Equation of state of an ideal gas with nonergodic behavior in two connected vessels

D. M. Naplekov, V. P. Semynozhenko, and V. V. Yanovsky
Institute for Single Crystals, NAS Ukraine, Lenin Avenue 60, Kharkov 31001, Ukraine

(Received 23 July 2013; revised manuscript received 27 December 2013; published 29 January 2014)

We consider a two-dimensional collisionless ideal gas in the two vessels connected through a small hole. One of them is a well-behaved chaotic billiard, another one is known to be nonergodic. A significant part of the second vessel's phase space is occupied by an island of stability. In the works of Zaslavsky and coauthors, distribution of Poincaré recurrence times in similar systems was considered. We study the gas pressure in the vessels; it is uniform in the first vessel and not uniform in second one. An equation of the gas state in the first vessel is obtained. Despite the very different phase-space structure, behavior of the second vessel is found to be very close to the behavior of a good ergodic billiard but of different volume. The equation of state differs from the ordinary equation of ideal gas state by an amendment to the vessel's volume. Correlation of this amendment with a share of the phase space under remaining intact islands of stability is shown.

DOI: 10.1103/PhysRevE.89.012920 PACS number(s): 05.45.Pq, 05.10.−a

I. INTRODUCTION

The model of collisionless ideal gas is one of the most frequently used in physics. Many different physical phenomena are reduced to the behavior of ideal gas. In this model, gas molecules considered to be noninteracting collide with the vessel's boundary considered to be absolutely elastic. It is clear that the behavior of such gas is determined by the properties of a single trajectory on sufficiently long observation times. It allows us to reduce the study of ideal gas to the movement of a single particle in billiards.

Mathematical billiard [1,2] is a model system where a particle moves inside a closed billiard boundary straightly and reflects strictly following the mirror law. Billiards are standard objects of study in the chaos theory. Well known are the billiards with strong chaos, such as Sinai billiard, where the good ergodic properties are met. However, also well known are weakly chaotic billiards with islands of stability in the phase space, which makes them *a priori* nonergodic. Two billiards with different types of chaos, connected through a small hole, have been studied in the works of Zaslavsky (see, e.g., Ref. [3]). The question of thermodynamic equilibrium in such systems was raised. The first billiard in the work [3] was the Sinai billiard. The second was a similar billiard with the central scatter in the form of Cassini's oval and with islands of stability in the phase space. A hole between billiards, through which the trajectory can both enter and leave, makes these billiards open.

For a small enough hole the consideration of such a system is generally reduced to the consideration of properties of separate open billiards and equilibrium between them. If properties of one of the billiards are well known, it is the way to understand and study the second, open billiard, through its influence on the first one. Open billiards have been recently intensively studied because of a large number of applications in acoustics, hydrodynamics, climatology, cosmology, optics, plasma physics, and so on [4–7]. And close associations with a large number of theoretical issues, such as justification of statistical physics or relationship between classical and quantum descriptions of a system [8–11]. Details about the relationship between open billiards and these topics can be found, for example, in Ref. [12]. Considering open billiards, attention is usually given to the distribution of particle residence times inside an open billiard. In general, for the chaotic behavior the exponential law of decay is typical, and for the regular dynamics this law is sedate (see, e.g., Ref. [13]). In the chaotic billiard's algebraic tail, in addition to the main, exponential decay is often observed. It is a trace of weaker chaotic system properties, like intermittent, quasiregular behavior [14]. As shown in Ref. [15], the same system with no islands of stability in the phase space may have or may not have an algebraic tail, depending on the system parameters. The presence of the tail also depends on the initial distribution of particles in the billiard. For example, for the Bunimovich stadium, with uniformly distributed initial trajectories in the phase space, the presence of such a tail is shown in Ref. [16]. But for the trajectories entering the same billiard through a hole, in Ref. [17] it was shown that for some hole positions an algebraic tail is absent.

Dealing with the ideal gas, usually the hypothesis of ergodicity of gas behavior and justice of the Gibb's distribution are accepted. However, in the theory of mathematical billiards the motion of a particle in some billiards is known to be nonergodic. Besides, for the Sinai and Bunimovich billiards it was shown theoretically [18] and experimentally [19] that replacement of the physically impossible case of a perfectly rigid border by a smooth potential leads to the emergence of islands of stability in the phase space, i.e., loss of ergodicity. Movement in a rectangular billiard is also nonergodic. For example, the pressure that trajectory produces on the walls of rectangular billiard is uneven and depends strongly on the choice of trajectory. Detailed discussion of possibility of transition to the thermodynamic limit in this case can be found in Ref. [20]. Also well known are systems [21,22], with sizes less than 10 nm in particular, for which the deviation from ergodic behavior has been found experimentally. For this reason, the effects of nonergodic behavior are important for many systems.

