Instability of subharmonic resonances in magnetogravity shear waves

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We study analytically the instability of the subharmonic resonances in magnetogravity waves excited by a (vertical) time-periodic shear for an inviscid and nondiffusive unbounded conducting fluid. Due to the fact that the magnetic potential induction is a Lagrangian invariant for magnetohydrodynamic Euler-Boussinesq equations, we show that plane-wave disturbances are governed by a four-dimensional Floquet system in which appears, among others, the parameter ε representing the ratio of the periodic shear amplitude to the vertical Brunt-Väisälä frequency N_3. For sufficiently small ε and when the magnetic field is horizontal, we perform an asymptotic analysis of the Floquet system following the method of Lebovitz and Zweibel [Astrophys. J. **609**, 301 (2004)]. We determine the width and the maximal growth rate of the instability bands associated with subharmonic resonances. We show that the instability of subharmonic resonance occurring in gravity shear waves has a maximal growth rate of the form $\Delta_m = (3\sqrt{3}/16)\varepsilon$. This instability persists in the presence of magnetic fields, but its growth rate decreases as the magnetic strength increases. We also find a second instability involving a mixing of hydrodynamic and magnetic modes that occurs for all magnetic field strengths. We also elucidate the similarity between the effect of a vertical magnetic field and the effect of a vertical Coriolis force on the gravity shear waves considering axisymmetric disturbances. For both cases, plane waves are governed by a Hill equation, and, when ε is sufficiently small, the subharmonic instability band is determined by a Mathieu equation. We find that, when the Coriolis parameter (or the magnetic strength) exceeds $N_3/2$, the instability of the subharmonic resonance vanishes.

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I. INTRODUCTION

Gravity or magnetogravity waves are ubiquitous in various geophysical and astrophysical systems. In the present paper, we perform an analytical study to characterize the response of these waves to a time-periodic forcing (i.e., a time-periodic shear). As in many physical systems submitted to a periodic forcing, the response of these waves can lead to the triggering of parametric instability.

In the context of astrophysical applications, Zaqarashvili *et al.* [1–3] suggested that the periodic shaking of the solar coronal magnetic field lines due to photospheric motions can generate a periodic shear, while in galactic disks they suggested that spiral density waves (propagating in these disks) can induce a periodic shear directed along the rotational axis. The interaction between the periodic shear and the magnetohydrodynamic waves can induce parametric instability that enhances the angular momentum transport in disks, heating of the solar atmosphere, and acceleration of the solar wind.

In the context of geophysical applications, the stability of time-periodic barotropic and baroclinic shear flows has been addressed in several studies. For instance, in the study of Pedlosky and Thomson [4], the baroclinic instability of the two-layer model on the beta plane in the classical model of Phillips [5] has been extended to include the effects of a time-periodic shear. Poulin [6] has studied stochastic baroclinic shear, in the context of the Phillips model, where the shear is the Kubo oscillator [7], to mimic realistic variations that occur in ocean currents such as the Antarctic circumpolar current. The study of Poulin extends the work of Poulin and Flierl [8], which focused on the stochastic Mathieu equation, into a model relevant to geophysical fluid dynamics. The parametric instability occurring in these shear flows can generate the formation of vortices observed in the atmosphere and in the ocean [9].

As shown in several studies, the use of the simple model, which consists of superimposing a base flow that is linear in the space coordinates and a disturbance flow that consists of plane waves with a time-dependent wave vector, allows one to characterize local instabilities in strained flows (see, e.g., Craik and Criminale [10]), such as the elliptical instability [11,12]. Bayly [12] was the first to realize that unbounded flows with elliptical streamlines and constant vorticity can sustain a Floquet-type instability of certain plane-wave modes, owing to their periodic distortion. Waleffe [13] clarified the physical mechanism of this instability as one of vortex stretching and determined analytically the instability growth rate which tends to a finite value as the wavelength along the vortex axis approaches zero (see Kerswell [14] for a review). This kind of disturbance flow is not usual in conventional hydrodynamic stability analysis, generally devoted to the search for normal modes; the modes of the disturbance flow are given below in Eq. (14) and are discussed at length afterwards. Later studies followed, including an additional Coriolis force [15,16], Coriolis force and density stratification [17], a magnetic field [18], a Coriolis force and magnetic field [19,20], and with time-dependent strains [21,22]. Such a model allows one to include the shearing sheet approximation (see Ref. [23]) used by many authors to study local instabilities in accretion disks such as the magnetorotational instability (MRI) [24,25]. This instability, which is due to the presence of a weak vertical magnetic field, is linear and local in that it does not rely on the presence of boundaries (see Ref. [26] for a review). In this context, Salhi and Cambon [27] considered unbounded stratified shear flows with basic flow combining vertical shear with a constant rate S_0, a vertical Coriolis force with a Coriolis parameter $f = 2\Omega$, and both vertical and horizontal density (or buoyancy) gradients with constant strengths $N_2^2 = -f S_0$ and N_3^2, respectively,

$$\nabla \times \boldsymbol{U} + 2\Omega = (0, S_0, f), \quad \Theta = -f S_0 x_2 + N_3^2 x_3. \tag{1}$$

