First-harmonic approximation in nonlinear chirped-driven oscillators

Raam Uzdin, Lazar Friedland, and Omri Gat
Racah Institute of Physics, Hebrew University, Jerusalem 91904, Israel
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Nonlinear classical oscillators can be excited to high energies by a weak driving field provided the drive frequency is properly chirped. This process is known as *autoresonance* (AR). We find that for a large class of oscillators, it is sufficient to consider only the first harmonic of the motion when studying AR, even when the dynamics is highly nonlinear. The first harmonic approximation is also used to relate AR in an asymmetric potential to AR in a "frequency equivalent" symmetric potential and to study the autoresonance breakdown phenomenon.

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I. INTRODUCTION

Driven oscillators are ubiquitous in physics and engineering. In the case of a linear oscillator driven by a fixed frequency the amplitude of oscillations is determined by the drive amplitude, oscillator-drive frequency difference, and the damping factor. Nonlinear (NL) oscillators offer a way to drive the oscillator to a high energy by using a weak drive. The drive frequency, however, must be varied in time. Under certain conditions, phase locking between the drive and the oscillator takes place. In NL oscillators the frequency of oscillations depends on the energy, so to follow the varying drive frequency, the NL oscillator must adjust its energy. This process is known as autoresonance (AR) (see Ref. [1] and references therein).

Autoresonance was first used in relativistic particle accelerators [2]. In recent years, AR was demonstrated and used as a robust method of excitation and control of nonlinear systems, ranging from atoms [3] and molecules [4] through plasmas [5,6] and fluids [7] to nonlinear optics [8] and superconducting Josephson junctions [9–11]. The most recent applications involved antihydrogen project at CERN [12]. Chirped-driven nonlinear oscillators have also been thoroughly studied in the quantum domain [10,13–16].

It should be noted that in all the above studies the term autoresonance refers to a drive that does not depend on the state of the oscillator. Typically it is of constant amplitude and linear chirp. In Ref. [17] the term autoresonance is used for a different nonlinear problem where the drive is chosen to fit the state of the oscillator so that a certain functional is minimized.

The goal of the present work is to simplify the AR equations and to get new results concerning AR dynamics. Yet the key simplification we introduce, the first harmonic approximation, is valid for other nonlinear scenarios.

In this work we exploit a result presented in the classical mechanics book of Landau and Lifshitz [18] to show that there are "equivalence classes" in AR. More specifically, we show that AR in asymmetric potentials can be studied using symmetric/even potentials [$V(x) = V(-x)$]. This equivalence, however, is not exact. It holds when the first harmonic of the oscillation dominates the action of the oscillator. At first, it may seem that the first harmonic can be dominant enough only in the weakly nonlinear regime where the potential is roughly quadratic. However, we show that for a large class of oscillators, it can hold even when the nonlinearity is extremely strong.

We start in Sec. II, by writing the evolution equations for a driven NL oscillator and giving the motivation for the first harmonic approximation, which is studied in Sec. III. In Sec. IV we use this approximation to relate AR in asymmetric potentials to AR in symmetric "frequency equivalent" potentials. In Sec. V the AR breakdown condition is derived from a "universal" AR Hamiltonian (even in the presence of friction). Finally, in Sec. VI we study a basic asymmetric potential and show that some AR phenomena cannot be studied by approximating the symmetric equivalent potential by a power series in x, due to the presence of a pole in the complex x plane. In Sec. VII we present our concluding remarks.

II. AUTORESONANCE IN ANGLE-ACTION VARIABLES

Angle-action variables are very useful in nonlinear dynamics. In particular, they are used for studying driven oscillators using the nondriven oscillators [19]. We are interested in a chirped frequency drive, so instead of the angle of the nondriven oscillators we use the phase mismatch between the drive and the oscillator. In AR this mismatch is roughly constant (phase locking). For this to happen, the oscillator must change its energy, so that the frequency of the oscillator will match the frequency of the drive. The Hamiltonian of the driven system is

$$H = p^2/2m + V(x) - \epsilon x \sin[\phi_d(t)], \tag{1}$$

where ϵ is the drive amplitude and ϕ_d is the drive phase:

$$\phi_d(t) = \int_{t_0}^{t} \omega_d(t')\,dt'. \tag{2}$$

For linear frequency chirp, the drive frequency is

$$\omega_d(t) = \omega_0 + \alpha t, \tag{3}$$

where α is the chirp rate and ω_0 is chosen to be the linear frequency at the bottom of the potential. The NL resonance equations [19] in angle-action variables for a chirped drive are

$$\frac{dI}{dt} = -\epsilon x_1(I)\sin\phi, \tag{4}$$

$$\frac{d\phi}{dt} = \omega(I) - \omega(t) - \epsilon\frac{dx_1(I)}{dI}\cos\phi, \tag{5}$$

where I is the action, ϕ is the phase mismatch between the drive and the oscillator, and $x_1(I)$ is the magnitude of the first

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harmonic of the position in the nondriven system. That is, for a given action (or energy), in the nondriven system, the position can be written as

$$x(t,I)=\sum_{n=0}^{\infty}x_n(I)\cos[n\omega(I)t+\varphi_n(I)]. \tag{6}$$

Since the NL resonance equations are based on the single resonance approximation (not to be confused with first harmonic approximation discussed later) only x_1 appears in the equations. In this work we do not consider dynamics close to separatrices, where this approximation is no longer valid. The main benefit of using the angle action variables for AR is universality. While AR can take place in very different systems (e.g., optics and mechanics), the governing equations are still given by Eqs. (4) and (5). The downside is that the two functions of the nondriven oscillator, $\omega(I)$ and $x_1(I)$, must be known. In principle, $\omega(I)$ can be directly evaluated using the relations

$$T=\frac{2\pi}{\omega}=2\int_{x_-}^{x_+}1/\sqrt{2m(E-V)}\,dx, \tag{7}$$

$$I=\frac{1}{\pi}\int_{x_-}^{x_+}\sqrt{2m(E-V)}\,dx, \tag{8}$$

where $x_\pm=x_\pm(E)$ are the turning points of the potential. The evaluation of the first harmonic $x_1(I)$, on the other hand, is considerably more complicated and often requires explicitly solving the equation of motion for various energies. The first harmonic approximation, presented next, offers a significant simplification by expressing x_1 in terms of ω and I.

