Characteristic times of biased random walks on complex networks

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We consider degree-biased random walkers whose probability to move from a node to one of its neighbors of degree k is proportional to k^{α}, where α is a tuning parameter. We study both numerically and analytically three types of characteristic times, namely (i) the time the walker needs to come back to the starting node, (ii) the time it takes to visit a given node for the first time, and (iii) the time it takes to visit all the nodes of the network. We consider a large data set of real-world networks and we show that the value of α which minimizes the three characteristic times differs from the value $\alpha_{\min} = -1$ analytically found for uncorrelated networks in the mean-field approximation. In addition to this, we found that assortative networks have preferentially a value of $\alpha_{\min}$ in the range $[-1, -0.5]$, while disassortative networks have $\alpha_{\min}$ in the range $[-0.5,0]$. We derive an analytical relation between the degree correlation exponent ν and the optimal bias value $\alpha_{\min}$, which works well for real-world assortative networks. When only local information is available, degree-biased random walks can guarantee smaller characteristic times than the classical unbiased random walks by means of an appropriate tuning of the motion bias.

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I. INTRODUCTION

In the past decade or so the quantitative analysis of networks having different origin and function, including social networks, the human brain, the Internet, and the World Wide Web, has revealed that all these systems exhibit comparable structural properties at different scales and are more similar to each other than expected [1,2]. It has been found that the structural complexity of networks from the real world usually has a significant impact on the dynamical processes occurring over them, including opinion dynamics [3], epidemics [4], and synchronization [5].

Random walks are the simplest way to explore a network and are one of the most widely studied class of processes on complex networks [6,7]. Different kinds of random walks have been used to implement efficient local search strategies [8,9] and also to reveal the presence of hierarchies and network communities [10,11]. Particular attention has been devoted to the study of the *characteristic times* associated to random walks, such as the *mean return times*, or the *mean first passage times*, respectively, the average time the walker takes to come back to the starting node or to hit a given node [12]. Such characteristic times can be determined analytically for random walks on regular lattices [13], but their calculation for graphs with heterogeneous structures is still the object of active research [14,15]. Recent results include the derivation of analytic expressions for the characteristic times of unbiased random walks on Erdös-Rényi random graphs [16], on fractal networks [17–20], and on particular classes of scale-free graphs [21]. To date, only approximate solutions are available for random walks on real networks [22–25].

A class of random walks which is particularly interesting to consider on heterogeneous networks is that of *degree-biased random walks*. In a degree-biased random walk, the probability to move from a given node to one of its neighbors, of degree k, is proportional to k^{α}, where α is a tuning parameter. According to the sign of the bias parameter α, the walkers preferentially move either towards hubs or towards poorly connected nodes [26]. Biased random walks have been recently employed for community detection [27] and to define new centrality measures [28,29]. Furthermore, analytical results on the characteristic times of degree-biased random walks have been obtained for specific classes of random graphs in the mean-field approximation [30]. However, the structure of real networks is far from being random, and empirical evidence suggests that the presence of degree-degree correlations can affect the dynamics of the walk [31]. For instance, the authors of Ref. [26] have shown that the value of entropy rate of biased random walks on real correlated networks substantially deviates from the prediction for the corresponding randomized graphs. Similarly, more recent works show that degree-biased random walks can approximate maximally entropic walks, but the quality of such approximation depends again on degree-degree correlations [32,33].

In this article we study, both numerically and analytically, three types of characteristic times for biased random walks, namely mean return times (MRT), mean first passage times (MFPT), and mean coverage times (MCT). We consider different synthetic graphs and a large data set of social, biological, and technological complex networks from the real world, and we study the effect of the bias parameter α on the characteristic times of the walk, focusing on the values $\alpha_{\min}$ that guarantee minimal return, first passage, and coverage times. Our main result is that the characteristic times of biased random walks on real-world networks sensibly deviate from those observed in uncorrelated graphs. In particular, we prove analytically that the minimum MRT in Erdös-Rényi and scale-free random graphs is always obtained for $\alpha_{\min} = -1$, while we show through numerical simulations that the minimum MRT in real-world networks is obtained for values of α that significantly deviate from -1. We find that the value $\alpha_{\min}$ depends on the presence and sign of degree-degree correlations in the network, and in particular that for assortative networks $-1 < \alpha_{\min} < -0.5$, while for

disassortative networks $-0.5 < \alpha_{\min} < 0$. We show that in the case of networks in which the expected degree $k_{\rm nn}(k)$ of the first neighbors of a node with degree k is a power law, i.e., when $k_{\rm nn}(k) \sim k^{\nu}$ as observed in many real-world networks, it is possible to derive an approximate relation between the optimal bias value $\alpha_{\min}$ and the exponent ν. This approximation works well for assortative networks in which, for any given value of ν, the predicted optimal value of $\alpha_{\min}$ is close to the real optimum. We also analyze the MRT for nodes of a given degree class k, and we derive a closed form, valid for uncorrelated scale-free graphs, to calculate the value of the bias $\alpha_{\min}(k)$ which minimizes the MRT for nodes of degree k. We also discuss the results found for MFPT and MCT, which suggest that the optimal value of α for MRT on a given network is a quite accurate approximation for the values of α which optimize the MFPT and the MCT on the same network.

The paper is organized as follows. In Sec. II we introduce degree-biased random walks and we provide the definitions of return, first passage and coverage times. In Sec. III we study how the MRT depends on the value of the bias parameter α, and we compare the analytical predictions of characteristic times, which assume the absence of degree correlations, with the numerical results obtained on a large data set of real-world networks. We also investigate the dependence of the MRT on the degree of the starting node. In Sec. IV and Sec. V we study, respectively, the behavior of MFPT and MCT on real-world networks, and we show that the relation between the sign of degree-degree correlations in a graph and the dynamics of the walkers on the graph are indeed similar to those found for MRT. In Sec. VI we provide a more detailed discussion of the results presented in the paper, we derive an analytical relation between ν and $\alpha_{\min}$, and we indicate possible applications to several problems connected with characteristic times of biased random walks. Finally, in Sec. VII we draw some conclusions and we suggest possible future directions of research in this field.

II. DEGREE-BIASED RANDOM WALKS

Let us consider an undirected and unweighted graph $G = (V,E)$ with $N = |V|$ nodes and $K = |E|$ edges. Denote as A the adjacency matrix of graph G, i.e., the symmetric $N \times N$ matrix whose entry a_{ij} is equal to 1 if an edge exists between node i and j and is 0 otherwise. We consider the following dynamical process occurring on the graph: a walker that at each time step moves from a node to one of its neighbors with a probability proportional to the α power of the degree of the target node. The process corresponds to a discrete-time Markov chain [34] on the state space V defined by the transition matrix Π, whose each entry π_{ji} is equal to the probability for a walker on node i to jump to one of its neighbors j and reads

$$\pi_{ji} = \frac{a_{ij}k_j^{\alpha}}{\sum_l a_{il}k_l^{\alpha}}. \tag{1}$$

The exponent α is the control parameter that allows us to tune the dependence of the process on the node degree. When $\alpha > 0$ the random motion is biased towards high-degree nodes (hubs), while when $\alpha < 0$ the walkers move with higher probability to neighbors with low degree. When $\alpha = 0$ the common (unbiased) random walk is recovered. The fundamental quantity to describe a random walk is the occupation probability distribution $p_i(t)$. Being $p_i(t)$ the probability that a walker is at node i at time t, then the probability $p_j(t+1)$ of being at node j at time $t+1$ is given by

$$p_j(t+1) = \sum_i \pi_{ji}\, p_i(t) \tag{2}$$

or, in vector notation, $\boldsymbol{p}(t+1) = \Pi\, \boldsymbol{p}(t)$. A fixed point solution $\boldsymbol{p^*}$ of the latter equation, such that $\boldsymbol{p^*} = \Pi\, \boldsymbol{p^*}$, is called *stationary distribution*. If the transition matrix Π is primitive, i.e., if the graph is connected and contains at least one odd cycle, the Perron-Frobenius theorem guarantees that $\boldsymbol{p^*}$ always exists, is unique, and

$$\lim_{t\to\infty} \Pi^t\ \boldsymbol{p}(0) = \boldsymbol{p^*} \quad \forall\ \boldsymbol{p}(0),$$

i.e., all initial occupation probability distributions $\boldsymbol{p}(0)$ converge to the stationary distribution $\boldsymbol{p^*}$ [35]. In particular, the stationary distribution associated to the transition matrix (1) of a degree-biased random walk is [26]

$$p_i^* = \frac{c_i k_i^{\alpha}}{\sum_\ell c_\ell k_\ell^{\alpha}}, c_i = \sum_j a_{ij}k_j^{\alpha}. \tag{3}$$

When $\alpha = 0$, Eq. (3) reduces to

$$p_i^* = \frac{k_i}{2K}, \tag{4}$$

which states that for unbiased random walks the number of walkers at a node i is proportional to the degree k_i, so the dynamic process is completely characterized by the degree sequence of the graph. Conversely, when $\alpha \neq 0$, the stationary distribution p_i^* depends not only on the degree k_i but also on the degrees of the first neighbors of node i, through the coefficient c_i. The stationary probability distribution $\boldsymbol{p}^*$ is therefore sensitive to the degree sequence and also to the presence of degree-degree correlations in the network. It is interesting to notice that the majority of real-world networks exhibit degree-degree correlations, meaning that their nodes are found to be preferentially connected with other nodes having either similar or dissimilar degree [36–38]. Consequently, in these networks the stationary probability distribution $\boldsymbol{p^*}$ can sensibly deviate from that observed on a random graph having the same degree distribution and no degree-degree correlations. Degree-degree correlations are fully described by the joint probability $P(k,k')$, which represents the likelihood that nodes with degree k and k' are connected through an edge or, equivalently, by the conditional probability distribution $P(k'|k)$, which represents the probability that a node of a given degree k is connected to a node of degree k'. The type of correlations is usually characterized by the average degree $k_{\rm nn}(k)$ of the nearest neighbors of nodes with degree k. This can be written in terms of the conditional probability distribution $P(k'|k)$ as [36]

$$k_{\rm nn}(k) = \sum_{k'} k' P(k'|k).$$

Networks are called assortative when $k_{\rm nn}$ is an increasing function of k and disassortative when $k_{\rm nn}$ is a decreasing function of k [36]. In many real-world networks the nearest

neighbors average degree is found to be a power-law function of k, i.e., $k_{nn}(k) \sim k^{\nu}$, so the exponent ν—often called degree correlation exponent—can be used to quantitatively characterize degree correlations. A positive exponent $\nu > 0$ indicates assortative correlation while a negative value $\nu < 0$ indicates disassortative ones.