In this paper, we consider two-dimensional vessels or billiards, which are the same things. However, the results obtained may be, for example, directly applied to the case of three-dimensional particle motion in a right prism with a base in the form of considered billiard. The hole between vessels will be small enough to allow typical ergodic distributions in the first vessel, so that macroscopic gas parameters can be introduced in the ordinary way. Pressure, for example, should be uniform there. We tried different first billiard choices, with positive and zero Lyapunov exponents, to make sure it does not affect the results. What we will study is how the nonergodicity of the second vessel affects the properties of gas in the first vessel.

II. CONSIDERED BILLIARDS AND THEIR CONNECTION

To describe the motion of a particle in the billiard in this case, it is convenient to use the Lagrange approach to billiards description. Birkhoff coordinates for billiards are more familiar, but this is an equal approach, good for analysis of the phase portrait of composite billiards. In this approach the phase portraits of subbilliards appear to be most separated. Boundary of the billiard in this formalism is described by the parameter $s \in S^1$. The main element determining the state of a billiards particle is a separate rectilinear segment of the trajectory. Each segment is uniquely determined by start and end points of the straight segment (s_1,s_2); these points belong to the border of the billiard. Each directed segment uniquely identifies the next segment of trajectory; this is enough for (s_1,s_2) to be phase variables. In general, not every rectilinear segment that begins and ends at the border of a billiard is a part of some valid trajectory. Therefore, a phase space of a billiard is a torus with holes. Detailed description of this approach can be found in Ref. [23]. In this paper we will use this approach to describe billiard trajectories.

The choice of both vessels is to a certain extent arbitrary. The first vessel can take any form that provides strong enough chaos. We tried billiards of different forms, including analogous to Sinai billiard, to make sure it does not affect the received results. In this paper, as a first vessel with required properties let us take a triangular billiard with angles incommensurable with π. Triangular billiards are not yet enough studied and the fact they suit our purposes is of some additional interest. Our results provide additional indirect numerical confirmation of the presence of mixing property in such billiards. Any trajectory of this billiard, not periodic and not falling in a vertex, completely fills all available phase space. Chaotic properties of such a billiard was studied numerically in Refs. [24,25]. It should be noted that this billiard implements weak chaos with zero Lyapunov exponent, unlike the scattering Sinai billiard in Ref. [3].

The second vessel must be of the form providing the existence of islands of stability in the phase space to guarantee the vessel's nonergodicity. We will choose as this vessel the billiard in the form of a curvilinear triangle, two curved sides of which are defined by the equation

$$y = \pm \frac{x}{2a}(2a - x) \quad x \in [0,b]. \tag{1}$$

The third side is flat (see Fig. 1). Billiards of such form belong to the one-parameter family with parameter $\sigma = \frac{b}{a}$; they can also be called parabolic triangular billiards.

Phase portraits of this billiard differ qualitatively for different values of σ. A share of the phase space volume under the chaotic component varies widely depending on the parameter σ. For many values of the parameter, the phase space is mostly occupied by islands of stability, with the chaotic component between them. But for $\sigma = 1$ the chaotic component of the phase space disappears completely.

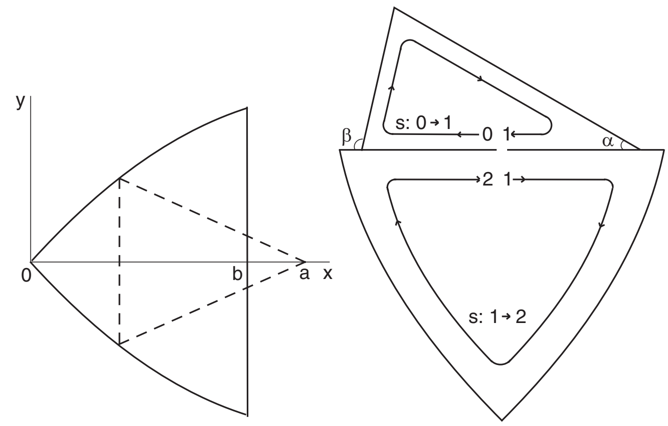

FIG. 1. General view of a curvilinear triangular billiard and parametrization of connected triangular and curvilinear triangular billiards. Coordinate of collision of the particle with a wall s is measured from the center of the hole between the billiards, first clockwise along the border of the triangular billiard, then along the border of the curvilinear triangular billiard.