III. THE FIRST HARMONIC APPROXIMATION, $I=\frac{1}{2}\omega(I)x_1^2$

Let us start by exploring the action of the nondriven oscillator. The motion is periodic with some frequency $\omega(I)$. Using (6), the action is given by

$$I=\frac{1}{2\pi}\oint p\,dx=\frac{m}{2\pi}\oint\left(\frac{dx}{dt}\right)^2dt=\frac{1}{2}\omega(I)\sum_{n=1}x_n^2n^2. \tag{9}$$

If the contribution of the x_1 is considerably larger than that of the other harmonics, then we get $I\cong\frac{1}{2}\omega(I)x_1^2$ or

$$x_1\cong\sqrt{\frac{2I}{\omega(I)}}. \tag{10}$$

At first sight, it may seem that this will work only in the weakly nonlinear limit where the frequency is roughly the linear frequency $\omega_0=\omega(I=0)$. Yet, in what follows we show that under a certain condition, this approximation holds even in the extreme nonlinear regime, where $\omega(I)$ is significantly larger than $\omega(I=0)$.

To quantify the first harmonic approximation (10), we need to evaluate the quantity

$$r=x_1\Big/\sqrt{\frac{2I}{\omega(I)}}=\frac{1}{\sqrt{\sum\left(\frac{x_n}{x_1}\right)^2n^2}}. \tag{11}$$

This quantity is smaller than one. Yet, our goal is to show that it is not too far from one. Although there are known bounds on the strength of the harmonics for large n depending on the smoothness of $x(t)$, there is very little that can be said on the first few harmonics. As will be seen shortly, it is the first few harmonic that matter the most.

To find a bound on the size of the $n\geqslant 2$ harmonics we use lemma 33 from Ref. [20]: If there is a periodic signal $y(t)$, which is positive for half a cycle and negative in the other half, then its harmonics satisfy

$$|y_n|\leqslant n|y_1|. \tag{12}$$

Obviously, this is completely useless if applied to $x(t)$. The sum in (11) will not even converge and the lower bound on r will be zero. Therefore, we will apply it to the third derivative of the position. First, it must be checked that d^3x/dt^3 satisfies the condition of the lemma. Notice that

$$d^3x/dt^3=\frac{d}{dt}[F(x)/m]=-\frac{1}{m}\frac{d^2V}{dx^2}\frac{dx}{dt}. \tag{13}$$

The velocity satisfies the condition of the lemma since the particle is moving from the left turning point to the right turning point with positive velocity in half a cycle and returns with a negative velocity in the other half of the cycle. Therefore, from (13) we see that if

$$\frac{d^2V}{dx^2}\geqslant 0, \tag{14}$$

then the validity condition of the lemma is satisfied. The condition $\frac{d^2V}{dx^2}\geqslant 0$ is trivially satisfied in the immediate vicinity of the equilibrium point and breaks down at the first inflection point of the potential. Applying the lemma to $y=d^3x/dt^3$ and using the relation $y_n=\omega^3n^3a_n$ we get

$$|\omega^3n^3a_n|\leqslant n|\omega^3a_1| \tag{15}$$

or

$$|a_n|\leqslant\frac{|a_1|}{n^2}. \tag{16}$$

Symmetric potentials have only odd harmonics, so we have

$$x_1\Big/\sqrt{\frac{2I}{\omega(I)}}\geqslant\frac{1}{\sqrt{\sum_{n=1}\frac{1}{(2n-1)^2}}}=\frac{\sqrt{8}}{\pi}\sim 0.9, \tag{17}$$

where the validity condition for this bound is given by (14). Notice that we did not assume weak nonlinearity in our derivation, so this bound is valid also in the extreme nonlinear regime. This lower bound can actually be reached in an infinite rectangular well that gives rise to a "sawtooth" solution, $x(t)$. In this case the harmonics x_n scale exactly like $\frac{1}{(2n-1)^2}$. In practice, when the potential is less sharp the approximation works much better. For example, for the Duffing potential, $V_{\mathrm{D}}=\frac{1}{2}x^2+\frac{\beta}{4}x^4$, $\beta>0$, we get $r\simeq 0.97$ even when $x\to\infty$ (i.e., in the most nonlinear part of this potential).

For the asymmetric case we get

$$x_1\Big/\sqrt{\frac{2I}{\omega(I)}}\geqslant\frac{1}{\sqrt{\sum_{n=1}\frac{1}{n^2}}}=\frac{\sqrt{6}}{\pi}\sim 0.78. \tag{18}$$

Here again, we can find a potential which satisfies this bound. Indeed, if the potential is infinite on the left side and linear on the right side, its harmonics satisfy: $a_n/a_1 = 1/n^2$ for $n = 1,2,3\ldots$. Therefore, $r = \frac{\sqrt{6}}{\pi}$ for this potential. Since we found an example that satisfies this bound, we conclude that there is no better lower bound of the same generality. Provided more information on the potential is available, we can find better bounds but this is outside the scope of this the current work.

As a first demonstration of the approximation $x_1 \simeq \sqrt{\frac{2I}{\omega(I)}}$, we consider the $V = x^{2n}$ potential studied in Ref. [21]. It was shown that $H = b_n I^{2n/(n+1)}$ and that $x_1 = 2\gamma_n I^{1/(n+1)}$. The b_n factors are known analytically but the γ_n were evaluated numerically. Using $\omega = \frac{\partial H}{\partial I} = \frac{2n}{n+1} b_n I^{\frac{n-1}{n+1}}$ we obtain that $x_1 = \sqrt{2I/\omega} = \sqrt{\frac{2I^{1+\frac{n-1}{n+1}}}{\frac{2n}{n+1}b_n}} = \sqrt{\frac{n+1}{nb_n}} I^{1/(n+1)}$. Our approximation yields the right scaling in I, but it also provides an analytical formula for γ_n: $\gamma_n = \sqrt{\frac{n+1}{4nb_n}}$. For $n = 2$, we obtain $\gamma_2 \simeq 0.3657$, which is only 0.9% away from the numerical value. For $V = x^6$, our estimate of γ_3 is 2% away from the numerical value. In the following sections, we show how this approximation can be used in studying AR phenomena.