In this paper we are interested in the typical times of degree-biased random walks. In particular, assuming that a walker is at node i at time $t = 0$ and moves according to Eq. (1), we consider the expected time that the random walker needs to

(i) come back to node i for the first time, referred to as mean return time (MRT) and denoted as r_i,

(ii) reach a node j ($j \neq i$) for the first time, referred to as mean first passage time (MFPT) and denoted as t_{ij},

(iii) visit all nodes in the network at least once, referred to as mean coverage time (MCT) and denoted as c_i.

In the following sections we explore how the three characteristic times defined above are affected by the bias in the random walk. In particular, we will focus on the value of the bias parameter α which, respectively, minimizes MRT, MFPT, and MCT. We use a data set consisting of many assortative and disassortative medium- to large-sized real-world networks, and we will show how degree-biased random walks can highlight assortativity or disassortativity from a dynamical point of view.

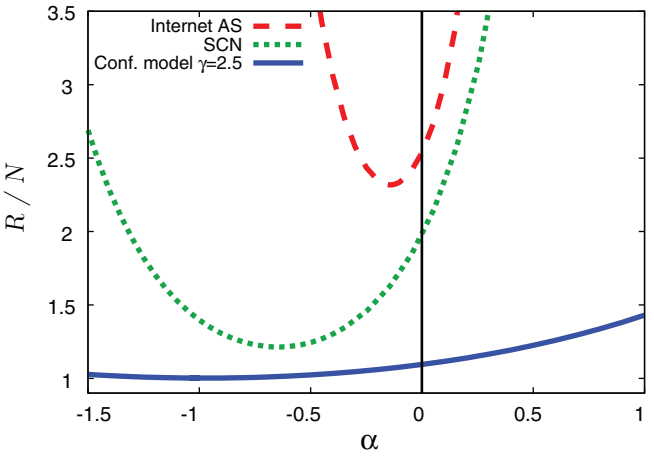

FIG. 1. (Color online) The graph mean return time R rescaled by the number of nodes N as a function of α for Internet AS (dashed red line), SCN (dotted green line), and an uncorrelated scale-free network (solid blue line). Due to the presence of correlations, R/N is a much narrower function of α in real-world networks than in synthetic networks, suggesting that mean-field approximations can adequately describe the dynamics of biased random walks only for uncorrelated graphs.

III. MEAN RETURN TIME

It is possible to prove that the mean return time r_i of a random walk on a graph is equivalent to the inverse of the stationary distribution of the walk [39],

$$r_i = 1/p_i^*. \tag{5}$$

In order to summarize in a single value the typical return time for the entire network, we define the *graph mean return time* R as the average of r_i over all nodes,

$$R = \langle r_i \rangle = \frac{1}{N} \sum_{i=1}^{N} r_i. \tag{6}$$

A. Empirical evidence

In the case of a degree-biased random walk, R depends on α because the stationary distribution depends on α as in Eq. (3). In Fig. 1 we show the graph mean return time R as a function of α for three networks, namely a scale-free network with $N = 10^4$ nodes, $P(k) \sim k^{-\gamma}$ with $\gamma = 2.5$, and average degree $\langle k \rangle = 46$, constructed by the configuration model [40], the scientific collaboration network of scientists in condensed matter (SCN) [41], having $N = 12\,722$ nodes and $K = 39\,967$ edges, and a sample of the Internet at the Autonomous System level (InternetAS) [36], having $N = 11\,174$ and $K = 23\,409$ edges. The values of R are rescaled by the network size N.

The networks reported in Fig. 1 are representative of the general behavior observed in the entire data set. In fact, for all the considered networks R is always a convex function of α, with a single minimum, denoted by $R_{\min}$, observed at a value of α denoted as $\alpha^R_{\min}$. For the uncorrelated scale-free network we find $\alpha^R_{\min} = -1$ and $R_{\min} \sim N$. The same result has been found for Erdös-Rényi random graphs and for other uncorrelated scale-free networks constructed through linear preferential attachment [42]. As shown in Table I, as the average degree $\langle k \rangle$ of a synthetic network increases, the corresponding value of $\alpha^R_{\min}$ approaches -1. Also the minimum return time becomes progressively more similar to the size of the network: $R_{\min} \sim N$. These results are in agreement with what has been found in Ref. [30]. We will give an analytical explanation of the fact that $\alpha^R_{\min} = -1$ for uncorrelated networks at the end of this section.

From Fig. 1 it is clear that the dynamical behavior of biased random walks on real-world networks considerably deviates from that observed in uncorrelated synthetic networks. In fact, if a network has degree-degree correlations, then the minimum of R always occurs for values of α larger than -1. In particular, for SCN we have $\alpha^R_{\min} \simeq -0.65$ while for InternetAS we have $\alpha^R_{\min} \simeq -0.15$ (refer to Table I for the values of $\alpha^R_{\min}$ in each of the real-world networks considered in this study). As we see in the figure, the value of R in real-world networks is also highly sensitive to the value of α, and $R_{\min}$ can be considerably lower than the value of MRT corresponding to an unbiased random walk ($\alpha = 0$) on the same network. For instance, in SCN the value of $R_{\min}$ is about half the value of R corresponding to an unbiased random walk. This result indicates that a carefully chosen value of the bias parameter can significantly reduce the characteristic times of degree-biased random walks.

In Fig. 2 we report the values of $R_{\min}$ and $\alpha^R_{\min}$ for each network in the data set. For those networks with $\alpha^R_{\min} < -0.5$ the minimum value $R_{\min}$ is only slightly greater than the size of the network N, while the differences are more pronounced in the region $\alpha > -0.5$. Notice that all the networks with clear assortative degree-degree correlations (indicated by circles and green labels) have a value of $\alpha^R_{\min} < -0.5$, while disassortative networks (indicated by triangles and red labels) have $\alpha^R_{\min} > -0.5$. This result indicates that the presence of degree-degree

TABLE I. Values of ν and of $\alpha_{\min}$ for MRT, MFPT, and MCT in synthetic and real-world complex networks. The mean-field approximation gives correct results for synthetic uncorrelated networks (i.e., Erdös-Rényi, configuration model and Barabási-Albert networks) with sufficiently large values of $\langle k \rangle$ and N. The values of ν are missing for those networks for which $k_{\rm nn}(k)$ is not a power-law.

Network	Nodes	Edges	$\langle k \rangle$	ν	$\alpha^R_{\min}$	$\alpha^T_{\min}$	$\alpha^C_{\min}$
			Synthetic models				
Erdös-Rény	10^4	2×10^4	4	–	-0.78	-0.46 ± 0.01	-0.90
ER	10^4	5×10^4	10	–	-0.89	-0.78 ± 0.01	-1.00
ER	10^4	2×10^5	40	–	-0.97	-0.96 ± 0.01	-1.00
Conf. Model ($\gamma = 3$)	10^3	4037	8	–	-0.87	-0.65 ± 0.01	-0.65 ± 0.05
Conf. Model ($\gamma = 3$)	5×10^4	21458	8	–	-0.86	–	–
Conf. Model ($\gamma = 3$)	10^3	8764	17.5	–	-0.94	-0.87 ± 0.01	-0.95 ± 0.03
Conf. Model ($\gamma = 3$)	10^3	28522	57	–	-0.99	-0.98 ± 0.01	-0.98 ± 0.02
Conf. Model ($\gamma = 2.5$)	10^4	232722	46.5	–	-0.98	-0.98 ± 0.01	-0.98 ± 0.02
Barabási-Albert ($m = 3$)	10^3	3×10^3	5.9	–	-0.78	-0.36 ± 0.02	-0.30 ± 0.05
BA ($m = 5$)	10^3	5×10^3	9.9	–	-0.98	-0.67 ± 0.01	-0.50 ± 0.05
BA ($m = 20$)	10^3	2×10^4	40	–	-1.02	-0.99 ± 0.01	-1.05 ± 0.05
BA ($m = 3$)	10^5	3×10^5	6	–	-0.76	–	–
BA ($m = 5$)	10^5	5×10^5	10	–	-0.99	–	–
BA ($m = 20$)	10^5	2×10^6	39.9	–	-1.02	–	–
BA ($m = 20$)	4×10^4	8×10^5	39.9	–	-1.02	–	–
			Real-world networks				
Gnutella (P2P) [50]	62561	147877	4.72	–	-0.91	-0.55 ± 0.02	-0.5 ± 0.05
PairsFSG [51]	10618	63787	12.01	–	-0.89	-0.68 ± 0.01	-0.60 ± 0.05
Email URV [52]	1133	5451	9.62	0.05	-0.76	-0.62 ± 0.01	-0.70
Jazz [53]	198	2742	29.01	0.11	-0.70	-0.70 ± 0.01	-0.90
amazon [54]	410236	2439437	11.89	–	-0.68	–	–
USPower [55]	4941	6593	2.66	-0.02	-0.66	0.17 ± 0.01	-0.12 ± 0.05
SCN [41]	12722	39967	6.28	0.18	-0.64	-0.32 ± 0.01	-0.5 ± 0.1
ca-CondMath [41]	21363	91286	8.54	0.16	-0.63	-0.43 ± 0.01	-0.47 ± 0.05
ca-HepTh [41]	8638	24806	5.76	0.19	-0.65	-0.37 ± 0.01	-0.47 ± 0.05
ca-AstroPh [41]	17903	196972	22.00	0.22	-0.62	-0.58 ± 0.01	-0.52 ± 0.05
ca-ASTRO [41]	13259	123838	18.68	0.34	-0.59	-0.54 ± 0.01	-0.60 ± 0.05
ca-HepPh [56]	11204	117619	20.99	0.54	-0.57	-0.51 ± 0.01	-0.43 ± 0.05
pgp [57]	10680	24316	4.55	–	-0.48	-0.20 ± 0.02	-0.25 ± 0.05
C. elegans [55]	279	2287	16.39	-0.15	-0.79	-0.68 ± 0.01	-0.7 ± 0.1
bio-Yeast [58]	2312	7165	6.20	-0.42	-0.44	-0.32 ± 0.01	-0.20 ± 0.02
www-Google [59]	855802	4291352	10.03	-0.42	-0.43	-0.33 ± 0.02	-0.25 ± 0.1
soc-Slashdot [59]	82168	582290	14.17	-0.78	-0.43	-0.38 ± 0.02	-0.16 ± 0.06
soc-Epinions [60]	75877	405739	10.69	–	-0.39	-0.34 ± 0.02	-0.24 ± 0.01
Actors [55]	374511	1222908	6.53	-0.23	-0.37	-0.35 ± 0.02	-0.28 ± 0.06
wordnet [61]	75609	120473	3.18	-0.41	-0.30	-0.11 ± 0.02	-0.20 ± 0.05
www-NotreDame [62]	325729	1090108	6.69	-0.84	-0.29	-0.1 ± 0.05	–
www-Stanford [59]	255265	1941926	15.21	-0.72	-0.23	-0.23 ± 0.07	-0.25 ± 0.08
www-BerkStan [59]	654782	6581870	20.10	-0.84	-0.25	-0.29 ± 0.05	–
caida [56]	26475	53381	4.03	-0.50	-0.15	-0.12 ± 0.01	-0.15 ± 0.05
InternetAS [36]	11174	23409	4.19	-0.52	-0.14	-0.12 ± 0.01	-0.11 ± 0.01
USairport [63]	1572	17214	21.90	–	-0.58	-0.55 ± 0.01	-0.6 ± 0.01
USairports500 [64]	500	2980	11.92	–	-0.50	-0.42	-0.40
netscience.net [65]	379	914	4.82	–	-0.67	-0.58	-0.20

correlations has a significant impact in the values of $\alpha^R_{\min}$ and, consequently, on the performance of a biased random walk on a graph in terms of exploration speed.

