the Jordan block: each block is composed of ℓ_b degenerate eigenvalues of X. The form of the transformation (B4) remains the same (although with a new matrix U) while (B7) becomes

$$\mathcal{L} = -\sum_{j=1}^{2n} x_j\, d_j^\times d_j - \sum_b \sum_{k=1}^{\ell_b-1} d_{b_k+1}^\times d_{b_k}, \tag{C2}$$

where b_k refers to the index of the kth element in the bth Jordan block. It is clear that the state (B10) is still a stationary state. Moreover, in [24] it has been shown that the spectrum of the Liouvillean is

$$\mathrm{Sp}(\mathcal{L}) = -\left\{ x_{\mathbf{n}} := \sum_b x_b n_b \,/\, n_b = 0,\ldots,\ell_b \right\}. \tag{C3}$$

Accordingly, $\Delta_{\mathcal{L}} = \Delta \equiv 2\min_b \mathrm{Re}[x_b]$. If $\Delta > 0$, the steady state (B10) is unique [24].

In the nondiagonalizable case, the last equation in Eq. (B17) is not satisfied. On the other hand, one can obtain the following:

Proposition 2.

$$\|\hat{X}^{-1}\|_\infty < \frac{1+p(\Delta^{-1})}{\Delta} \tag{C4}$$

for a certain polynomial $p()$.

Proof. We start by writing

$$\begin{aligned} \hat{X} &= \bigoplus_b J_{\ell_b}(x_b)\otimes \mathbb{1} + \bigoplus_b \mathbb{1}\otimes J_{\ell_b}(x_b) \\ &= \bigoplus_{b,d} [J_{\ell_b}(x_b)\otimes \mathbb{1}_{\ell_d} + \mathbb{1}_{\ell_b}\otimes J_{\ell_d}(x_d)] \\ &= \hat{x} + \bigoplus_{b,d} [J_{\ell_b}(0)\otimes \mathbb{1}_{\ell_d} + \mathbb{1}_{\ell_b}\otimes J_{\ell_d}(0)], \end{aligned} \tag{C5}$$

where $\hat{x}$ is the diagonal matrix with entries $x_i + x_j$ and where we used the decomposition $\mathbb{1} = \oplus_b 1_{\ell_b}$. Moreover, thanks to Lemma 3.1 of Ref. [24],

$$\begin{aligned} \hat{X} &= \hat{x} + \bigoplus_{b,d} \bigoplus_{r=1}^{\min\{\ell_b,\ell_d\}} J_{\ell_b+\ell_d-2r+1}(0) \\ &= \hat{x}\left[\mathbb{1} + \bigoplus_{b,d} \bigoplus_{r=1}^{\min\{\ell_b,\ell_d\}} \frac{J_{\ell_b+\ell_d-2r+1}(0)}{x_b+x_d}\right]. \end{aligned} \tag{C6}$$

As J is nilpotent,

$$\hat{X}^{-1} = \hat{x}^{-1}\left[\mathbb{1} + \bigoplus_{b,d} \bigoplus_{r=1}^{\min\{\ell_b,\ell_d\}} \sum_{m=1}^{\ell_b+\ell_d-2r} \left(-\frac{J_{\ell_b+\ell_d-2r+1}(0)}{x_b+x_d}\right)^m\right]$$

and

$$\begin{aligned} \|\hat{X}^{-1}\|_\infty &\leqslant \|\hat{x}^{-1}\|_\infty \left[1 + \max_{b,d}\max_r \sum_{m=1}^{\ell_b+\ell_d-2r} \frac{1}{|x_b+x_d|^m}\right] \\ &= \|\hat{x}^{-1}\|_\infty \left[1 + \max_{b,d} \sum_{m=1}^{\ell_b+\ell_d-2} \frac{1}{|x_b+x_d|^m}\right] \\ &\leqslant \frac{1}{\Delta}\left[1 + \max_{b,d} \sum_{m=1}^{\ell_b+\ell_d-2} \frac{1}{\Delta^m}\right]. \end{aligned} \tag{C7}$$