IV. AUTORESONANCE IN ASYMMETRIC POTENTIALS

In this section we use the first harmonic approximation to establish a relation between AR in an asymmetric potential to AR in symmetric "frequency equivalent" potential.

A. Symmetric frequency equivalent potentials

It is shown in the Landau and Lifshitz book on classical mechanics [18] that if the period of a one-dimensional (1D) oscillator as a function of energy, $T(E)$, is given, then the distance between the turning points of the potential is

$$x_+(V) - x_-(V) = \int_0^V \frac{T(E)}{\sqrt{V-E}}\, dE, \tag{19}$$

provided the potential has only one minimum. This formula shows that there are infinitely many frequency-equivalent potentials. If, however, the potential is symmetric then $T(E)$ uniquely defines the potential. Let $V_{\rm as}(x)$ be some asymmetric potential with a single minimum. At each energy, one can shift both walls of the potential by the same amount so that the turning point is centered at $x = 0$. The shift needed is a function of the energy (of V) so we can write the symmetric frequency equivalent potential (SFE) as

$$V_{\rm sfe}(x) = V_{\rm as}[x - s(V_{\rm as})], \tag{20}$$

where the shift function $s(V_{\rm as})$ is the average position of the turning points:

$$s(V_{\rm as}) = \frac{x_+(V_{\rm as}) + x_-(V_{\rm as})}{2}. \tag{21}$$

Clearly, $V_{\rm sfe}(x)$ satisfies $V_{\rm sfe}(x) = V_{\rm sfe}(-x)$. In general, it is not simple to invert the given asymmetric potential and obtain $x_\pm(V_{\rm as})$. Moreover, even when possible, it is not always possible to write an explicit expression for $V_{\rm sfe}$. In Sec. VI, we show a convenient method to obtain $V_{\rm sfe}$ for small s. Next, we wish to combine this result with the result of the previous section.

From the Landau and Lifshitz construction (19), it follows that $\omega_{\rm as}(E) = \omega_{\rm sfe}(E)$. In Appendix A, we show that there are other quantities, which are invariant. In particular, $I_{\rm as}(E) = I_{\rm sfe}(E)$. Combing the two results we get

$$\omega_{\rm as}(I) = \omega_{\rm sfe}(I). \tag{22}$$

Therefore, the only element that distinguishes AR dynamics [see Eqs. (4) and (5)] in asymmetric potential from AR in the symmetric frequency equivalent potential is the coefficient $x_1(I)$. Although in general $x_{1,\rm as} \neq x_{1,\rm sfe}$, we have shown in the previous section that, under a certain condition, they are roughly equal. This leads to an important conclusion, which is the subject of the next section:

To a good approximation, AR in an asymmetric potential can be mapped to an AR in a symmetric potential

B. Studying AR threshold using the SFE potentials

The threshold phenomenon is one of the most important aspects of the AR effect. As it turns out, there is a limitation of how small the driving force can be. It must exceed a certain threshold to give rise for drive-oscillator phase locking. This threshold is determined by the chirp rate and by the nonlinearity of the potential [21,22]. The faster the chirp, the stronger the drive must be. Weaker nonlinearity (e.g., a smaller quartic term in Duffing) lead to higher thresholds since AR is a nonlinear effect and, as such, it must vanish (i.e. the threshold goes to infinity) in the linear limit.

To test our statement that AR in an asymmetric potential can be approximated by AR in some symmetric potential, we first compare the AR thresholds in the two potentials. We expect the threshold for AR to be almost the same for $V_{\rm as}$ and for $V_{\rm sfe}$. To make an accurate comparison, it is preferable to consider a potential where both $V_{\rm as}$ and $V_{\rm sfe}$ can be written analytically. We choose the Duffing oscillator $V_{\rm sfe} = \frac{1}{2}x^2 + \frac{\beta}{4}x^4$ with $\beta = 1/6$ and "Landau shift" it by rescaling the x coordinate differently for $x > 0$ and $x < 0$. The resulting antisymmetric potential is given by

$$V_{\rm as} = \begin{cases} V\left(\frac{x}{1-\rho}\right) & x < 0 \\ V\left(\frac{x}{1+\rho}\right) & x > 0 \end{cases}, \tag{23}$$

where ρ, the asymmetry parameter, controls the degree of the shift. This particular form of coordinate shift does not change the distance between the turning points. The potential is symmetric for $\rho = 0$, while for $\rho = 1$, the left wall becomes completely vertical, i.e., $\rho = 1$ is the largest shift possible before the potential becomes a multivalued function. Figure 1 shows $V_{\rm as}$ and its $V_{\rm sfe}$ for $\rho = 0.5$. Using $\alpha = 0.001$ we plotted the evolution of the particle's position in $V_{\rm as}$ (black) and in its SFE potential (red). Although the asymmetry is very significant, the threshold value, $\epsilon_{\rm th}$, in both potentials differs by only 4% ($\epsilon_{\rm th} = 0.02109$ for the asymmetric oscillator). In Sec. VI, we get 0.7% deviation for a smaller and smoother shift of the Duffing oscillator.

To the best of our knowledge, the potentials studied so far in the literature in the context of AR have been analytical. In this example, however, AR can easily be observed even though $V'_{\rm as}$ is not differentiable at $x = 0$. This can be understood from the NL resonance equations. For AR, $\omega(I)$ must be well defined