The relation between the degree-correlation exponent ν and the value of $\alpha^R_{\min}$ is shown in Fig. 3. The values corresponding to real-world networks lie almost exclusively in the top-left and in the bottom-right quadrants, respectively, corresponding to ($\alpha^R_{\min} < -0.5, \nu > 0$) and ($\alpha^R_{\min} > -0.5, \nu < 0$). Figure 3 shows very clearly that the value of $\alpha_{\min}$ is always in the interval $[-1.0, -0.5]$ for assortative networks and larger than -0.5 for disassortative ones.

To further investigate the special role played by the bias parameter $\alpha = -0.5$ we have considered a large set of synthetic networks in which we tuned the level and sign of degree-degree correlations through the edge-swapping procedure described in Ref. [43]. This procedure, discussed in detail in the appendix, starts from an uncorrelated network and artificially introduces a prescribed amount of either assortative

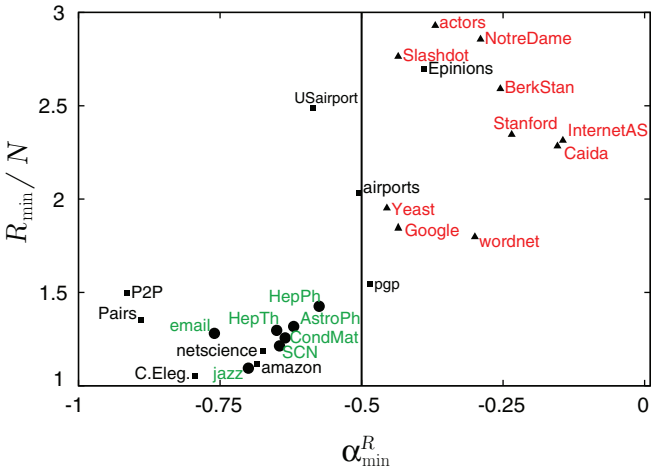

FIG. 2. (Color online) The minimum value of the normalized graph mean return time $R_{\min}/N$ and the corresponding $\alpha^R_{\min}$ for all the networks in the considered data set. Assortative networks (circles, green labels) have $-1 < \alpha^R_{\min} < -0.5$ and $R_{\min}/N \in [1.0, 1.5]$, while for disassortative networks (triangles, red labels) $-0.5 < \alpha^R_{\min} < 0$ and $R_{\min}/N > 1.5$. Black squares indicate those networks for which $k_{nn}(k)$ is not a power-law function of k.

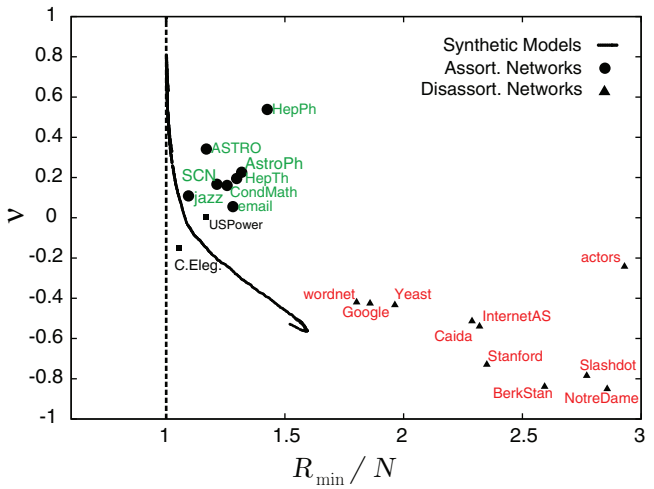

FIG. 4. (Color online) The degree-correlation exponent ν and the normalized graph mean return time $R_{\min}/N$ for the same set of synthetic networks in Fig. 3 (solid black line) and for real-world assortative (circles) and disassortative (triangles) networks. Notice that in the $\nu - R_{\min}/N$ plane all real-world networks lie on the right of the curve, corresponding to synthetic correlated networks.

or disassortative degree-degree correlations by rewiring the edges of the graph without modifying the degree sequence. As a result, this algorithm allows us to investigate the relation between the value of ν and $\alpha^R_{\min}$ of a network by varying continuously the correlation exponent ν while preserving the degree sequence.

The black curve in Fig. 3 has been obtained by starting with a configuration model scale-free network with $N = 10^4$ nodes, $k_{\max} = 300$, and $\gamma = 3$ [44] and by running the

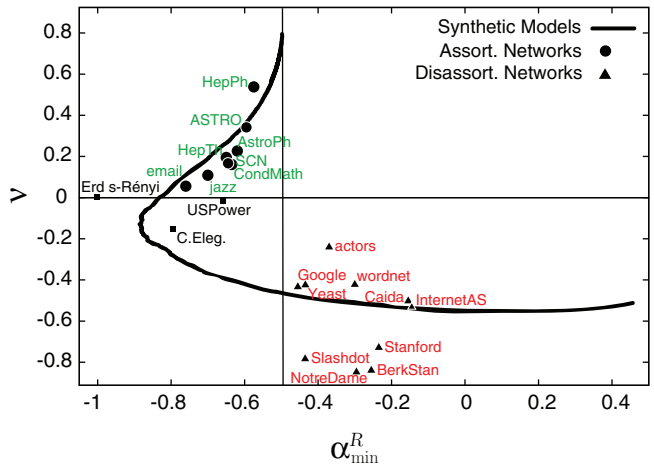

FIG. 3. (Color online) The degree correlation exponent and the value $\alpha^R_{\min}$ that minimizes the graph mean return time in real-world networks (black points) and in synthetic networks with a tunable value of ν (solid black line). Notice that assortative networks (circles) are confined in $-1 < \alpha^R_{\min} < -0.5$ while almost all the disassortative networks (triangles) lie in the region $-0.5 < \alpha^R_{\min} < 0$. For comparison we also report the value of $\alpha^R_{\min}$ for an Erdös-Rényi random graph with $N = 10^4$ and $\langle k \rangle = 40$, which is equal to $\alpha = -1$ as predicted by the theory for uncorrelated networks.

swapping procedure to introduce assortative or disassortative correlations. We notice that by performing assortative swaps the value of ν increases considerably, while $\alpha^R_{\min}$ remains asymptotically confined below -0.5. Conversely, few disassortative swaps are enough to determine a fast change on $\alpha^R_{\min}$, which enters the region $\alpha > -0.5$ where the majority of real-world disassortative networks lie.

In Fig. 4 we report as a solid line the values of $R_{\min}$ as a function of the degree-correlation exponent ν for the same set of synthetic networks considered in Fig. 3. Filled points represent the values obtained on real-world networks. We observe that $R_{\min}/N$ is considerably larger than 1 for disassortative networks, while it is closer to 1 for assortative networks. Notice that the MRT of the synthetic network with tunable degree correlations (solid black line) is consistently smaller than that of any real-world network, with the only exception being the *Caenorhabditis elegans* neural network.

B. Analytical arguments

The numerical analysis of MRT suggests that, for uncorrelated networks, $\alpha_{\min} = -1$, so the deviations from this value observed in real-world networks should be due to the presence of degree-degree correlations. Here we provide an analytical proof of the fact that $\alpha_{\min} = -1$ for uncorrelated graphs in the mean-field approximation, and we compare this analytical prediction with numerical results on real-world networks. In the mean-field approximation a graph is described by the annealed adjacency matrix,

$$\langle a \rangle_{ij} = \frac{k_i k_j}{2K}, \tag{7}$$

where the value $\langle a \rangle_{ij}$ represents the probability to find an edge connecting node i and node j, having degrees k_i and k_j, if the nodes are connected uniformly at random. In fact, let us imagine a network where each node i has k_i stubs to be paired

with some of the stubs of other nodes. If K is the total number of links there are $2K$ of such free stubs. Among these $2K$ stubs, only k_j are incident on node j. Therefore, there are k_j ways a stub of node i can be connected with node j over a total of $2K$ possible pairings with other nodes. One obtains the expression for $\langle a \rangle_{ij}$ in Eq. (7) by observing that node i has k_i different stubs to connect with one of the stubs of j. If we plug Eq. (7) into Eq. (3) we obtain

$$p_i^* = \frac{k_i^{\alpha+1}}{N\langle k^{\alpha+1}\rangle}, \tag{8}$$

which gives

$$r_i(k_i) = N\langle k^{\alpha+1}\rangle k_i^{-\alpha-1} \tag{9}$$

and

$$R = N\langle k^{\alpha+1}\rangle\langle k^{-\alpha-1}\rangle, \tag{10}$$

in agreement with the result found in Ref. [30].