Let us now consider the connection of the curvilinear and usual triangular billiards described above. Their integrated border again may be parametrized by a single natural parameter s. The chosen parametrization of the resulting billiard is shown in Fig. 1. The hole between billiards lies on the common section of the border, in the center of the parabolic billiard's flat side. Through this hole a trajectory can get from one billiard into another. While this hole is closed, billiards are separated from each other and their typical phase portraits in chosen parametrization are shown at the top in Fig. 2.

After the opening of the hole, these billiards turn to one single closed billiard with obstacles inside. Phase portrait of this billiard is shown in Fig. 2. In addition to trajectory segments lying entirely inside one of billiards, there will appear segments passing through the hole, so that a start of segment is in one billiard and the end is in another. These segments form the "mustache" in the 2nd and 4th quarters of the phase portrait. Each point in the 2nd quarter corresponds to a transition of particle from the triangular billiard into the parabolic one, and vice versa for the 4th quarter.

Important is the fact that trajectories of the chaotic sea and some islands of stability in parabolic triangular billiard now cease to be separated. A particle located on a trajectory from chaotic sea can now move to the triangular billiard and return on the trajectory that belongs to the island of stability in the closed parabolic triangular billiard. Those islands of stability, whose trajectories were partly on the removed site of the border, are destroyed. As a result, the greater part of the phase space is available for the trajectory of one particle; it fills previously inaccessible places of the phase space and the result looks like a single chaotic sea.

However, the properties of trajectories still substantially depend on the characteristics of "destroyed" islands of stability. Before the connection of billiards there were two main types of trajectories: purely chaotic and regular. This situation

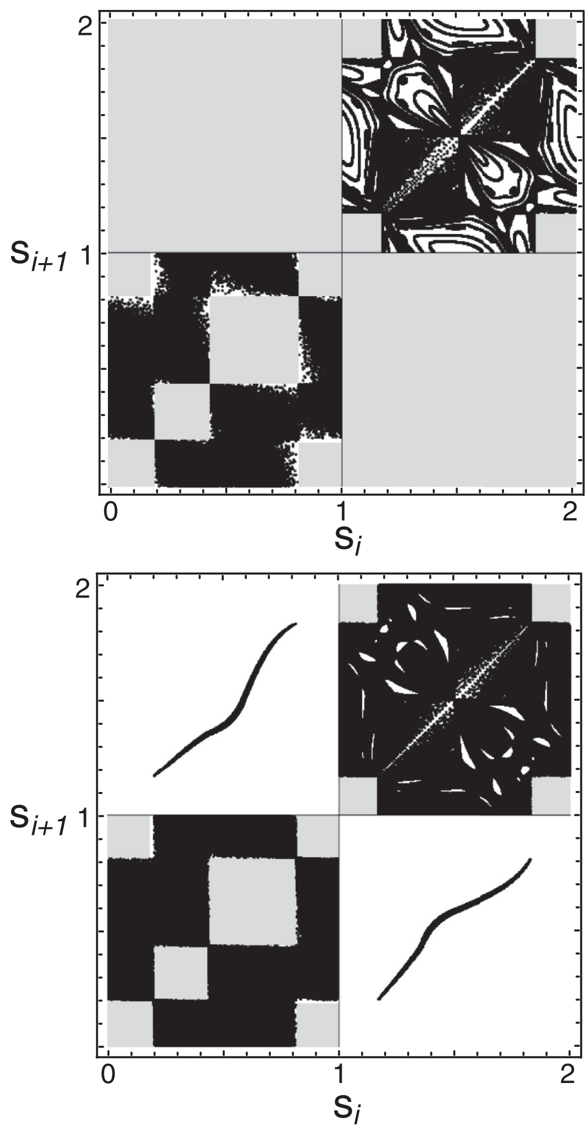

FIG. 2. On the top are the phase portraits of trajectories of triangular and curvilinear triangular billiards with a closed hole between them. Bottom left: the typical trajectory of triangular billiard with angles $\alpha = 0.63$ and $\beta = 1.91$. Top right: phase portrait of the curvilinear triangular billiard with parameter $\sigma = 0.8$. The forbidden zones arise due to the inability of a trajectory to move between billiards and impossibility of two successive collisions with the same flat side of border. On the bottom is the phase portrait of the trajectory in connected billiards with the same parameters. Points in the 2nd and 4th quarters correspond to the transitions of trajectory from one billiard to another. It is obvious that the islands of stability of the curved triangular billiard are mostly destroyed, but they are still areas of regular motion.