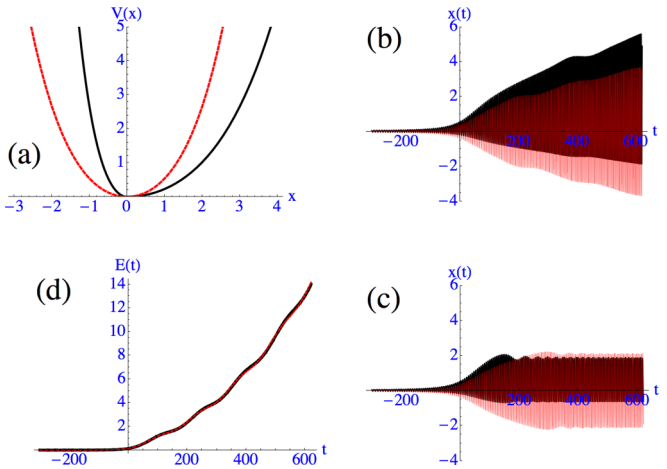

FIG. 1. (Color online) (a) The black curve shows the potential of an asymmetric Landau shifted Duffing oscillator, and the red (light gray) shows its symmetric frequency equivalent potential. (b) AR in the original asymmetric oscillator and in the symmetric equivalent just above the drive threshold amplitude. Figure (c): The same just below the threshold. The symmetric equivalent potential predicts the threshold in the asymmetric problem to an accuracy of 96%. (d) The energy plot of the oscillators shows that the actual dynamics is very similar in both oscillator at all times. The drive amplitude used in (d) is 50% above the threshold.

and monotonic. Since the frequency and the action can be written as integrals that involve the potential [see (7) and (8)], the nonanaliticity has very little impact on AR dynamics. Another aspect of nonanaliticity in AR is discussed in Sec. VI.

The striking similarity of the thresholds in $V_{\rm as}$ and $V_{\rm sfe}$ is a test for our main claim in the weakly nonlinear regime, as typically the system gets phase-locked when the frequency of the oscillator and of the drive are rather close to the linear frequency determined by $\partial_x^2 V_{\rm as}$ at the bottom of the potential. At the point where the trapping starts, the nonlinear frequency shift typically does not exceed 10% of the linear frequency. In principle, AR goes way beyond this regime. The larger the desired amplitudes of oscillations, the larger the nonlinear frequency shift must be. This does not go on forever, and at a certain moment the nonlinear phase locking mechanism fails and the amplitude of the oscillator stops growing. From this point (the "breakdown" hereon), the oscillator and the drive are no longer synchronized. The breakdown can take place at a very large nonlinear frequency shift (it can be several multiples of the linear frequency) depending on the chirp rate and frequency. As such, it serves as a good object to test the equivalence of $V_{\rm as}$ and $V_{\rm sfe}$ for studies of AR. This is the subject of the next section.

V. EXPLICIT AR BREAKDOWN CONDITION

We turn now to another application of the $x_1 = \sqrt{2I/\omega}$ approximation. As will be shown, the breakdown takes place when the nonadiabatic effects prevail over the phase confining potential created by the drive. By slowing down the chirp of the drive, it is possible to reduce the nonadiabatic effects and to push the breakdown to higher amplitudes. This is no longer true when friction is included. Friction introduces a new scale to the problem, and it is no longer possible to restrain the breakdown effect by reducing the chirp.

So far this phenomenon received little attention [21,23], even though it is just as fundamental as the threshold phenomenon (in fact, the two are intimately related).

Obviously the breakdown phenomenon is highly important for applications and demonstrations of AR as it sets a limit on the size of the effect. Moreover, as mentioned before, when friction is included the breakdown can take place even for small oscillation amplitudes, and it cannot be avoided by changing the chirp.

The breakdown condition (without friction) was written in Ref. [21] for the potential $V = x^4/4$. The effect of friction on the breakdown was first studied in Ref. [23] for three-oscillator interactions. Here, we wish to write the condition for more general potentials, to include friction, and to apply the $x_1(I)$ approximation.

In the Appendix B we derive a general AR Hamiltonian for the phase-locked regime:

$$H_{\rm AR} = \frac{\delta I^2}{2\left(\frac{dI_0}{d\omega}\right)} - \epsilon x_1(I_0)\cos\phi + \alpha\frac{dI_0}{d\omega}\phi = \frac{P^2}{2M(t)} + V_{\rm AR}(t,\phi), \tag{24}$$

where $I_0 = I_0(\omega)$ is the action of the nondriven oscillator as a function of frequency, and δI is the deviation of the actual action from the nondriven one: $\delta I = I - I_0$. ϕ is the drive-oscillator phase mismatch as before. Thinking of δI as momentum, P, and ϕ as a coordinate, the first term can be viewed as the kinetic energy [with time-dependent mass, $M(t)$] and the two other terms can be considered as potential energy, $V_{\rm AR}(t,\phi)$, since they depend only on the coordinate and time.

At each instant, the potential $V_{\rm AR}(t,\phi)$ has the shape of a "washboard potential." The $\cos\phi$ term, which is proportional to the drive amplitude ϵ, creates the minima in the potential that enable the phase confinement. The term proportional to ϕ creates a tilt in the potential that makes the minima of the potential less deep. This term is proportional to the chirp rate α, and it is only this term that contains the information on how fast the system parameters change in time. Hence, all the nonadiabatic effects originate from this term. As long as the washboard potential has minima, trapping may take place. Since the washboard parameters change in time the minima completely disappear, at a certain stage (the potential wells "open up"), and phase locking is no longer possible. The potential opens up when the minimum and the nearest maximum coalesce, i.e., $\partial_\phi^2 V_{\rm AR}(t,\phi) = 0$. From this, we get $\phi_{\rm breakdown} = -\pi/2$ and the breakdown condition:

$$\epsilon x_1(I_0) = \alpha\frac{dI_0}{d\omega}. \tag{25}$$

Next we want to generalize (25) to the case of AR with a damping force. It is shown in Appendix B that by using a certain transformation it is possible to incorporate the effect of a friction force of the form $\Gamma\frac{dx}{dt}$ into an effective Hamiltonian that generates the new equations of motion. This Hamiltonian

also has a washboard form potential. By repeating the same steps as for the frictionless case, we get that the breakdown condition is now

$$\epsilon x_1(I_0) = \alpha \frac{dI_0}{d\omega} + 2\Gamma I_0. \tag{26}$$

By using the $x_1 = \sqrt{2I/\omega}$ approximation we obtain the following breakdown condition:

$$\epsilon = \left(\frac{\alpha}{2} + \Gamma\omega\right)x_1 + \alpha\omega\frac{dx_1}{d\omega}. \tag{27}$$

Even without any knowledge of the relation between x_1 and ω, it becomes clear that the Γ term alone will satisfy the breakdown condition when x_1 is large enough. Simply stated, a friction-related breakdown takes place when the friction force becomes stronger than the driving force. To get a more explicit form, the relation $\omega(x_1)$ is needed. Naturally, this relation is oscillator dependent. For simplicity, we consider once again the Duffing oscillator.