It is straightforward to verify that $R = N$ when $\alpha = -1$. Moreover, one can easily verify that for Erdös-Rényi graphs the minimum value of R is obtained for $\alpha = \alpha_{\min} = -1$. In order to see this fact, we replace the average over nodes $\langle \ldots \rangle$ in Eq. (10) with an integral over degree classes $\int_1^\infty \ldots P(k)\, dk$. We denote with $P_{\rm ER}(k)$ the degree distribution of Erdös-Rényi graphs (this distribution is binomial and can be approximated by a Poisson distribution for large N). Differentiating with respect to α to find the minimum value of R we have the following:

$$\begin{aligned} 0 &= \frac{dR}{d\alpha} \\ &= N\frac{d}{d\alpha}\left[\int_1^\infty P_{\rm ER}(k)k^{\alpha+1}\, dk \int_1^\infty P_{\rm ER}(z)z^{-\alpha-1}\, dz\right] \\ &= N\int_1^\infty P_{\rm ER}(k)log(k)k^{\alpha+1}\, dk \int_1^\infty P_{\rm ER}(z)z^{-\alpha-1}\, dz \\ &\quad + -N\int_1^\infty P_{\rm ER}(k)k^{\alpha+1}\, dk \int_1^\infty P_{\rm ER}(z)log(z)z^{-\alpha-1}\, dz. \end{aligned}$$

The latter expression is equal to 0 when $\alpha = -1$, since we have $k^{\alpha+1} = 1 = z^{-\alpha-1}$ and the last two terms are equal and opposite in sign. Analogously, we can derive the minimum value of R also for uncorrelated networks with power-law degree distribution $P(k) \sim k^{-\gamma}$,

$$\begin{aligned} R &\sim N\int_1^\infty k^{-\gamma}k^{\alpha+1}dk \int_1^\infty k^{-\gamma}k^{-\alpha-1}dk \\ &= N\left[\frac{1}{\gamma-\alpha-2}\right]\left[\frac{1}{\gamma+\alpha}\right], \end{aligned} \tag{11}$$

where the integrability conditions are satisfied if α is in the range $[-2,0]$ and $2 < \gamma < 4$, which is compatible with the values of γ observed in real-world networks.

Differentiating Eq. (11) with respect to α we get again the value $\alpha_{\min}^R = -1$, while the second derivative is always positive, as expected. It is worth noticing that the result $\alpha_{\min}^R = -1$ is independent from the value of the scaling exponent γ of the degree distribution and from the maximum degree in the network, $k_{\max}$.

The quantity R is an average over all graph nodes. However, Eq. (9) allows us also to compute the value $\alpha_{\min}^r(k)$ that minimizes the return time $r(k)$ for nodes having a certain degree k. In the case of Erdös-Rényi graphs a large number of nodes have the same degree because the degree distribution is picked around $\langle k \rangle$ and, as a result, the values of return times are very similar for most of the nodes. For real-world networks, instead, the degree distribution is often heterogeneous and the the return time sensibly depends on the degree of the starting node. Differentiating Eq. (9) with respect to α we get

$$0 = \frac{d}{d\alpha}r(k) = C'_\alpha k^{-\alpha-1} - C_\alpha k^{-\alpha-1}\log(k), \tag{12}$$

with $C_\alpha = N\langle k^{\alpha+1}\rangle$.

Replacing the average over nodes $\langle \ldots \rangle$ with the integral over degree classes, and considering networks with power-law degree distributions $P(k) \sim k^{-\gamma}$ and with minimum and maximum degree k_m and k_M, we get

$$C_\alpha \sim N\int_{k_m}^{k_M} k^{-\gamma}k^{\alpha+1}\, dk. \tag{13}$$

Integrating Eq. (13) and plugging in Eq. (12) we obtain

$$\begin{aligned} &\left[\left(k_M^\beta \ln k_M - k_m^\beta \ln k_m - \left(k_M^\beta - k_m^\beta\right)\ln k\right)\beta + k_M^\beta + k_m^\beta\right]k^{-\alpha-1} \\ &\quad = 0, \end{aligned} \tag{14}$$

where $\beta = -\gamma + \alpha + 2$. The return time $r(k)$ for nodes of a given degree class k takes its minimum at the value of α, which satisfies the previous equation. Excluding the indeterminate case $\beta = 0$, Eq. (14) has only one solution for each value of k.

In the three top panels of Fig. 5 we report the return time $r(k)$ as a function of α for different degree classes (solid lines) compared with the average return time R of the same graph (black dotted lines). The three panels correspond, respectively, to [Fig. 5(a)] a configuration model scale-free graph with $\gamma = 3$, [Fig. 5(b)] InternetAS, and [Fig. 5(c)] SCN. These plots show that a wrong choice of the biased parameter can result in a large increase of the return time. For instance, in Fig. 5(c) the minimum return time $r_{\min}(17)$ for the degree class $k = 17$ occurs for $\alpha = 0.5$ and is about 4 times smaller than the return time $r(17)$ obtained at $\alpha = -1$ (refer to the vertical dashed lines for guidance).

In the three bottom panels of Fig. 5 we report, as a function of k, the value $\alpha_{\min}^r(k)$, which optimized the MRT for nodes having degree k. The black crosses are the numerical results, while the solid blue line is the prediction in mean field obtained from the zeros of Eq. (14). We notice an excellent agreement between the numerical results and the mean-field solution in the case of the uncorrelated scale-free graph [Fig. 5(d)], while for real-world networks [Figs. 5(e) and 5(f)] we observe considerable deviations from the analytical prediction, evidently due to the presence of degree-degree correlations. From the point of view of network exploration, Eq. (14) turns out to be useful when an agent is sent through the network in order to collect information and then has to come back to its starting point [45]. In fact, this equation gives insight about how to fine-tune the bias parameter in order to increase or decrease the time required (on average) by the agents to come back to the starting nodes with the collected information. It is worth noting that small changes in α can produce large variations in the return times, thus highlighting the importance of a proper tuning of the motion bias.

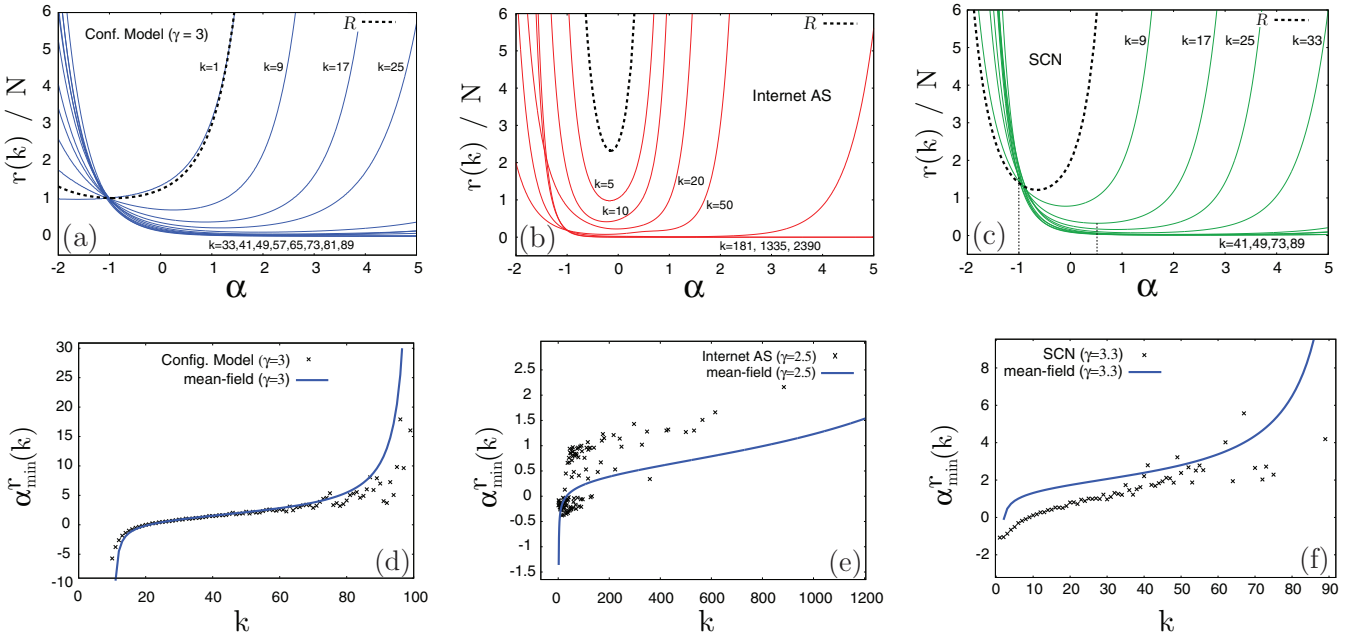

FIG. 5. (Color online) Top panels: return time $r(k)$ for nodes of degree class k as a function of α (solid lines, each curve correspond to a value of k), respectively, for (a) a configuration model scale-free graph with $\gamma = 3$ (blue), (b) Internet AS (red), and (c) SCN (green). The dotted black line in each panel is the graph mean return time R in Eq. (6). Bottom panels: the value $\alpha^r_{\min}(k)$ which minimizes $r(k)$ as a function of k (crosses) for the three networks considered in the top panels. The solid blue line is the mean-field prediction of Eq. (14) where γ is chosen equal to the exponent of the degree distribution of the corresponding network.

IV. MEAN FIRST PASSAGE TIME

In this section we focus on the mean first passage time, showing that the interplay between degree correlations and the dynamics of biased random walks produces qualitatively similar results to those found for the mean return time.

We denote as t_{ij} the expected time needed for a random walker to reach node j for the first time when starting from node i. If the transition matrix Π of the walker is primitive, it is possible to determine t_{ij} by using the fundamental matrix of the Markov chain associated to the random walk [39]. The fundamental matrix Z is defined as

$$Z = (I - \Pi^{\top} + W)^{-1}, \tag{15}$$

where each row of W is equal to the stationary probability distribution $\boldsymbol{p}^*$ and I is the identity matrix. The mean first passage time t_{ij} is then equal to

$$t_{ij} = \frac{z_{jj} - z_{ij}}{p^*_j}, \tag{16}$$

where z_{jj} and z_{ij} are the entries of the fundamental matrix Z. Notice that, in general, $t_{ij} \neq t_{ji}$. We define the *graph mean first passage time* T as the average of the first passage time over all possible node pairs,

$$T = \frac{1}{N(N-1)} \sum_{i,j} t_{ij}. \tag{17}$$

Notice that, in general, the calculation of the fundamental matrix in Eq. (15) is computationally intensive, since it requires the inversion of a $N \times N$ matrix and is practically unfeasible for large networks. For this reason, we used the fundamental matrix Z only to compute the mean first passage time for relatively small networks ($N \lesssim 10^4$), while we resorted to agent-based simulation for larger networks (see the appendix for a description of the employed agent-based algorithm).