changes significantly after billiards connection. That islands of stability, whose trajectories never reach the hole, remain unchanged. All the other trajectories cease to be purely regular or chaotic and acquire universal intermittent behavior. Indeed, former regular trajectory goes into the triangular billiard once it reaches the hole and becomes chaotic. After some time, this trajectory leaves the triangular billiard and can get to either the area of chaotic motion or another trajectory from the ex-island of stability of the parabolic billiard. In the first case, the chaotic behavior changes its type, in the second case the chaotic behavior is replaced by the laminar motion phase. It should be emphasized that the quasiperiod of motion in this part of trajectory coincides exactly with the quasiperiod of the appropriate trajectory in closed billiard. Regular motion phase may be quite durable depending on the hole size and characteristics of the island. Thus, all former chaotic trajectories and most of the regular ones after connection of billiards acquire universal intermittent behavior.

III. ESTABLISHMENT OF A STATIONARY STATE

Let us consider the establishment of equilibrium in the gas of not colliding particles in the union of two billiards. In order to do this, we must deal with a single long trajectory of a particle. Any particle switches the billiard of its residence strictly in consecutive order. A number of transitions from one billiard to another may differ from a number of returns by no more than one time, so for times much greater than the time of Poincaré recurrence to the hole, average fluxes of particles through the hole in both directions necessarily become equal. As the magnitude of the particle's velocity is constant, this also applies to a flow of energy and, assuming identical distributions of movement directions, to a flow of momentum through the hole. In other words, establishment of equilibrium means that for the average number of particles in each of the billiards, their total momentum and energy are constant and all flows through the hole in both directions are equal. However, a flow of momentum through a hole is not the same thing as the pressure on a real wall, which is a momentum passed to the wall per unit of time per unit of area.

To understand the difference between them, let us recall one well-known "paradox" [26] from the probability theory, which is typically formulated as follows: if busses pass a bus stop once in 20 minutes on average, the average time of waiting at the bus stop could be one year and more, from 10 minutes to infinitely long. It depends not only on how many buses per unit of time are passing the bus stop, but also on how regular they go. Figure 3 qualitatively shows possible options: (a) in clusters, (b) uniformly. All buses contribute equally in the average number of passing buses, but the buses that follow the first one in a cluster almost does not reduce the time of waiting at the bus stop.

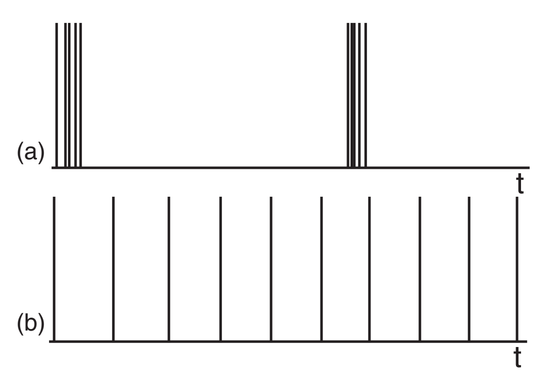

FIG. 3. The possible variants of events arrangement is qualitatively shown: clusters on the top, uniform distribution on the bottom.

Similar is the considered situation. We can imagine that the hole is closed by some virtual wall and count, how much momentum will be received from reflected particles in the case of the wall reflecting them. But it is a momentum flow, not a real pressure; we will further call it "virtual pressure." The difference between real and "virtual" pressures can be easily seen. Suppose that collisions of particles with the real wall occur in clusters; i.e., the transfer of momentum to the wall by individual collisions is similar to the one shown in Fig. 3(a). This picture takes place for the exponential form of the corresponding distribution of recurrence times. However, for the virtual wall this order of collisions is impossible in principle, because after the first "collision" a particle leaves the billiard and starts moving in another one, where the flight time to the nearest wall can be arbitrarily large. Therefore, only the first collision from the cluster will contribute to the "virtual pressure," while to the real pressure—all cluster collisions. Depending on the average number of collisions in clusters, same "virtual pressure" in the hole can match different real pressures on the real nearby walls.

Therefore, in equilibrium only "virtual pressures" on the hole are for sure equal, and whether real pressures on the nearby walls are equal or not also depends on the correlation functions of collisions with walls. If the collisions in both billiards are equally correlated, the pressures will be the same. It is the case of macroscopic vessels, where the gas has the same Maxwell's distributions of velocities, etc. But for the considered billiards it is not true. The triangular billiard has a typical exponential distribution of residence times, i.e., a lot of small times and rarely large clusters. In the parabolic billiard, the distribution function is substantially different. In particular, this distribution has clearly defined characteristic times between collisions, there are series of collisions with a regular interval between them. That is connected with destroyed islands of stability. As the result, virtual pressures are the same, but due to different correlation functions, the real pressures are different even in the vicinity of the hole. It is interesting to note that according to the well-known Pascal law, equal local pressures should be expected there. However, in our case of nonergodic gas behavior the Pascal law fails.