The equation of motion for the symmetric Duffing oscillator with friction is $\frac{d^2x}{dt^2} = -\Gamma\frac{dx}{dt} - x - \beta x^3$. We consider here $\beta > 0$ and accordingly $\alpha > 0$. To get an explicit expression for the breakdown amplitude we use the approximation $\omega^2 = 1 + \frac{3}{8}\beta x_1^2$ [18]. Using this in the breakdown formula (27) we find that the breakdown amplitude, $x_{\rm bd}$, for the symmetric Duffing oscillator is

$$x_{\rm bd} = \frac{\sqrt{6\alpha[3\alpha - \sqrt{9\alpha^2 + 2\Gamma(\sqrt{3\beta}\epsilon + \Gamma)}] + 2\Gamma(\sqrt{3\beta}\epsilon - \Gamma)}}{\Gamma\sqrt{3\beta}}. \tag{28}$$

Since the first term in the numerator is negative, the second one must be positive, so that friction will not completely prevent AR from taking place. That is, $\epsilon > \Gamma/\sqrt{3\beta}$. In the limit of very slow chirp, $\alpha \to 0$, and nonzero Γ, we get

$$x_{\rm bd}(\alpha \to 0) = \sqrt{\frac{2(\sqrt{3\beta}\epsilon - \Gamma)}{3\beta\Gamma}}. \tag{29}$$

In the frictionless case (28) reduces to

$$x_{\rm bd}(\Gamma = 0) = \frac{1}{3}\sqrt{\frac{\epsilon^2}{\alpha^2} - \frac{12}{\beta}}. \tag{30}$$

All these breakdown formulas cease to be valid when the breakdown amplitude is roughly equal to the threshold amplitude since the approximation $\delta I \ll I_0$ breaks down. For example, the value $\epsilon = 12\alpha/\sqrt{\beta}$, which yields $x_{\rm bd}(\Gamma = 0) = 0$, is well below the threshold for phase locking.

The existence of friction does not affect the relation established above regarding AR in $V_{\rm as}$ and in $V_{\rm sfe}$. Therefore, Eq. (28) holds also for Landau shifted Duffing oscillators. The only difference is that in the asymmetric case, the center of oscillations is no longer at $x = 0$. Nonetheless, the breakdown amplitude around the new zero still follows (28). In Fig. 2 we see formula (28) at work. We use the same potential as in Fig. 1 and the same chirp rate. The drive amplitude, though, is now $\epsilon = 0.04$, which is roughly twice the threshold value that is used in Fig. 1.

VI. FINDING THE SYMMETRIC FREQUENCY EQUIVALENT POTENTIAL

In the previous sections, we used a simple potential whose asymmetry is such that both the symmetric and the asymmetric potentials are known analytically. However, in practice only $V_{\rm as}(x)$ is known. To get the shift function $s(V)$, explicit expressions for the turning points $x_\pm(V)$ are needed. Even for polynomial nonlinearity $x_\pm(V)$ is either not soluble analytically or too cumbersome to work with. In Appendix C, we present a method for obtaining $V_{\rm sfe}(x)$ directly from $V_{\rm as}(x)$ provided the shift s is small enough with respect to the turning points. The first order in our scheme yields

$$s^{(1)} = \frac{V_{\rm as}(x) - V_{\rm as}(-x)}{V'_{\rm as}(x) - V'_{\rm as}(-x)}, \tag{31}$$

To avoid confusion, $V'_{\rm as}(-x) \triangleq \frac{\partial V_{\rm as}}{\partial x}|_{-x}$. The first order approximation for the SFE potential is

$$V^{(1)}_{\rm sfe}(x) = \frac{V_{\rm as}\left(x - s^{(1)}\right) + V_{\rm as}\left(-x + s^{(1)}\right)}{2}. \tag{32}$$

In the Appendix C, we write the expression for the second order in s as well. Notice that even small Landau shift can lead to significant changes in the potential depending on the steepness of the potential. Therefore, a small shift is not necessarily a weak perturbation with respect to the original potential.

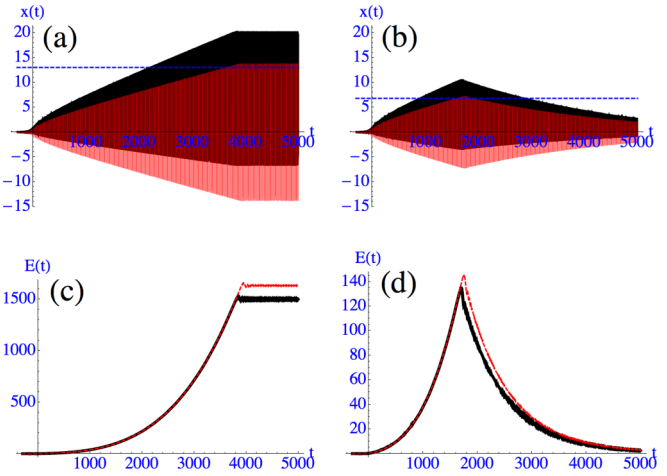

FIG. 2. (Color online) (a) Breakdown created by nonadiabatic effect. The black curve shows AR evolution for the same potential as in as in Fig. 1 (see text for parameters), and the red (light gray) curve shows AR in the symmetric frequency equivalent potential. (b) Breakdown generated by friction. The horizontal lines in both plots show the theoretical estimation for the breakdown amplitude. In (c) and (d), we plot the energy of the oscillators with and without friction, respectively.