As found for the global mean return time R, T also is a convex function of the bias parameter α with a single minimum at $\alpha^T_{\min}$. This is illustrated in Fig. 6(a). Again, the position of the minimum is at $\alpha = -1$ only for uncorrelated networks (see Table I). We also notice that for disassortative real-world networks $-0.5 < \alpha^T_{\min} < 0$, as already found in the case of the mean return time. Conversely, some assortative networks can have a value $\alpha^T_{\min}$, which is not in the range $[-1, -0.5]$. It is worth noting that the minimum value $T_{\min}$ in real-world networks is significantly smaller than the MFPT for unbiased ($\alpha = 0$) random walks or, for the case of uncorrelated networks, ($\alpha = -1$). In Fig. 6(b) we plot, for all the networks in the data set, the minimum value of the graph first passage time $T_{\min}$ rescaled by the number of nodes N. Despite the fact that there is no clear separation at $\alpha = -0.5$ between assortative and disassortative networks, as observed for the MRT, the behavior is similar to that shown in Fig. 2: The farther $\alpha^T_{\min}$ gets from -1, the more $T_{\min}/N$ deviates from 1.

A comparison between the values of $\alpha_{\min}$ for MFPT and MRT is shown in Fig. 6(c). Excluding the network of the U.S. power grid (indicated in the figure as USPower), the value of the Pearson's linear correlation coefficient between $\alpha^R_{\min}$ and $\alpha^T_{\min}$ is $r = 0.87$. Despite the fact that the two values of $\alpha_{\min}$ are not equal for all networks, the strong positive correlation we observe is quite remarkable. We notice that the U.S. power grid is the only spatially embedded network in the data set, so its exceptional values of $\alpha_{\min}$ can be due to spatial constraints, which are absent in the other networks studied. The existence

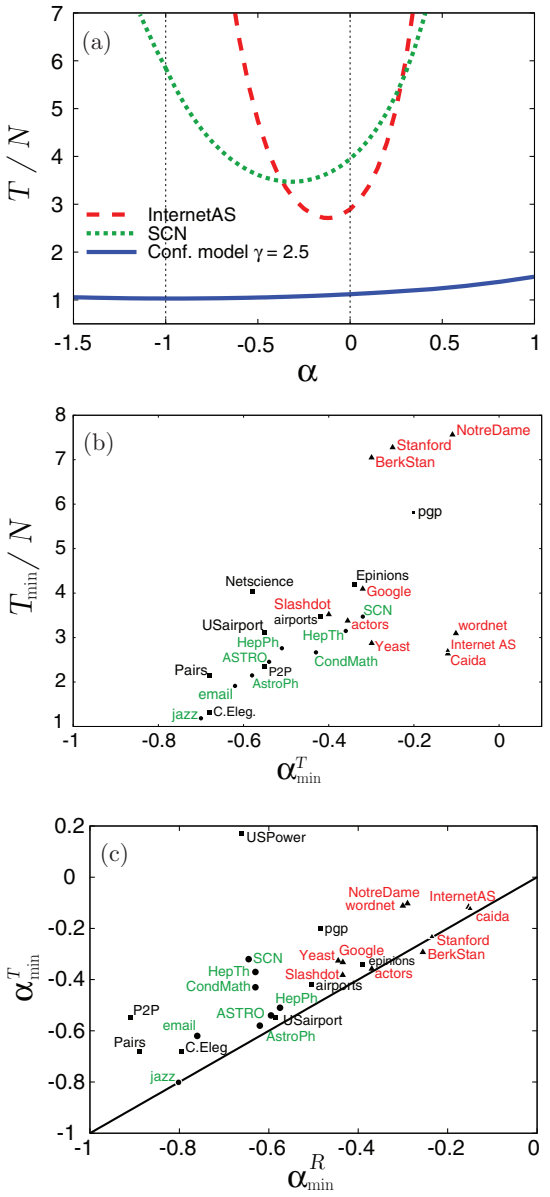

FIG. 6. (Color online) (a) The graph mean first passage time T, rescaled by the number of nodes N, plotted as a function of α for SCN (dotted green line), InternetAS (dashed red line), and an uncorrelated scale-free graph with $N = 10^4$ and $\gamma = 2.5$ (solid blue line). (b) The value $\alpha^T_{\min}$ and the corresponding minimum value of global mean first passage time $T_{\min}/N$ for all the networks in the data set. (c) There is a strong positive correlation between the two values of the bias α which minimize respectively MRT and MFPT. The solid line corresponds to $\alpha^R_{\min} = \alpha^T_{\min}$. The value of the Pearson's linear correlation coefficient is $r = 0.87$ (the U.S. power-grid network is excluded).

of a relatively strong positive correlation between $\alpha^R_{\min}$ and $\alpha^T_{\min}$ could have interesting practical applications. In fact, in order to obtain a walk having a small graph MFPT on a large network, it is possible to use $\alpha^R_{\min}$ as an approximation of $\alpha^T_{\min}$, so one can obtain a quasioptimal biased random walk with respect to MFPT without the need to invert the fundamental matrix of the graph, which is practically impossible for large networks.

V. MEAN COVERAGE TIME

The last characteristic time under investigation is the mean coverage time (MCT) c_i, defined as the expected number of time steps required for the walker to visit all the nodes of the graph at least once when starting from node i. We also study the *graph mean coverage time* C, defined as an average of c_i over all the graph nodes,

$$C = \frac{1}{N}\sum_{i}^{N} c_i . \tag{18}$$

We have computed the graph mean coverage time C for all but two networks in the data set by means of an agent-based simulation and by averaging over many realizations of the walk as described in the appendix. The asymptotic lower bound on the coverage time for the unbiased ($\alpha = 0$) random walk on a generic graph is given by [46]

$$c_i \geqslant (1 + O(1))\, N\, \ln(N) \quad (\alpha = 0), \tag{19}$$

where the equality is satisfied for the complete graph of N nodes, i.e., the graph in which there is a link between each pair of nodes. The inequality (19) implies the following lower bound for the global mean coverage time,

$$C \geqslant (1 + o(1))\, N\, \ln(N) \quad (\alpha = 0). \tag{20}$$

We therefore normalize the obtained values of C by the quantity $N\ \ln(N)$.

In Fig. 7(a) we report such normalized quantity as a function of the bias parameter for a configuration model scale-free network, SCN and InternetAS. The mean coverage time is a convex function of α with a single minimum at $\alpha^C_{\min}$. As for MRT and MFPT we notice that the minimum of the global mean coverage time for the uncorrelated scale-free graph occurs at $\alpha^C_{\min} = -1$ and that the minimum value $C_{\min}$ is very close to the lower bound given by Eq. (20). Real-world networks have instead values of $C_{\min}$ that are significantly higher than the lower bound.

We notice that the MCT is more sensitive to α than MRT and MFPT [the typical concavity of MCT in Fig. 7(a) is narrower than the ones observed for MRT and MFPT, respectively, in Fig. 1 and Fig. 6(a)]. For instance, in SCN the minimum mean coverage time, $C_{\min} \simeq 7Nln(N)$, is about 1.7 times smaller than the mean coverage time obtained for $\alpha = 0.0$ or for $\alpha = -1$ on the same graph, which is $C_{(\alpha=0)} \simeq C_{(\alpha=-1)} \simeq 12Nln(N)$. Instead, disassortative networks like InternetAS have a minimum value of the coverage time that is similar to that for the unbiased case, while differing markedly from the value at $\alpha = -1$.

In Fig. 7(b) we report the values of $\alpha^C_{\min}$ and $C_{\min}$ for all the networks in the considered data set. The results are qualitatively similar to those reported in Fig. 2 and Fig. 6(b). Although for a given network the minimum of C occurs at $\alpha^C_{\min} \neq \alpha^R_{\min}$, it is evident from Fig. 7(c) that the two values are positively correlated (the Pearson's linear correlation coefficient is $r = 0.77$).

We have investigated the differences between the optimal values of α for the three characteristic times comparing $\alpha^R_{\min}$, $\alpha^T_{\min}$, and $\alpha^C_{\min}$ for a set of synthetic networks generated through the swapping procedure (the starting network in this case is a

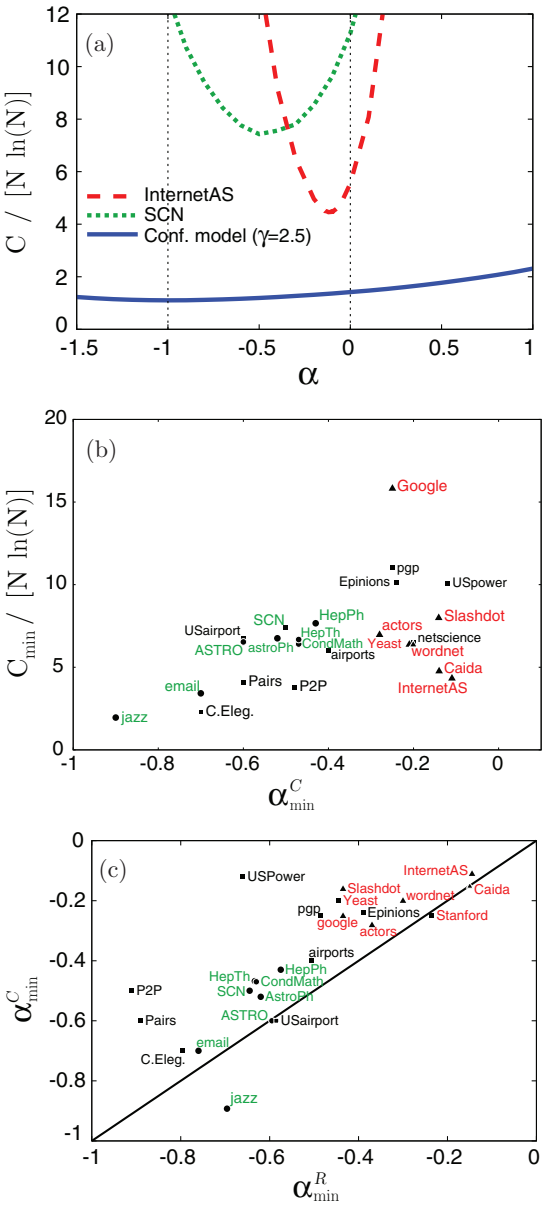

FIG. 7. (Color online) (a) The graph mean coverage time C, rescaled by the lower bound $N\ln(N)$, plotted as a function of α for the same networks as in Fig. 6(a). (b) The minimum value $\alpha^C_{\min}$ and the corresponding coverage time $C_{\min}$ for all the networks in the data set. (c) There is a positive correlation between the two values of the bias α which minimize respectively MRT and MCT. The solid line corresponds to $\alpha^R_{\min} = \alpha^C_{\min}$. The value of the Pearson's linear correlation coefficient is $r = 0.77$ (the U.S. power-grid network is excluded).

configuration model with $\gamma = 2$, $N = 1000$ and $\langle k\rangle = 14.6$). The results (figure not shown) suggest that synthetic assortative networks have equal optimal bias values ($\alpha^R_{\min} = \alpha^T_{\min} = \alpha^C_{\min}$), so the deviations from the bisector in Figs. 6(c) and 7(c) might be due only to fluctuations in the pattern of degree correlations of real-world networks. Instead, in the case of synthetic disassortative networks, we observe deviations from the bisector of the same order of those observed in real-world networks.