The arguments above have been verified by numerical calculations. Individual trajectories with a start in a triangular billiard have been considered, long enough to have all the distributions established. Reflecting from the border of the billiard, particle provides it some momentum, thus creating pressure. This pressure was calculated as

$$P = \lim_{\Delta t \to \infty} \frac{1}{l} \frac{\sum_i \triangle p_i}{\Delta t}, \tag{2}$$

where $\triangle p_i$ is the momentum transferred to the wall by the ith collision, l is the length of the part of the border under consideration, and Δt is the observation time. Reflections strictly follow the mirror law, so overall local pressure and all of $\triangle p_i$ individually are obviously normal to the border of billiard.

As a result of numerical simulation the distributions of pressure along the borders of triangular and curvilinear triangular billiards were calculated. Established after sufficiently long time distribution of pressure depends only on the choice of billiards parameters and does not change with time or depend on the initial data of trajectory. The resulting pressure on the common part of boundary is shown in Fig. 4, where top gray is the pressure on the common wall from the side of normal triangle, and below it, on the same wall from the side of parabolic billiard. It is visible that in the regular triangle the pressure on the walls is almost uniform and its value is greater than the maximum for the curvilinear triangle. Shown in the area of the hole, "virtual pressures" on both sides of the hole are equal and considerably different from the pressure on the walls. It is also visible that the pressure on the site of the border directly adjacent to the hole in the triangular billiard is higher than pressure in the parabolic billiard. For the parameters above, the difference is 1.2%, which is much greater than the calculations inaccuracy. Thus, the numerical simulation confirms our arguments.

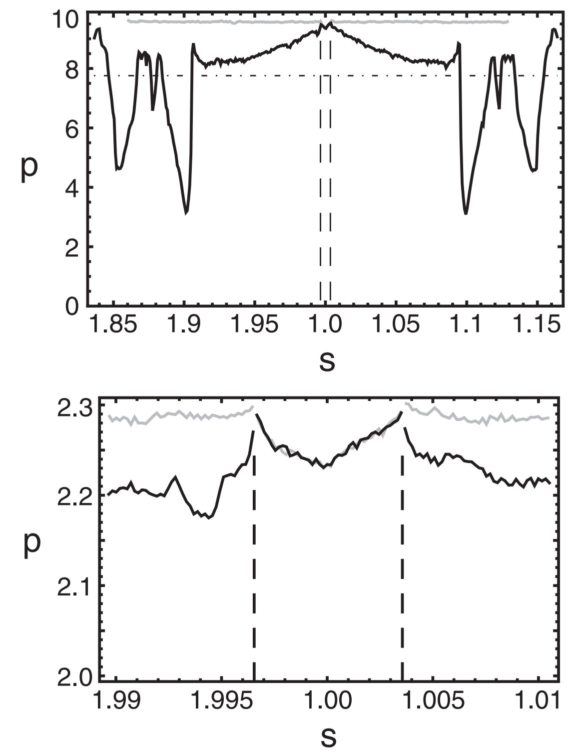

FIG. 4. On the top is the pressure on the common flat wall of billiards with parameters $\sigma = 0.8, \alpha = 0.63$, and $\beta = 1.91$, generated by one trajectory that reaches $N = 2.5 \times 10^9$ collisions with the walls of triangular billiard. Mean pressure value, equal for all taken separately sides of parabolic billiard, is shown by horizontal dotted line. Top gray: pressure on the common wall from the side of triangular billiard; below it, on the same wall from the side of parabolic one. In the hole between billiards, allocated by dotted lines, the value of "virtual pressure" is shown. The bottom graph shows the received separately distribution of pressure in the vicinity of the hole.

For both billiards, the distributions of directions from which the trajectory approaches to the hole and to the adjacent site of the border were also built (Fig. 5). It is visible that the distributions for the hole are absolutely identical, while distributions for the border are somewhat different. In the parabolic billiard, particles do not approach the border from some directions.