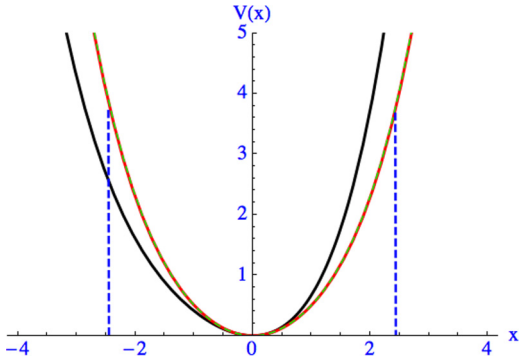

FIG. 3. (Color online) The black curve is the original asymmetric potential. In red (light gray), we plot our first-order approximation to the exact symmetric frequency potential. The second-order correction is shown in the dashed green curve. The first and second orders are hardly distinguishable from each other or from the exact symmetric potential (not shown here). Yet when comparing thresholds in the asymmetric potential and in the symmetric potentials it is found that the first-order approximation predicts the threshold to an accuracy of 97% while the second order is 99.3% accurate. The symmetric frequency potential is not an analytical function, and the blue vertical lines show the radius of convergence of a Taylor series.

To test this procedure, we consider the potential $V_{\rm as} = \frac{1}{2}x^2 + \frac{1}{3}\lambda x^3 + \frac{1}{4}\beta x^4$, which was used in Ref. [24] to study subharmonic AR. This potential is equivalent to adding a uniform force term to the Duffing oscillator. Figure 3 shows $V_{\rm as}(x)$, $V_s^{(1)}(x)$, and $V_s^{(2)}(x)$ for $\beta = 1/6$ and $\lambda = 4/10$.

Although the difference between $V_s^{(1)}(x)$ and $V_s^{(2)}(x)$ is hardly detectable in this plot, the tiny difference changes the value of the threshold quite a bit. The threshold of $V_s^{(1)}(x)$ is 97% of the threshold of $V_{\rm as}$ while the threshold of $V_s^{(2)}(x)$, is 99.3% of the $V_{\rm as}$ threshold. Although the exact analytic expression is available, it is instructive to expand it in powers of λ:

$$V_{\rm sfe}^{(2)} = \frac{1}{4}(\beta x^4 + 2x^2) + \frac{\lambda^2(-3\beta x^6 - 5x^4)}{18(\beta x^2+1)^2} + \frac{\lambda^4(-3\beta x^8 - 28x^6)}{324(\beta x^2+1)^4} + O(\lambda^6). \quad (33)$$

In the example studied above the same accuracy is obtained even if only order λ^4 in (33) is kept.

Order λ^2 is the same for $V_{\rm sfe}^{(1)}$ and $V_{\rm sfe}^{(2)}$ but they differ at order λ^4 and higher. Interestingly, (33) has a pole in the complex x plane. This pole sets the radius of convergence of a Taylor expansion in x to be $x_{\rm max} = 1/\sqrt{\beta}$. In the current example, this point is not too far away from the threshold amplitude (about 50% larger from the threshold amplitude). This suggests that the threshold phenomenon cannot be accurately studied by Taylor expanding the potential around the minimum. The breakdown phenomenon, for example, cannot be studied at all regardless of the number of elements in the Taylor expansion around the minimum.

Using Witham's averaging method it is possible to get an effective symmetric potential of the form $V_{\rm eff} = \frac{1}{2}x^2 + \frac{1}{4}(\beta - \frac{10}{9}\lambda^2)x^4$ [24]. When expanding our SFE potential around zero in powers of x we find $V_{\rm sfe}^{(1,2)} = \frac{1}{2}x^2 + \frac{1}{4}(\beta - \frac{10}{9}\lambda^2)x^4 + O(x^6, \ldots)$. In this expansion, $V_{\rm sfe}^{(1)}$ and $V_{\rm sfe}^{(2)}$ start to differ at $O(x^6)$. Note that $V_{\rm eff}$ can be used only for small λ and small amplitudes (not too close to the radius of convergence). Yet, for small amplitudes where $V_{\rm eff}$ is valid, we get a perfect consistency with our SFE potentials.

VII. CONCLUDING REMARKS

In this work we identified a certain class of oscillators where one can neglect the higher harmonics of the nondriven motion in studying AR dynamics, even when the system is highly nonlinear. This first harmonic approximation of the action allows a simplification of the equation of motion of the driven oscillator. In particular, it was shown that to a good approximation, AR in an asymmetric potential can be studied by analyzing the frequency equivalent symmetric potential. These "symmetric frequency equivalent potentials" may not be analytic even when the original asymmetric is. It was demonstrated that the main features of AR, the threshold and the breakdown, can be accurately obtained using these potentials (the accuracy depends on the validity of the first harmonic approximation in a given problem). The breakdown phenomenon was studied in detail using the associated Hamiltonian formulation.

Interestingly, we have found cases where the first harmonic approximation works well even when condition (14) no longer holds. Therefore, it is likely that a better validity condition exist.

Finally, we point out that the first harmonic approximation may also have use in the quantum domain. In studying quantum AR in the semiclassical limit, the matrix element $\langle n|x|n+1\rangle$ needs to be evaluated [13]. According to the correspondence principle in the semiclassical limit (e.g., Ref. [25]): $\langle n|x|n+1\rangle = x_1$. Using the first harmonic approximation, $x_1 \sim \sqrt{2I/\omega(I)}$, and the Bohr-Sommerfeld quantization rule, $I = n\hbar$ [26], one can obtain $x_1(n)$ without knowing explicitly the form of the states $|n\rangle$and $|n+1\rangle$.

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APPENDIX A: FREQUENCY AND ACTION EQUIVALENCE

In their derivation, Landau and Lifshitz obtained a formula from which the turning points period can be constructed, given the dependence of the period on energy $T(E)$. For our purposes, the actual construction is not needed. We just want to show the degree of freedom in the construction. Using this freedom, it is possible to match a symmetric frequency equivalent potential to every single minimum 1D asymmetric potential. We start by showing the frequency (or period) invariance between $V_{\rm as}$ and $V_{\rm sfe}$ and then show that the same proof can be used to obtain the invariance of many other quantities. In particular, we show that the action is invariant under Landau shifts. We start from the standard formula for the period:

$$T_{\rm as} = 2\sqrt{m/2}\int_{x_-}^{x_+} \frac{dx}{\sqrt{E - V_{\rm as}(x)}}. \quad (A1)$$