VI. DISCUSSION

In this section we discuss in detail some of the results reported in the paper, we provide a mechanistic explanation of the variations of $\alpha_{\min}$ observed in real-world networks, and we outline possible applications to practical problems.

A. Deviations from $\alpha_{\min} = -1$

The results reported in Fig. 2 confirmed that the value of α which minimizes the MRT in real-world networks sensibly deviates from the value $\alpha_{\min} = -1$ predicted for the uncorrelated graph and that this deviation seems to depend on the sign and magnitude of degree-degree correlations. We notice that, in the absence of degree correlations, the stationary distribution of walkers is given by Eq. (8), which for $\alpha = -1$ corresponds to a uniform distribution of walkers across the nodes of the network, i.e., $p^*_i = 1/N$. Consequently, the minimum value of MRT for uncorrelated graphs is obtained for a uniform distribution of walkers and is equal to $R_{\min} = N$ [see Eq. (10)]. We argue that the minimum of MRT in a generic network is always obtained for a value of α which induces the distribution of walker that is the closest possible to a uniform one.

We start by noticing that, according to Eq. (6), R is the harmonic mean of the stationary distribution $\boldsymbol{p}^*$. By using Jensen's inequality [47], it is possible to prove that any stationary distribution $\boldsymbol{p}^*$ which is not uniform produces a value of the mean return time which is larger than (or at most equal to) that obtained from a uniform $\boldsymbol{p}^*$ (which is equal to N),

$$R = \frac{1}{N}\sum_i \phi(p^*_i) \geqslant \phi\left(\frac{\sum_i p^*_i}{N}\right) = N, \qquad (21)$$

where $\phi(x) = 1/x$. We observe that if a graph is not uncorrelated, and especially if the graph has assortative degree correlations, then the stationary distribution of the biased random walk obtained for $\alpha = -1$ is generally far from being uniform, while the stationary distribution corresponding to $\alpha = \alpha_{\min}$ is usually very close to a uniform one. And in fact, Fig. 2 confirms that for assortative networks the value of $R_{\min}$ is very close to N, despite the fact that larger deviations are observed for disassortative networks. Thus we assume that, for a given network, the discrepancy between the observed value of $\alpha_{\min}$ and the prediction $\alpha = -1$ for uncorrelated networks is indeed due to the necessity to obtain a stationary distribution as close as possible to a uniform one.

If this hypothesis is correct, it should be possible to determine the value of $\alpha_{\min}$ by imposing that the resulting stationary occupation probability distribution is as close as possible to $p_i = 1/N$. Let us consider the case of assortative networks and assume that the expected degree of the first neighbors of a node having degree k is a power law, i.e.,

$$k_{\rm nn}(k) = Dk^{\nu}, \quad \nu > 0, \qquad (22)$$

where D is a normalization constant. Let us also make the assumption that the fluctuations in the degree of the neighbors of a node with degree k are negligible, so if j is a first neighbor of node i we can write

$$k_j \simeq k_{\rm nn}(k_i) = Dk_i^{\nu}.$$

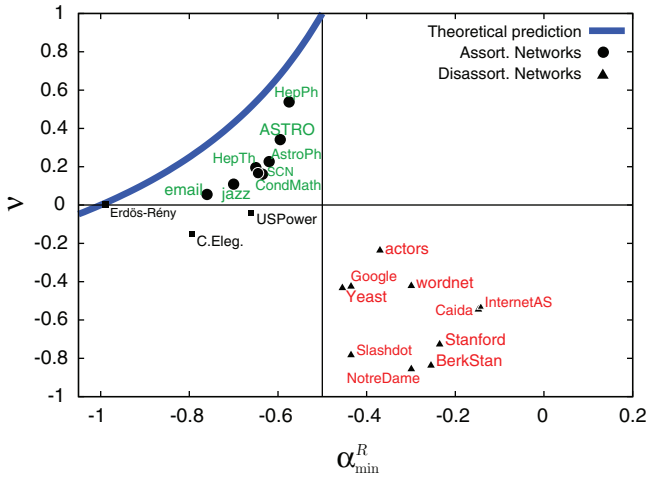

FIG. 8. (Color online) The theoretical prediction (solid blue curve) of $\alpha_{\min}$ given by Eq. (24) works well for assortative networks ($\nu = 1$ and $\alpha_{\min} = -0.5$) and uncorrelated networks ($\nu = 0$ and $\alpha_{\min} = -1$). The theoretical curve falls outside the plot for negative values of ν and does not approximate well the optimal bias value for disassortative networks.

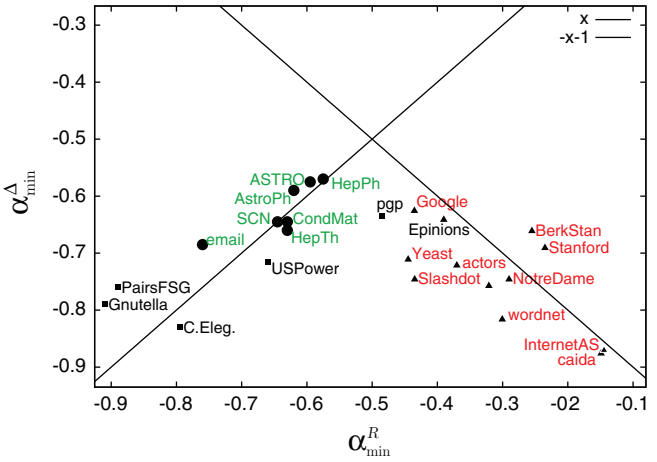

FIG. 9. (Color online) The value $\alpha^\Delta_{\min}$ of the motion bias whose corresponding stationary distribution is the closest one to a uniform distribution is correlated with $\alpha^R_{\min}$. In particular, in assortative networks (circles) the two values are positively correlated ($\alpha^R_{\min} \simeq \alpha^\Delta_{\min}$), while in disassortative (triangles) networks we have $\alpha^\Delta_{\min} \simeq -1 - \alpha^R_{\min}$.

By plugging Eq. (22) in Eq. (3) we get

$$c_i \sim k_i^{\alpha\nu} \sum_j a_{ij} = k_i^{\alpha\nu+1}, \qquad p_i^* \sim \frac{k_i^{\alpha\nu+\alpha+1}}{\sum_\ell k_\ell^{\alpha\nu+\alpha+1}}. \tag{23}$$

Imposing that $\boldsymbol{p}^*$ is a uniform distribution, i.e., that $p_i^* = p_j^*, \forall\, i,j = 1,\ldots,N$, we obtain that α should satisfy the following equation:

$$\alpha\nu + \alpha + 1 = 0, \quad \nu > 0. \tag{24}$$

In Fig. 8 we show the curve $\alpha\nu + \alpha + 1 = 0$ (solid blue line) together with the values $(\alpha_{\min}, \nu)$ corresponding to real-world networks. Notice that for uncorrelated networks, i.e., when $\nu = 0$, we obtain the analytical prediction $\alpha_{\min} = -1$, while for maximally assortative networks, i.e., for $\nu = 1$, we get $\alpha_{\min} = -0.5$. Also, the values of $(\alpha_{\min}, \nu)$ for real-world assortative networks are close—but admittedly not identical—to the prediction of Eq. (24). The observed discrepancies between theory and observations are due to the fact that in real-world networks the fluctuations in the degree of the nearest neighbors of a node with degree k are not negligible, despite the fact that we usually have $k_{\rm nn}(k) \sim k^\nu$. Therefore, if j is a neighbor of i, then $k_j \neq k_{\rm nn}(k_i) \sim k_i^\nu$, and, consequently, the second of the two assumptions used in the derivation of Eq. (23) and Eq. (24) does not hold.

The case of disassortative networks is a bit more cumbersome, due to the constraints introduced by a discrete degree sequence (namely a node cannot have a degree smaller than 1 or larger than $k_{\max}$). In particular, it is possible to define

$$k_{\rm nn}(k) \sim k^\nu k_{\max}, \quad \nu < 0, \tag{25}$$

obtaining an equation similar to Eq. (24). Unfortunately, such an equation has a discontinuity at $\nu = -1$ and does not match the values of $(\alpha_{\min}, \nu)$ observed in real-world disassortative networks.

In the absence of an analytical argument for disassortative networks, we computed numerically the value of α which minimizes the variance $\Delta = \langle p_i^{*2}\rangle - \langle p_i^*\rangle^2$ of the stationary occupation probability distribution. In fact, Δ provides a rough estimation of how far the distribution is from a uniform one (for which $\Delta = 0$). The results are reported in Fig. 9, in which we show, for each network in the considered data set, the value of $\alpha^R_{\min}$ which minimizes the MRT and the value $\alpha^\Delta_{\min}$ which minimizes the variance of $\boldsymbol{p}^*$. Notice that for assortative networks we have a strong positive correlation between $\alpha^R_{\min}$ and $\alpha^\Delta_{\min}$, with $\alpha^\Delta_{\min} \simeq \alpha^R_{\min}$. Conversely, for disassortative networks the two values are negatively correlated and seem to be connected by the relation $\alpha^\Delta_{\min} \simeq -1 - \alpha^R_{\min}$. These results confirm that there is indeed an intimate relation between the variance of the stationary probability distribution obtained for a given value α of the motion bias and the corresponding MRT and suggest that the optimization of the mean return time for assortative networks is obtained for a value of α which guarantees a stationary distribution as close as possible to the uniform one.

B. Applications

The results shown in this paper can have interesting applications in several different contexts, from the control of diffusion processes to the successful advertisement of products and services on online social networks. A typical example is that of congestion control in communication and transport systems, such as the Internet, the WWW, P2P networks, and road networks. In these systems traffic is usually modeled through simple packet generation and routing algorithms. At each time step, a certain number R of new packets is introduced in the system, and each packet is assigned a source and a destination node. The nodes of the network route packets according to a certain policy (which might be a biased random walk), and have a fixed delivery capacity D. When a packet arrives at its destination node, it is removed from the system.