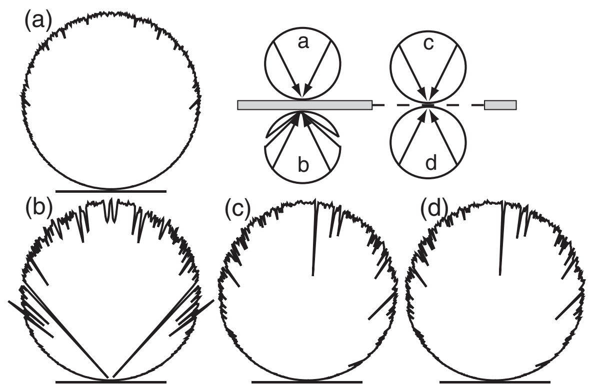

FIG. 5. The distribution of directions, from which the trajectories approach to the hole between billiards and to the adjacent site of the border. Billiards parameters are $\sigma = 0.8$, $\alpha = 0.63$, and $\beta = 1.91$; the trajectory was built until it reaches $N = 2.5 \times 10^9$ collisions with the walls of triangular billiard (2.5×10^7 returns). On the charts (a) and (c) are distributions in the triangular billiard; (b) and (d) are the same in the curvilinear one. It is visible that in the curvilinear billiard there are some directions, from which a particle never comes to the border.

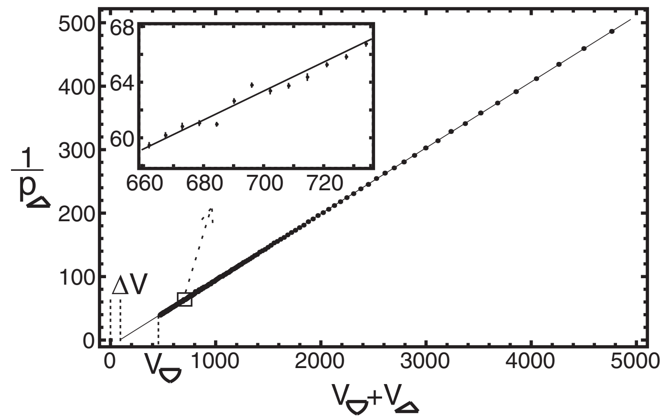

FIG. 6. Dependence of reverse pressure in the triangular billiard on the volume of both billiards is shown. It is almost linear, but has some local deviations. Example is shown in the insert. Volume of the curvilinear triangular billiard with parameter $\sigma = 0.8$ has been fixed, only the volume of triangular billiard has been changed. The pressure was created by a single trajectory, evaluated till $N = 5 \times 10^8$ collisions with the walls of triangular billiard. The "mustache" shows standard deviation from a mean pressure value. Values on both axes in conventional units.

IV. THE EQUATION OF STATE

Since the discussed above distributions in the triangular billiard in general do not differ from the distributions typical for strongly chaotic behavior, it is possible to implement the macroscopic gas parameters in the ordinary way. Pressure, for example, will be constant and isotropic anywhere in the triangular billiard, unlike the curvilinear billiard. Now, changing the parameters and scaling of the triangular billiard and, accordingly, its volume, we obtain the dependence of pressure on a total volume of triangular and parabolic triangular billiards. This dependence was built for the fixed volume of the curvilinear billiard; all the change in volume was achieved due to the triangular billiard. The trajectory also begun in the triangular billiard; i.e., the question was what would the pressure be after the opening of a hole between billiards if at first all the gas was in the triangular billiard. In this case, the trajectory never gets on the remaining islands of stability and pressure is proportional to the number of particles in the gas. For initial conditions that allow particles to be in the islands of stability, some particles may never get into the triangular billiard and do not contribute to the pressure in it.

Thus, obtained dependence of reverse pressure on a volume is shown in Fig. 6. In general, this dependence was found to be almost linear, although there are deviations from the linear law beyond the error of calculation. As the curvilinear billiard approaches to the regular dynamics ($\sigma \to 1$), these deviations are growing to a significant level. In addition, the line of linear approximation misses the coordinate's origin. Such dependence, assuming it to be linear, would have an ideal gas in the not considered nonergodic case, but usual ergodic billiards of smaller volume:

$$p = kT \frac{N}{V - \Delta V}. \quad (3)$$

In other words, from the viewpoint of the triangular billiard, considered curvilinear billiard effectively behaves like a normal ergodic billiard, but with volume smaller than the real one, 20% smaller for $\sigma = 0.8$.

The dependencies of pressure in triangular billiard on a volume were constructed numerically for different values of curvilinear billiard parameter σ. For all parameters, this dependence was close to a similar dependencies for two ergodic billiards. Significant difference was only that the volume of curvilinear billiard in the equation of state does not coincide with its actual volume. The received amendment to the real billiard's volume is shown in Fig. 7; for $\sigma < 0.5$ this amendment was indistinguishable from zero.