Therefore, the tendency for the horizontal basic buoyancy gradient to generate streamwise vorticity was balanced by twisting the system vorticity via the $S_0 f$ term (see also Pieri *et al.* [28]). This effect is often called the geostrophic adjustment, or the thermal wind effect, in the geophysical community [29]. A similar basic shear flow configuration was studied by Salhi and Cambon [30]. This corresponds to an unbounded flow subjected to spatial uniform density stratification and shear rate that is time dependent. In addition to vertical stratification with constant strength (N_3^2), the base flow also includes an additional, horizontal, time-dependent density gradient. The exact solution of the Euler-Boussinesq equations found in Ref. [30] is time periodic with period $T = 2\pi/N_3$ for stable vertical stratification ($N_3^2 > 0$),

$$\nabla \times \boldsymbol{U} = (S_0 \cos\tau, 0, 0), \quad \Theta = (N_3 S_0 \cos\tau) x_2 + N_3^2 x_3, \tag{2}$$

where τ is a dimensionless time. Therefore, the time derivative of the vorticity in the spanwise direction plays a similar role in equilibrating the buoyancy term $\nabla \times (\Theta e_3)$.

The Floquet system governing plane waves for the base flow (2) has been computed numerically for particular orientations of the initial wave vector [30]. One aspect of the present paper is to extend the latter study by performing an asymptotic analysis for small $\varepsilon = S_0/N_3$ by considering an arbitrary orientation of the initial wave vector. We derive analytical results characterizing the bandwidth of the instability and its maximal growth rate Δ_m. Similarities with the elliptic instability are discussed. A second aspect of our study is to investigate the effect of a horizontal magnetic field on the gravity shear waves. Lastly, for potential geophysical and astrophysical applications of the present analytical study, we also investigate similarities between the effect of the Coriolis force and the effect of a vertical magnetic field on the gravity shear waves.

The paper is organized as follows. The formulation of the Floquet system is given in Sec. II. In Sec. III, we present an asymptotic analysis of the Floquet system by using Lebovitz and Zweibel's method [18] considering a horizontal magnetic field. Section IV deals with similarities between the the magnetogravity waves and gravity-Coriolis waves when they are excited by a vertical periodic shear. Our concluding remarks are given in Sec. V.

II. FORMULATION

We consider a stratified flow of an inviscid and nondiffusive fluid. The governing equations are the Boussinesq-Euler equations of fluid dynamics in a frame rotating with constant rate Ω about the vertical axis (x_3),

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla \boldsymbol{u} = -\nabla p^* - 2\Omega \boldsymbol{e}_3 \times \boldsymbol{u} + \Theta \boldsymbol{e}_3 + (\nabla \times \boldsymbol{B}) \times \boldsymbol{B}, \tag{3a}$$

$$\partial_t \Theta + \boldsymbol{u}\cdot\nabla\Theta = 0, \tag{3b}$$

and the induction equation

$$\partial_t \boldsymbol{B} + \boldsymbol{u}\cdot\nabla \boldsymbol{B} = \boldsymbol{B}\cdot\nabla \boldsymbol{u}, \tag{3c}$$

with $\nabla\cdot\boldsymbol{u} = 0$ and $\nabla\cdot\boldsymbol{B} = 0$. Here, $\boldsymbol{e}_3$ is a vertical upward unit vector and $\Theta = -(g/\varrho_0)\varrho$ is the buoyancy scalar, ϱ is the fluid density (and ϱ_0 is a reference density), g is the gravity acceleration, and

$$p^* = \frac{p}{\varrho_0} - \frac{1}{2}\Omega^2\left(x_1^2 + x_2^2\right),$$

in which p is the pressure and (x_1, x_2, x_3) is the Cartesian coordinate system referred to as "azimuthal," "radial," and "axial" (or vertical) directions, respectively, in reference to the shearing sheet approximation (see Sec. IV A 3). The magnetic field is rescaled as $\boldsymbol{B} \to (1/\sqrt{\varrho_0 \chi_0})\boldsymbol{B}$, where χ_0 is the magnetic permeability. Recall that the Boussinesq approximation filters out the higher-frequency acoustic waves. This arises from the incompressibility part of the approximation. On other words, in the Boussinesq approximation, density (or temperature) can fluctuate but the velocity field is assumed to be strictly solenoidal (divergence free).