An interesting quantity that characterizes the emergence of congestion is the critical packet generation rate R_c, defined as the number of new packets above which the number of packets removed from the system per unit time is smaller than the number of new packets per unit time introduced in the network. Under the condition $R > R_c$ the number of packets flowing in the network keeps increasing with time, leading to congestion. In Ref. [48] it has been shown that, for a routing strategy based on biased random walk, R_c and the graph mean first passage time T are related by the following equations:

$$R_c(i) = \frac{D}{p_i^* T}, \tag{26}$$

$$R_c = \min_i \{R_c(i)\}, \tag{27}$$

where D is the delivering capacity. Equation (26) shows that the value of R_c depends on α through both T and p_i^*. This implies that in order to maximize R_c one has to minimize the product $\max_i\{p_i^*\}\,T$. By noticing that, in assortative networks, for $\alpha = \alpha^R_{\min}$ we obtain a $\boldsymbol{p^*}$ which is the closest possible to a uniform distribution (according to the results shown in Fig. 9) and, at the same time, we get an almost minimal value of T (due to the strong positive correlation between $\alpha^R_{\min}$ and $\alpha^T_{\min}$); therefore we can conclude that a good approximation for the value of the bias which maximizes R_c can be obtained by setting $\alpha = \alpha^{\Delta}_{\min} \simeq \alpha^R_{\min} \simeq \alpha^T_{\min}$. Unfortunately, this reasoning does not work in disassortative networks, for which the value of α that minimizes $\max_i\{p_i^*\}$ does not coincide with the value of α which minimizes T. In this case, the optimal critical packet rate depends on the trade-off between the homogeneity of the stationary occupation probability distribution and the corresponding value of T. This is an example of how the correlation between the optimal values of $\alpha_{\min}$ and the sign and magnitude of degree-degree correlations can be used to avoid congestion and improve transport performance on a given network.

The results of this paper might also find application in the field of optimal network crawling, i.e., the exploration of the structure of a graph by means of agents performing random walks over it. Examples include the sampling of online social networks (e.g., Facebook and copurchasing networks) and online communication networks (e.g., the World Wide Web and Twitter). In particular, exploring the network at the fastest possible speed corresponds to minimizing the MCT. As we have seen, this is achievable by using a degree-biased random walk with $\alpha = \alpha^R_{\min}$, since there is a pretty strong correlation between $\alpha^R_{\min}$ and $\alpha^C_{\min}$. If the network is assortative, which is actually the case for the majority online social networks, the value of $\alpha^R_{\min}$ which optimizes the coverage time will lie in $[-1,0.5]$ and can be obtained using Eq. (24), where the correlation exponent ν can be measured from a relatively small sample of the graph of interest. If, instead, the network is disassortative, as usually happens for online communication networks, then the value of α should be chosen in the range $[-0.5,0]$ and a good hint is provided by the value $-1-\alpha^{\Delta}_{\min}$ (see Fig. 9). Such a value can be computed taking into account a small representative sample of the degree sequence of the graph. In both cases, an appropriate tuning of the bias parameter α will outperform the standard unbiased random walk.

Another interesting application of the relationship between assortativity and optimal graph traversal could be that of information retrieval. In a recent work [49] it has been shown that the biased random walk on the directed network of Wikipedia pages can be used to implement an algorithm able to retrieve professional skills from an arbitrary text (e.g., a curriculum vitae). The authors have shown that the performance of the system can be optimized by means of an appropriate tuning of the motion bias α. The results reported in Table 2 of Ref. [49] show that the best performance of the retrieval system are achieved for α between -1 and 0 and in particular for $\alpha \simeq -0.4$, which is a reasonable optimal value of the motion bias considering that the undirected version of the network of Wikipedia pages is known to be disassortative. Therefore, the generalization of the present study to the case of directed networks could provide theoretical insights and guidelines for the optimal choice of the bias parameter in skill retrieval systems.

Finally, another possible application of these results concerns social-marketing campaigns. At present the advertising of products and services is more often conveyed through online social networking platforms. Customers are promised a reward if they promote a certain range of products to their online friends, and usually they get an equal reward for each friend who adopts the product or service. If we assume that the diffusion of the advertising can be regarded as a random motion, then promising equal rewards is not the best diffusion strategy, because customers will not have any reason to preferentially advertise the product to any of their neighbors in particular and will therefore choose one of their friends at random with equal probability. If we look to the advertisement as a walker which jumps from one customer to another, this strategy would correspond to an unbiased random walk ($\alpha = 0$). Our results about mean coverage time suggest instead that the diffusion speed (i.e., the number of advertised users per unit time) can be increased if the customer is rewarded proportionally to a biased transition probability, i.e., if the customer receives a reward proportional to the α power of the degree of the friend who has adopted the suggested product or service. The optimal bias parameter can be directly computed if the network topology is entirely known or, given that social networks are often assortative, it can be guessed using Eq. (24). This also disproves the intuitive idea that the best strategy is to always advertise the highly connected users.

VII. CONCLUSIONS

Random walks are the simplest way to visit a network, and degree-biased random walks, which make use of information about the degree of destination nodes, are particularly suited to highlight the presence of degree-degree correlations. In this paper we have focused on the typical times of biased-random walks, namely on the expected time that a walker needs to come back to its starting node (MRT), to hit a given node (MFPT), or to visit all the nodes of the network (MCT). We have studied how such characteristic times depend on the value of the motion bias α. We have proved analytically that, in the mean-field

approximation, the value $\alpha_{\min}$ that minimizes the characteristic times in uncorrelated networks is equal to -1. This corresponds to a walk in which the probability to move to a node is inversely proportional to its degree. As shown by numerical simulations, the mean-field approximation works pretty well for uncorrelated networks. However, real-world networks are characterized by nontrivial degree-degree correlations and, as a result, the characteristic times of degree-biased random walks on real-world networks deviate from those obtained by using the mean-field approach, as we have shown in the paper by studying a large data set of medium to large real-world networks.

In particular, the value of $\alpha_{\min}$ sensibly differs from -1 in a way that depends on the sign of the degree-degree correlations. We have found that optimal values of the bias parameter α lie between -1 and 0 for a large number of real-world networks. In addition to this, we have shown that the minimum characteristic times occur preferentially for α in the range $[-1, -0.5]$ for assortative network and for α in the range $[-0.5,0]$ for disassortative ones. We have derived an approximate analytical relation between $\alpha_{\min}$ and the degree-correlation exponent ν, which might be useful to refine the choice of the optimal bias for assortative networks, and we have shown numerically that the value of the mean return time obtained for a given value of α is related with the heterogeneity of the corresponding stationary probability distribution of the walk.

By discussing several different possible applications of these results, we have stressed the fact that the minimization of characteristic times may be useful in many domains, from mitigation of network congestion to successful product advertisement in online social networks. In general, when only local information is available, degree-biased random walks can achieve better exploration performance than unbiased random walks by appropriately tuning the bias parameter α according to the global structural properties of the graph at hand.

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APPENDIX

We describe here the algorithm we have used to generate graphs with tunable degree-degree correlations and the agent-based approach used to estimate the mean coverage time and the mean first passage time in large graphs.

1. Swapping algorithm

In Fig. 3 and Fig. 4 we have reported the values of the degree-correlations exponent ν and the motion bias which minimizes the return time $\alpha^R_{\min}$ for a set of graphs with the same degree sequence of a chosen starting graph and tunable degree-degree correlations. An increasing amount of assortative or disassortative correlations is introduced by repeatedly applying the edge swapping procedure described in Ref. [43] to an initially uncorrelated graph. Each swap is performed as follows. Two edges connecting four different nodes are randomly selected and the nodes at the ends are ordered according to their degree $k_1 \leqslant k_2 \leqslant k_3 \leqslant k_4$. The two edges are then removed. Positive assortative correlations are introduced by connecting the two nodes with the smaller degrees and the two nodes with the larger degrees. Instead, disassortative correlations are introduced by connecting the node with the smallest degree with the node with the largest degree and the two remaining nodes with intermediate degrees. In order to preserve the degree sequence, all swaps that produce parallel edges are not allowed. Figure 10 and Fig. 11 illustrate the two types of swaps.

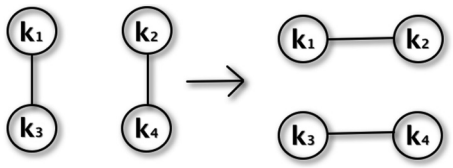


FIG. 10. Assortative swap.

2. Agent-based simulation for MFPT and MCT

The MFPT and MCT are estimated by means of an agent-based simulation. In both cases we simulated a walker which moves across the nodes of the network according to the transition probability given in Eq. (1). The simplest way to compute the characteristic times is to wait until the walker, started at a randomly selected node, explores all the nodes at least once. At that point the value of c_i is given by the total number of time steps spent by the walker to visit all the nodes, while t_{ij} can be obtained by storing in memory the first passage times to all other nodes during the simulation. However, despite the fact that this procedure is pretty simple to implement, it is not suitable to obtain robust results in a reasonable amount of time. In fact, in order to have an estimate of c_i and t_i, we need to average over a sufficiently large number of walks starting at node i, and the same procedure should be repeated for all the starting nodes. However, the heterogeneity of the degree distribution of real-world networks induces heterogeneity in the number of visits on nodes with different degree. Just to make an example, in the unbiased case ($\alpha = 0$) the walker visits a node with degree 1 only once every $k_{\max}$ visits on the node with the maximum degree. As a result, most of the computation time is wasted by repeated visits to highly connected nodes. A value of $\alpha \neq 0$ can either accentuate or mitigate the disproportion in the number of visits. To overcome this problem, we implemented a smarter strategy. The key point of our method is to consider each hop as the starting point of a new walk and to store the entire sequence of node labels in an array we call *Tape*. As soon as all nodes

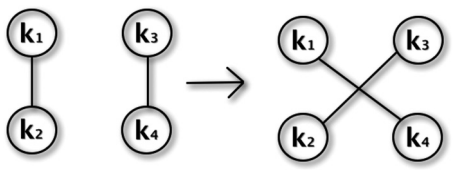


FIG. 11. Disassortative swap.

have been visited at least once, both t_{ij} and c_i can be calculated (here i is the node label at the beginning of *Tape*). Then the first entry of *Tape* is removed, and the computation of the mean first passage and coverage time is performed for the new node which now occupies the first entry of *Tape*. If, after a removal of the first entry, a node label is no longer contained in *Tape*, new walker hops are simulated until all missing nodes are visited.