APPENDIX D: UPPER BOUNDS

To derive some bounds to the fidelity metric ds^2, let us express Eq. (1) in a convenient form thanks to the vectorization isomorphism. As $\mathrm{Ad}_C(X) = (L_C \circ R_C)(X)$, one has $\phi\circ(L_C\circ R_C)\circ\phi^{-1} = C\otimes C^T = -C^{\otimes 2}$ and Eq. (1) becomes

$$ds^2 = \langle(\mathbf{1}+C^{\otimes 2})^{-1}(d\tilde{C}), d\tilde{C}\rangle = \|(\mathbf{1}+C^{\otimes 2})^{-1/2}(d\tilde{C})\|^2, \tag{D1}$$

where $d\tilde{C} = \phi(dC)$. Using the Cauchy-Schwarz inequality and the definition of operator norm, one obtains

$$\begin{aligned} ds^2 &\leqslant \|(\mathbf{1}+C^{\otimes 2})^{-1}(d\tilde{C})\|\,\|d\tilde{C}\| \leqslant P_C\,\|d\tilde{C}\|^2 \\ &\leqslant 2n P_C\,\|dC\|_\infty^2, \end{aligned} \tag{D2}$$

where we have exploited the fact that, by construction, $\|\tilde{A}\| := \|\phi(A)\| = \|A\|_2$ and $\|A\|_2 \leqslant \sqrt{2n}\|A\|_\infty$. Now $\mathrm{Sp}(C^{\otimes 2}) = \{c_i c_j \,/\, c_i, c_j \in \mathrm{Sp}(C)\}$ and, from $C = -C^T$, the spectrum of C is invariant under $c_i \to -c_j$, it follows that $\|(\mathbf{1}+C^{\otimes 2})^{-1}\|_\infty = (1+\min_{i,j} c_i c_j)^{-1} = (1-\max_i c_i^2)^{-1} = (1-\|C\|_\infty^2)^{-1}$. The bound (D2) is not specific to dissipative quadratic Liouvillean. To connect Eq. (D2) with the properties of the Liouvillean (B7), we differentiate Eq. (B14),

$$d\tilde{C} = \hat{X}^{-1}d\tilde{Y} - \hat{X}^{-1}d\hat{X}\tilde{C}. \tag{D3}$$

As $d \equiv \sum_\mu d\lambda_\mu \partial_\mu$, the above equation can be conveniently calculated via

$$X\,(\partial_\mu C) + (\partial_\mu C)\,X^T = \partial_\mu Y - (\partial_\mu X)\,C - C\,(\partial_\mu X^T), \tag{D4}$$

i.e., the matrices $\partial_\mu C$ entering in (A26) can be obtained by solving a new Sylvester equation where the matrices $X, Y, \partial_\mu X, \partial_\mu Y$ are given by the model. Taking norms in $(\mathbf{C}^{2n})^{\otimes 2}$,

$$\begin{aligned} \|d\tilde{C}\| &\leqslant \|\hat{X}^{-1}\|_\infty(\|d\tilde{Y}\| + \|d\hat{X}\|_\infty\|\tilde{C}\|) \\ &= \|\hat{X}^{-1}\|_\infty(\|dY\|_2 + \|d\hat{X}\|_\infty\|C\|_2) \\ &\leqslant \sqrt{2n}\|\hat{X}^{-1}\|_\infty(\|dY\|_\infty + \|d\hat{X}\|_\infty\|C\|_\infty) \\ &\leqslant \sqrt{2n}\|\hat{X}^{-1}\|_\infty(\|dY\|_\infty + \|d\hat{X}\|_\infty), \end{aligned} \tag{D5}$$

where, among other things, we used the inequality $\|C\|_\infty \leqslant 1$, which follows from the ansatz (A5).

In summary, we have the following upper bound on the squared Hibert-Schmidt norm of dC in terms of the control parameters and their differentials, i.e., X, dX and Y, dY:

$$\|d\tilde{C}\|^2 \leqslant 2n\|\hat{X}^{-1}\|_\infty^2(\|dY\|_\infty + 2\|dX\|_\infty)^2, \tag{D6}$$

where we also used $\|d\hat{X}\|_\infty = \|dX \otimes \mathbf{1} + \mathbf{1} \otimes dX\|_\infty \leqslant 2\|dX\|_\infty$. Plugging the above equation in (D2) and using Proposition 1 one then obtains the bound (7).

Note that in the nondiagonalizable case, there is a correction to Eq. (7) due to the polynomial p in (C4). However, this correction does not alter the main conclusion of bound (7): a superextensive behavior of ds^2 implies the closing of the Liouvillean gap.

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