Now, using (20), we write

$$T_{\rm as} = \sqrt{2m}\int_{x_-(E)}^{x_+(E)} \frac{dx}{\sqrt{E - V_{\rm sfe}[x - s(V)]}}, \tag{A2}$$

where the shift function is

$$s(V) = \frac{x_+(V_{\rm as}) + x_-(V_{\rm as})}{2}. \tag{A3}$$

We define a new coordinate

$$x_{\rm new} = x - s(V), \tag{A4}$$

and therefore

$$s[V_{\rm as}(x)] = s\{V_{\rm sfe}[x - s(V)]\} = s[V_{\rm sfe}(x_{\rm new})]. \tag{A5}$$

The differential is

$$dx = dx_{\rm new}\left(1 + \frac{ds}{dV_{\rm sfe}}\frac{dV_{\rm sfe}}{dx_{\rm new}}\right). \tag{A6}$$

Then

$$T_{\rm as} = \sqrt{2m}\int_{-\Delta x(E)/2}^{\Delta x(E)/2} \frac{dx_{\rm new}}{\sqrt{E - V_{\rm sfe}(x_{\rm new})}} + \sqrt{2m}\int_{-\Delta x(E)/2}^{\Delta x(E)/2} \frac{ds}{dV_s}\frac{dV_{\rm sfe}}{dx_{\rm new}}\frac{1}{\sqrt{E - V_{\rm sfe}(x_{\rm new})}}dx_{\rm new}. \tag{A7}$$

Notice that $\frac{dV_{\rm sfe}(x_{\rm new})}{dx_{\rm new}}$ is odd in $x_{\rm new}$ and that $1/\sqrt{E - V_{\rm sfe}(x_{\rm new})}$ is even. $ds/dV_{\rm sfe}$ is a function of $V_{\rm sfe}$ which is even in x, therefore $ds/dV_{\rm sfe}$ is even as well. As a result, the second term is zero. The first term is exactly the period of the frequency equivalent symmetric potential, so eventually we get $T_{\rm as} = T_{\rm sfe}$. Repeating the exact same steps, it is easy to show that for the same energy the action is the same, $I_{\rm as}(E) = I_{\rm sfe}(E)$:

$$I_{\rm as} = \frac{1}{2\pi}\oint pdq = \frac{1}{\pi}\int_{x_-}^{x_+} \sqrt{2m[E - V_{\rm as}(x)]}\,dx = \frac{1}{\pi}\int_{-\Delta x(E)/2}^{\Delta x(E)/2} \sqrt{2[E - V_{\rm s}(x)]}\,dx = I_{\rm sfe}. \tag{A8}$$

Similarly, this procedure will work for any quantity of the form $\int_0^T G(V[x(t)],|\dot{x}|)\,dt$.

APPENDIX B: THE AR HAMILTONIAN

The Hamiltonian of the fundamental chirped NL resonance equations (4) and (5) is

$$H_c = H_0(I) - \omega_d(t)I - \epsilon x_1 \sin\phi. \tag{B1}$$

We write the action as $I = I_0(t) + \delta I$. As we shall see later I_0 will be the action of the nondriven oscillator for a given frequency $\omega = \omega_d(t)$ where t is regarded as a parameter, so that $I_0(t) = I_0[\omega(t)]$. Note that while $\{I,\phi\}$ satisfy the Hamilton's equations, the variables $\{\delta I,\phi\}$ do not. To fix this we make a canonical transformation and find the Hamiltonian that generates the equations for δI and ϕ:

$$H_{\delta I} = H_0(I_0 + \delta I) - \omega_d(t)\delta I - \epsilon x_1\cos\phi + \frac{dI_0}{dt}\phi, \tag{B2}$$

where $\omega_d(t)I_0(t)$ was dropped, as it does not affect the equation of motion. In AR, after passing the linear resonance at $t = 0$, the potential energy of the drive is small in comparison to the kinetic and potential energy of the oscillator. This suggest that the action at this state is roughly given by the action of the nondriven oscillator. That is, we can use $I_0 \gg \delta I$ and Taylor expand the Hamiltonian around I_0:

$$H_{\delta I} \cong \left.\frac{dH_0}{dI}\right|_{I_0}\delta I + \left.\frac{d^2H_0}{dI^2}\right|_{I_0}\frac{\delta I^2}{2} \tag{B3}$$

$$- \omega_d(t)\delta I - \epsilon x_1\cos\phi + \frac{dI_0}{dt}\phi. \tag{B4}$$

To cancel the first order in δI we set

$$\left.\frac{dH_0}{dI}\right|_{I_0} = \omega_d(t). \tag{B5}$$

The LHS of (B5) is $\omega(I_0)$, and therefore this equation defines implicitly $I_0(t)$. In other words, $I_0(t)$, changes in time so that the nondriven frequency will match the changing drive frequency. Next, notice that $\frac{dI_0}{dt} = \frac{dI_0}{d\omega}\alpha$ and that $\frac{d^2H_0}{dI^2} = (\frac{dI_0}{d\omega})^{-1}$. x_1 depends on $I_0 + \delta I$, but since it is multiplied by the small ϵ, we can write $x_1(I_0)$. Finally, we obtain a "universal" autoresonance Hamiltonian:

$$H_{\rm AR} = \frac{\delta I^2}{2\left(\frac{dI_0}{d\omega}\right)} - \epsilon x_1(I_0)\cos\phi + \alpha\frac{dI_0}{d\omega}\phi = \frac{P^2}{2M(t)} + V_{\rm AR}(t,\phi). \tag{B6}$$

Without the α term this is similar to a pendulum Hamiltonian with time dependent coefficients (P is the momentum, ϕ is the coordinate). The α plays a destabilizing role. When it becomes too large, the $\cos\phi$ term is no longer capable of trapping the phase.