The share of the phase space of curvilinear billiard, still occupied by the islands of stability after the connection of billiards, has also been calculated. To do this we divide the

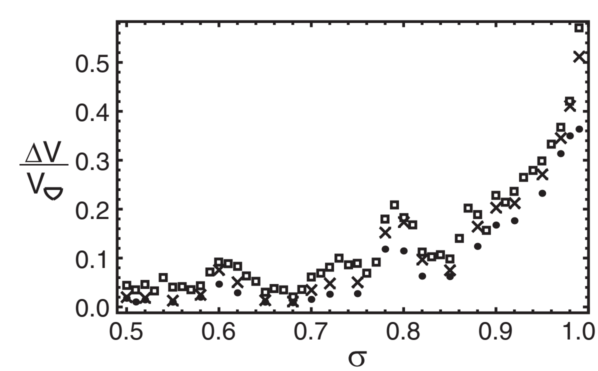

FIG. 7. Squares show the associated with nonergodicity amendment ΔV to the real volume of the curvilinear billiard, obtained from the analysis of numerically calculated equations of state for different parameters σ. Circles and crosses show the ratio of area under remaining islands to the area of phase space associated with the curvilinear billiard in Lagrange and Birkhoff coordinates, respectively.

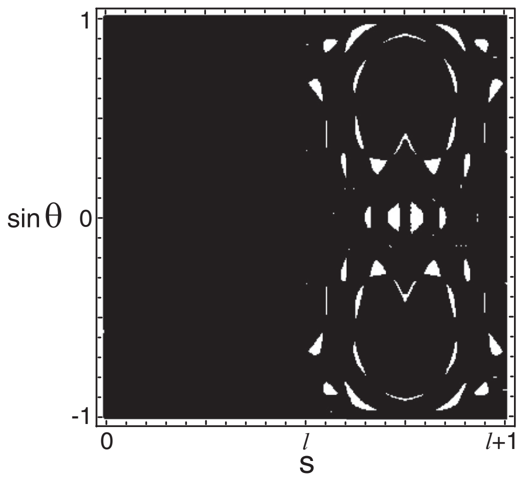

FIG. 8. Phase portrait of connected billiards in Birkhoff coordinates is shown. Billiard parameters are $\sigma = 0.8$, $\alpha = 0.63$, and $\beta = 1.91$, border parametrization by coordinate s is shown on Fig. 1, θ is an angle of reflection, l is the perimeter of the triangular billiard.

phase space into cells and then diminish discretization until the calculated volume is stabilized. In this way we receive the approximate value, which is actually a lower estimate of the volume of islands. Volume of an island on phase portrait is constant in any coordinates, unlike the volume of a phase drop. Received ratio of area under islands to area of phase space of lower billiard is shown in Fig. 7. Well visible is that the amendment to the billiard volume correlates with this share of a phase space. It is interesting to note that nevertheless the trajectory fills available phase space highly uneven. Thus, qualitatively an amendment to a volume in equation of state can be obtained by a simple analysis of the phase portrait of united billiards.

Received via pressure analysis amendment to the volume in equation of state obviously does not depend on the choice of coordinates. Unlike this, the share of the phase space under islands of stability may depend on the choice of the phase variables. However, if there are no islands in a phase space, they will be absent with any choice of coordinates. Analogically, if islands exist, they will always be. For this reason, if a correlation is observed in one of the coordinates, we may expect its traces to be observed in any other reasonable choice of phase coordinates. Only independent on the choice of coordinates results are physically reasonable. No islands, no amendment; more islands, the greater the amendment is—this should be universal.

To ensure that we also build and analyze the phase portraits of considered system in standard Birkhoff coordinates. Typical phase portrait in this coordinates is shown in Fig. 8. In Fig. 7, the share of the phase space under islands of stability in these coordinates is shown by crosses. It is visible that in these coordinates a good correlation is also observed. Nevertheless, excluded phase volume is not the only reason that defines pressure in triangular billiard. Another important factor is not even fulfilling of the phase space; for example, some trajectories get caught in the Kolmogorov-Arnold-Moser region around the islands and spend large times there. But the discovered correlation with the volume of remaining islands shows that excluded volume is one of the most important factors defining the amendment to the volume in equation of state.

V. CONCLUSIONS

In this paper a two-dimensional ideal gas of not colliding particles in two connected vessels has been studied. It is equivalent to a question of movement of one particle in two connected open mathematical billiards. One of them has the form of a triangle with all angles incommensurable with π. In the closed form this billiard is weakly chaotic and has no islands of stability in the phase space. The second was the billiard in the form of a curvilinear triangle. Its form is specially chosen so that a significant part of the phase space is occupied by islands of stability. This billiard is obviously not ergodic. After connection of these billiards, the islands of stability were mainly destroyed. All trajectories acquire regular or intermittent character. Trajectories belonging to the chaotic sea in the closed form of the curvilinear billiard can now leave to another billiard and return on a former island of stability. Some of the islands, however, remain undisturbed.