A. Base flow

We briefly review the derivation of the exact solution of the Boussinesq-Euler equations found by Salhi and Cambon [30]. It consists of considering unbounded flows subjected to spatial uniform density stratification and shear rate that are time dependent,

$$\begin{aligned} \boldsymbol{U}(\boldsymbol{x},t) &= S(t) x_2 \boldsymbol{e}_3, \\ \Theta(\boldsymbol{x},t) &= N_1^2(t) x_1 + N_2^2(t) x_2 + N_3^2(t) x_3, \\ \boldsymbol{\Omega} &= \Omega \boldsymbol{e}_3, \end{aligned} \tag{4}$$

where $S(t)$ is the shear rate, $N_i^2(t)$ $(i = 1,2,3)$ is the buoyancy frequency in the x_i direction, and Ω is a constant. For example, if we consider the case without a magnetic field and we apply the curl to Eq. (3a), we obtain the equation for the absolute vorticity $\boldsymbol{w} = \nabla \times \boldsymbol{u} + 2\boldsymbol{\Omega}$, (see, e.g., Ref. [31]),

$$\partial_t \boldsymbol{w} + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{w} = (\boldsymbol{w}\cdot\nabla)\,\boldsymbol{u} + \nabla \times (\Theta \boldsymbol{e}_3). \tag{5}$$

From Eq. (5), we deduce both the absolute vorticity $\boldsymbol{W} = \nabla \times \boldsymbol{U} + 2\boldsymbol{\Omega} = S(t)\boldsymbol{e}_1 + 2\Omega \boldsymbol{e}_3$ and $\nabla \times (\Theta \boldsymbol{e}_3) = -N_2^2(t)\boldsymbol{e}_1 + N_1^2(t)\boldsymbol{e}_2$. Then, the substitution of these forms into Eq. (5) leads to

$$\frac{dS}{dt} = N_2^2(t), \tag{6}$$

and $N_1^2(t) = 0$. This means that the basic buoyancy gradient must be perpendicular to the basic vorticity which splits with the x_1 direction, and hence $\Theta = N_2^2(t) x_2 + N_3^2(t) x_3$. On the other hand, the substitution of the latter form (for the buoyancy scalar) into Eq. (3b) leads to

$$\left(\frac{dN_2^2}{dt} + S(t)N_3^2(t)\right)x_2 + \left(\frac{dN_3^2}{dt}\right)x_3 = 0.$$

Because the latter relation must be valid for any $\|\boldsymbol{x}\|$, we deduce that the vertical buoyancy frequency N_3 must be time independent, while the horizontal one, N_2, satisfies the following differential equation:

$$\frac{dN_2^2}{dt} + S(t)N_3^2 = 0. \tag{7}$$

Consider stable vertical stratification, i.e., $N_3^2 > 0$. Then the solution of the coupled equations (6) and (7) exhibits an oscillatory behavior,

$$\begin{aligned} N_2^2(t) &= N_{20}^2\cos(N_3 t) - A_0 N_3 \sin(N_3 t),\\ S(t) &= A_0\cos(N_3 t) + \frac{N_{20}^2}{N_3}\sin(N_3 t), \end{aligned} \tag{8}$$

where $N_{20}^2 = N_2^2(0)$ and $A_0 = S(0)$. Finally, by setting $S_0\cos\phi_0 = N_{20}^2/N_3$ and $S_0\sin\phi_0 = A_0$ or

$$S_0 = \sqrt{\frac{N_{20}^4}{N_3^2} + A_0^2}, \quad \tan\phi_0 = \frac{A_0 N_3}{N_{20}^2},$$

the solution can be rewritten as

$$\begin{aligned} \boldsymbol{U} &= [0,0,S(\tau)x_2], \quad S(\tau) = N_3\varepsilon\sin\tau,\\ \Theta &= N_2^2 x_2 + N_3^2 x_3 = \left(N_3^2\varepsilon\cos\tau\right)x_2 + N_3^2 x_3, \end{aligned} \tag{9}$$

where $\varepsilon = S_0/N_3 \geqslant 0$ is a small parameter corresponding to the ratio of the amplitude of the oscillating shear to the buoyancy frequency, and $\tau = N_3 t + \phi_0$ is a dimensionless time where ϕ_0 is a constant.