Here we describe separately the two algorithms for MCT and MFPT despite the fact that the simulation could in principle be performed simultaneously.

3. Algorithm for the mean coverage time

We randomly select a starting node and we simulate the walk according to the transition probability of Eq. (1) for a given value of α. We dynamically add the labels of the nodes visited at the end of an array referred to as *Tape*. An array *number-of-visits[i]* of length N keeps track of the number of visits on each node i. A *counter* stores the number of unique nodes visited: When all nodes have been visited at least once the *counter* is equal to N. Finally, a variable L stores the number of hops between the node at the first entry of *Tape* and the node at the end, i.e., the length of *Tape* minus 1. The steps of the algorithm are reported in the following:

(0) Initialize all variables to zero and choose a node i at random. Set *number-of-visits*[i] and *counter* equal to 1.

(1) Jump to a successive node, say node j, and add the node label j as new element at the end of *Tape* (push-back operation). Increase L and *number-of-visits*[j] by 1. If the new value of *number-of-visits*[j] is equal to 1, also increase *counter* by 1.

(2) If *counter* is equal to N proceed to step (3), otherwise go to step (1).

(3) The current vale of L is the estimate of the coverage time c_i relative to the node in the first entry of *Tape* (say, i). Store the value c_i and the corresponding node label i.

(4) Consider again the first entry i of *Tape* and decrease L and *number-of-visits*[i] by 1. If the new value of *number-of-visits[i]* is equal to zero, also decrease *counter* by 1.

(5) Remove the first entry i of *Tape* and free the memory (pop-front operation) and then go to step (2).

The simulation ends when the estimated values of c_i are averaged over at least 1000 realizations for each node i. Consequently, in the unbiased case the value c_j for a node j with degree k_j will be averaged over $1000 * k_j / k_{\min}$ realizations. In Fig. 12 we illustrate the basic principle of the algorithm. The loop (1)–(2) performs the walker motion and adds the node labels in *Tape*. When all nodes have been visited at least once the algorithm enters the (3)–(5) loop, where the estimates of the coverage time are calculated and stored. If the *number-of-visits[i]* for a certain node i is equal to zero, then this node i is no longer contained in *Tape* and the algorithm goes back to the (1)–(2) loop.

4. Algorithm for the mean first passage time

We notice that the estimation of the mean first passage time does not require the computation of each entry of the matrix

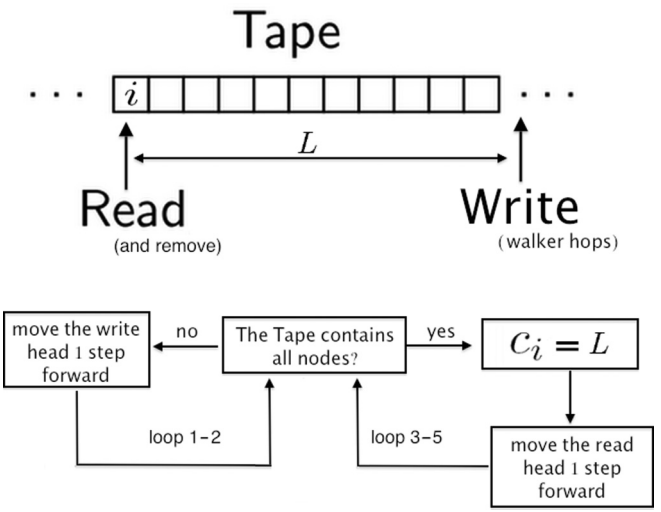


FIG. 12. The flowchart illustrates the core principle of the algorithm for the estimation of the mean coverage time. *Tape* is an array whose length changes dynamically. In steps (1)–(2) new node labels are written at the end of *Tape*, while in steps (3)–(5) nodes are removed from the beginning of *Tape*.

Z but just the average of its rows,

$$t_i = \frac{1}{(N-1)} \sum_{j \neq i} t_{ij}, \tag{A1}$$

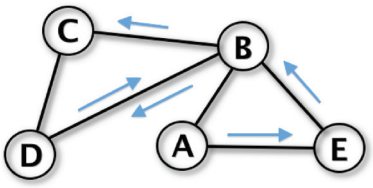
that is, the average MFPT from node i to all the other nodes. We randomly select the starting node and we simulate the walk according to the transition probability of Eq. (1) for a given value of α. As before, we add the labels of visited nodes at the end of *Tape*. An array of dimension N keeps track of the *number-of-visits*[i] on each node i, and a *counter* stores the number of unique nodes visited. We use the variable L to keep track of the total number of hops during the entire walk and in this case this value will not be reduced when we pull off nodes from the beginning of *Tape*. Indeed, we use a second variable L^{old} to store the number of nodes pulled off from *Tape*. Moreover, for each node i we initialize an array *not-first-passage[i]* that stores the times, i.e., the values of L, at which the walker visits a node already previously visited. At later stages of the algorithm these values will be used to rapidly compute the first passage time for a given walker path. Finally, a variable FPTs temporarily accumulates the sum of the values of the first passages times t_{ij} in order to calculate t_i in Eq. (A1). Its role will be clear later. The algorithm consists of the following steps:

(0) Initialize all variables to zero and choose a node i at random. Set *number-of-visits*[i] and *counter* equal to 1.

(1) Jump to a successive node (say, j), add the node label j as new element at the end of *Tape* (push-back operation), and increase L by 1.

(2) If *number-of-visits*[j] is equal to zero, go to step (3) and otherwise go to step (4).

(3) Add the value (L -L^{old}) to the variable FPTs. Increase the *counter* and *number-of-visits*[j] by 1. Then go to step (5).



Node Sequence (*Tape*)	A	E	B	D	B	C	...
Time Passages (L)	0	1	2	3	4	5	...

FIG. 13. (Color online) A walk on a network of $N = 5$ nodes. The passage on node **B** at time $L = 2$ is a genuine first passage relative to the walk starting at nodes **A** and **E**. The passage on node **B** at time $L = 4$ is a genuine first passage only when the first three nodes are removed from *Tape* and we consider the walk starting on node **D**.

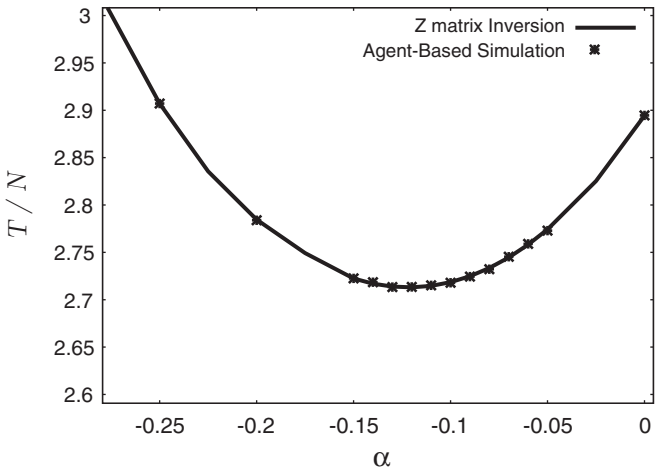


FIG. 14. The value of the graph mean first passage time T normalized by the number of nodes N as a function of α for the InternetAS network. The agent-based simulation (crosses) provides a very good approximation of the exact values obtained using the fundamental matrix Z (solid line).

(4) Add the current value of L as new element at the end of the *not-first-passage*[j] array and increase *number-of-visits*[j] by 1. Then go to step (5).

(5) If *counter* is equal to N go to step (6) and otherwise go to step (1).

(6) Consider the first entry of *Tape* (let us say it is node i). The current vale of FPTs divided by $N - 1$ is the first passage time t_i of Eq. (A1) relative to node i.

Store t_i and the corresponding the node label i. Remove the first entry of *Tape* (but keep in memory the label i). Decrease the *number-of-visits*[i] and L^{old} by 1.

(7) If *number-of-visits*[i] is equal to zero go to step (8) and otherwise go to step (9).

(8) Decrease *counter* by 1 and FPTs by $(N - 1)$. Then go to step (5).

(9) Select the value L^* in the first entry of the array *not-first-passage*[i]. Set

$$\text{FPTs} = \text{FPTs} - (N - 1) + (L^* - L^{\text{old}}).$$

Remove the first entry of the array *not-first-passage*[i]. Go to step (5).

Steps (1)–(4) perform the walk motion and add the sequence of visited nodes in *Tape*. In step (2) we check if the node j has not yet been visited and, if so, in step (3) we store the first passage time $t_{ij} = L - L^{\text{old}}$ in the variable FPTs. When all nodes have been visited at least once the algorithm enters the loop (5)–(9). Steps (5)–(9) repeatedly remove the entries at the beginning of *Tape* and compute, after each removal, the mean first passage time t_i of Eq. (A1) relative to each removed node i. If a node label is no longer contained in *Tape*, the algorithm goes back to the (1)–(4) loop until all nodes have been visited at least once. The advantage of this strategy is that the estimated mean first passage time t_i for a certain node i can be computed using the mean first passage time t_ℓ of the node ℓ that precedes the node i in *Tape* as described by the recursive equation in step (9). The numerical simulation is left running until the estimate of t_i is averaged over 1000 realizations for each node i.

To further clarify the key strategy used in the algorithm, let us give an example on a small graph with $N = 5$ nodes and a walker path illustrated in Fig. 13.

The second passage on node **B** at time $L = 4$ is excluded in the computation of $t_{\mathbf{A}}$ because the first passage on node **B** has already occurred at the second hop ($L = 2$). However, the value $L = 4$ is added at the end of the array *not-first-passage*[**B**] to be used later (let us call this value $L^* = 4$). Indeed, when the first three entries of *Tape* are removed [step (6)] and we consider the walk starting on node **D**, the second passage in node **B** that occurred at $L = 4$ is now a genuine first passage. At this time, because we have removed three entries from *Tape*, we have $L^{\text{old}} = 3$ and the correct number of hops between node **D** and the first passage on node **B** is given by $L^* - L^{\text{old}} = 4 - 3 = 1$. The value $L^* - L^{\text{old}}$ is then used in the computation of $t_{\mathbf{D}}$.

In Fig. 14 we show a validation of our agent-based simulation by comparing it with the result of the inversion of the Z matrix for a medium-sized network.

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