The gas of particles located in such billiards exerts some pressure on the walls of billiards. In the triangular billiard the pressure distributes uniformly along its border. This allows the implementation of a single macroscopic gas pressure. In the curvilinear billiard, provided that initially all the gas was in the triangular billiard, pressure is highly uneven. Magnitude of the local pressure is everywhere less than the pressure in the triangular billiard. Even in the vicinity of the hole, "virtual pressure" on which is equal from the both sides, the pressure on the walls differs for about one percent.

Owing to the uniform pressure in the triangular billiard, the question of equation of gas state in it, i.e., how this pressure changes with volume, was considered. The dependence of the reverse pressure on a total volume of both billiards was calculated and found to be almost linear, but not passing through the coordinate's origin. Thus, while general billiard is not ergodic, functionally its equation of state is similar to a normal equation of ideal gas state. There are, however, some deviations from linear law, especially significant near the regular dynamics regime of the curvilinear billiard. In other words, the obtained equation of state would have a usual ideal gas in a normal vessel but of different volume. Value of the amendment correlates with the share of the phase space under undestroyed islands of stability. Thus, in terms of equation of state, nonergodic curvilinear billiard effectively behaves like a billiard with smaller volume and good ergodic properties.

[1] E. Gutkin, Chaos **22**, 026116 (2012).
[2] S. Tabachnikov, *Geometry and Billiards* (American Mathematical Society, Providence, RI, 2005).
[3] G. M. Zaslavsky and M. Edelman, Phys. Rev. E **56**, 5310 (1997).
[4] I. Tuval, J. Schneider, O. Piro, and T. Tel, Europhys. Lett. **65**, 633 (2004).

[5] J. Nagler, Phys. Rev. E **71**, 026227 (2005).
[6] S. Shinohara, T. Harayama, T. Fukushima, M. Hentschel, S. Sunada, and E. E. Narimanov, Phys. Rev. A **83**, 053837 (2011).
[7] J. S. Portela, I. L. Caldas, and R. L. Viana, Eur. Phys. J. Spec. Top. **165**, 195 (2008).
[8] L. A. Bunimovich and C. P. Dettmann, Europhys. Lett. **80**, 40001 (2007).
[9] H. Buljan and V. Paar, Phys. Rev. E **63**, 066205 (2001).
[10] B. Chirikov and D. L. Shepelyansky, Physica D **13**, 395 (1984).
[11] P. Jacquod and C. Petitjean, Adv. Phys. **58**, 67 (2009).
[12] E. G. Altmann, J. S. E. Portela, and T. Tel, arXiv:1208.0254.
[13] W. Bauer and G. F. Bertsch, Phys. Rev. Lett. **65**, 2213 (1990).
[14] N. Chernov and H-K. Zhang, Nonlinearity **18**, 1527 (2005).
[15] A. J. Fendrik, A. M. F. Rivas, and M. J. Sanchez, Phys. Rev. E **50**, 1948 (1994).
[16] H. Alt, H.-D. Gräf, H. L. Harney, R. Hofferbert, H. Rehfeld, A. Richter, and P. Schardt, Phys. Rev. E **53**, 2217 (1996).
[17] C. P. Dettmann and O. Georgiou, Phys. Rev. E **83**, 036212 (2011).
[18] V. J. Donnay, J. Stat. Phys. **96**, 1021 (1999).
[19] A. Kaplan, N. Friedman, M. Andersen, and N. Davidson, Phys. Rev. Lett. **87**, 274101 (2001).
[20] V. V. Kozlov, *Thermal Equilibrium According to Gibbs and Poincaré* (Institute of Computer Studies, Moscow, 2002).
[21] X. Brokmann, J. P. Hermier, G. Messin, P. Desbiolles, J.-P. Bouchaud, and M. Dahan, Phys. Rev. Lett. **90**, 120601 (2003).
[22] D. Krapf, Phys. Chem. Chem. Phys. **15**, 459 (2013).
[23] S. V. Naidenov and V. V. Yanovsky, Theor. Math. Phys. **127**, 500 (2001).
[24] G. Casati and T. Prosen, Phys. Rev. Lett. **83**, 4729 (1999).
[25] T. Araújo Lima, S. Rodríguez-Pérez, and F. M. de Aguiar, Phys. Rev. E **87**, 062902 (2013).
[26] G. J. Szekely, *Paradoxes in Probability Theory and Mathematical Statistics* (Reidel, Dordrecht, 1986).