Next, we want to find a new Hamiltonian that includes the effect of a friction force of the form $-\Gamma\frac{dx}{dt}$. The friction modifies Eq. (4), and it now reads [22]

$$\frac{dI}{dt} = -\epsilon x_1(I)\sin\phi - 2\Gamma I. \tag{B7}$$

Equation (5) remains unchanged. We define a new variable,

$$I = Je^{-2\Gamma t}, \tag{B8}$$

then

$$\frac{dJ}{dt} = -\epsilon e^{+2\Gamma t}x_1(Je^{-2\Gamma t})\sin\phi, \tag{B9}$$

$$\frac{d\phi}{dt} = \omega(Je^{-2\Gamma t}) - \omega_d(t) - \epsilon e^{+2\Gamma t}\frac{\partial x_1(Je^{-2\Gamma t})}{\partial J}\cos\phi. \tag{B10}$$

Accordingly, the corresponding Hamiltonian for J and ϕ is

$$H(J,\phi,t) = e^{+2\Gamma t}H_0(Je^{-2\Gamma t}) - \omega_d(t)J - \epsilon e^{+2\Gamma t}x_1(Je^{-2\Gamma t})\cos\phi. \tag{B11}$$

Writing $J = J_0 + \delta J$ and Taylor expanding, we get the Hamiltonian for δJ:

$$H_{\delta J} = e^{+2\Gamma t}\left[H_0(I_0) + \delta J e^{-2\Gamma t}\partial_{I_0}H_0(I_0) + \delta J^2 e^{-4\Gamma t}\partial_{I_0}^2 H_0(I_0)/2\right] - \omega_d(t)(J_0 + \delta J) - \epsilon e^{+2\Gamma t}x_1(Je^{-2\Gamma t})\cos\phi + \frac{dJ_0}{dt}\phi. \tag{B12}$$

For convenience, we returned back to I_0 in factors, which do not depend on δJ. Setting the linear term in dJ to be zero, we obtain

$$\partial_{I_0} H_0(I_0) = \omega_d(t), \tag{B13}$$

just as before. Using this in (B12) we have

$$H_{\mathrm{AR,fric}} = \delta J^2 e^{-2\Gamma t} \partial_{I_0}^2 H_0(I_0)/2 - \epsilon e^{+2\Gamma t} x_1(I_0)\cos\phi + \frac{dJ_0}{dt}\phi. \tag{B14}$$

For the AR breakdown discussion in Sec. V it will be more convenient to write the potential terms using $J_0 = I_0 e^{2\Gamma t}$. Finally, we obtain that the AR Hamiltonian with friction is given by

$$H_{\mathrm{AR,fric}} = e^{-2\Gamma t}\frac{\delta J^2}{2\left(\frac{dI_0}{d\omega}\right)} + e^{+2\Gamma t}\left[-\epsilon x_1(I_0)\cos\phi + \left(\alpha\frac{dI_0}{d\omega} + 2\Gamma I_0\right)\phi\right]. \tag{B15}$$

Comparing this to (B6), we see that even in the presence of friction the Hamiltonian still keeps the same form: a kinetic energy term with time-dependent mass and a potential energy term, which consists of $\cos\phi$ and ϕ terms (a "washboard" potential).

APPENDIX C: SMALL ASYMMETRY EXPRESSION FOR V_{sfe}

In this Appendix we use a perturbative approach to derive V_{sfe} from V_{as} for small shifts. We start with a Taylor expansion of Eq. (20):

$$V_{\mathrm{sfe}}(x) = V_{\mathrm{as}}[x - s(V_{\mathrm{as}})] = V_{\mathrm{as}}(x) - V'_{\mathrm{as}}(x)s + V''_{\mathrm{as}}(x)s^2/2\cdots, \tag{C1}$$

where prime designates differentiation with respect to x. Assuming the shift s is small (with respect to $\Delta x = x_+ - x_-$), we can neglect higher order term and obtain an approximate expression for V_{sfe}. We start with $O(s^1)$. $V_s^{(1)}(x) = -V'_{\mathrm{as}}(x)s$. This potential has to be symmetric, i.e., the asymmetric part is zero:

$$0 = V_{\mathrm{sfe}}^{(1)}(x) - V_{\mathrm{sfe}}^{(1)}(-x) = [V_{\mathrm{as}}(x) - V_{\mathrm{as}}(x)] - [V'_{\mathrm{as}}(x) - V'_{\mathrm{as}}(x)|_{-x}]s, \tag{C2}$$

so

$$s^{(1)} = \frac{V_{\mathrm{as}}(x) - V_{\mathrm{as}}(-x)}{V'_{\mathrm{as}}(x) - V'_{\mathrm{as}}(-x)}. \tag{C3}$$

As before, we use the notation $V'(x)|_{-x} \equiv V'(-x)$. Now, there are two alternatives to obtain V_s. The first is to use $s^{(1)}$ in the first order approximation of V_s and get an expression for V_s. The second option is to use $s^{(1)}$ in Eq. (20). The advantage of the second alternative is that the first order approximation is used only once and not twice. The problem with the second procedure is that $V_{\mathrm{as}}(x - s^{(1)})$ is not exactly symmetric. To overcome the problem, we symmetrize the potential and write

$$V_{\mathrm{sfe}}^{(1)}(x) = \frac{V_{\mathrm{as}}\left(x - s^{(1)}\right) + V_{\mathrm{as}}\left(-x + s^{(1)}\right)}{2}, \tag{C4}$$

where we have used the fact that $s^{(1)}(-x) = -s^{(1)}(x)$. This procedure has been considerably better than the first alternative mentioned above. For some purposes $V_s^{(1)}(x)$ will be enough, yet for others higher accuracy is needed. A quadratic equation for $s^{(2)}$ can be obtained by keeping order s^2 in (C1) and by forcing condition (C2). Although with mathematical software pacakges the obtained expression is completely practical to work with, we wish to obtain a more compact (yet less accurate) expression for $s^{(2)}$ by writing $s^{(2)} = s^{(1)} + ds$ and keeping only order ds and $(s^{(1)})^2$. After some algebra, the result is

$$s^{(2)} \cong s^{(1)} + \frac{[V_{\mathrm{as}}(-x) - V_{\mathrm{as}}(x)]^2[V''_{\mathrm{as}}(-x) - V''_{\mathrm{as}}(x)]}{2[V'_{\mathrm{as}}(-x) - V'_{\mathrm{as}}(x)]^3}, \tag{C5}$$

and finally

$$V_{\mathrm{sfe}}^{(2)}(x) = \frac{V_{\mathrm{as}}\left(x - s^{(2)}\right) + V_{\mathrm{as}}\left(-x + s^{(2)}\right)}{2}. \tag{C6}$$

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