The base flow (9) is not barotropic since the buoyancy gradient $\nabla\Theta$ and the pressure gradient ∇P are not collinear for all times,

$$\nabla P \times \nabla\Theta = (\Theta\boldsymbol{e}_3 - \partial_t\boldsymbol{U}) \times \nabla\Theta = -\left(N_3^4 x_3\varepsilon\cos\tau\right)\boldsymbol{e}_2,$$

and it rather remains similar to a baroclinic flow (see, e.g., Ref. [29]). In fact, at a given time such that $\tau \neq \ell\pi$ ($\ell \in \mathbb{N}$) there are differences between the surfaces of constant scalar buoyancy (or density) and of constant pressure in the fluid.

Consider now the case where a spatial uniform magnetic field $\boldsymbol{B} = (B_1,B_2,B_3)$ is also present. The substitution of the basic velocity form (9) into the induction equation (3c) leads to

$$\partial_\tau B_1 = 0, \quad \partial_\tau B_2 = 0, \quad \partial_\tau B_3 = B_2\varepsilon\sin\tau,$$

or, after integration,

$$B_1 = B_{01}, \quad B_2 = B_{02}, \quad B_3 = B_{03} + B_{02}\varepsilon(1-\cos\tau), \tag{10}$$

where $\boldsymbol{B}_0 = (B_{01},B_{02},B_{03})$ is the magnetic field at $\tau = 0$. Note that the potential vorticity $\Pi_\theta = \boldsymbol{W}\cdot\nabla\Theta$ is no longer a Lagrangian invariant for the magnetohydrodynamic (MHD) Boussinesq-Euler equations, while its counterpart, the magnetic induction potential

$$\Pi_m = \boldsymbol{B}\cdot\nabla\Theta, \tag{11}$$

constitutes a Lagrangian invariant for a nondiffusive fluid (see Ref. [32]). It is easier to verify that, for the fields (9) and (10), Π_m takes the form $\Pi_m = N_3^2(B_{03} + \varepsilon B_{02})$.

B. The perturbed system

Let $\boldsymbol{u}, \boldsymbol{B}, p^*, \Theta$ be replaced by $\boldsymbol{U} + \boldsymbol{u}, \boldsymbol{B} + \boldsymbol{b}, P^* + p, \Theta + \theta$ in Eqs. (3a)–(3c), and linearize. The resulting perturbation equations are

$$\begin{aligned} \dot{u}_i &= -N_3^{-1}\partial_{x_i}p - (\varepsilon\sin\tau)\,u_2\delta_{i3} - 2\,[\Omega\boldsymbol{e}_3\times\boldsymbol{u}]_i\\ &\quad + N_3^{-1}\theta\delta_{i3} + N_3^{-1}\,[(\nabla\times\boldsymbol{b})\times\boldsymbol{B}]_i\,,\\ N_3^{-1}\dot{\theta} &= -(\varepsilon\cos\tau)u_2 - u_3,\\ \dot{b}_i &= N_3^{-1}B_j\partial_{x_j}u_i + (\varepsilon\sin\tau)\,b_i, \end{aligned} \tag{12}$$

with the condition for $\boldsymbol{u}$ and $\boldsymbol{b}$ to be solenoidal, i.e., $\nabla\cdot\boldsymbol{u} = 0$ and $\nabla\cdot\boldsymbol{b} = 0$. Here,

$$\dot{\psi} = \left(\partial_\tau + N_3^{-1}U_j\partial_{x_j}\right)\psi = (\varepsilon\sin\tau)\,x_2\partial_{x_3}\psi$$

denotes the Eulerian derivative. Moreover, the linearized part of the magnetic induction potential perturbation takes the form

$$N_3^{-2}\pi_m = b_2\varepsilon\cos\tau + b_3 + N_3^{-2}B_i\partial_{x_i}\theta. \tag{13}$$

Solutions of the system (12) are sought in terms of Fourier modes with a time-dependent wave vector (see Refs. [15,33])

$$\left[\boldsymbol{u},\boldsymbol{b},p,N_3^{-1}\theta\right] = [\hat{\boldsymbol{u}}(\tau),\hat{\boldsymbol{b}}(\tau),\hat{p}(\tau),\hat{\theta}(\tau)]\exp\imath\,[\boldsymbol{k}(\tau)\cdot\boldsymbol{x}]\,. \tag{14}$$

For instance, the substitution of the above form into the first equation of the perturbed system (12) yields

$$\begin{aligned} \dot{\hat{u}}_i &= -\,N_3^{-1}\imath\left[\dot{k}_j + k_m\left(\partial_{x_j}U_m\right)\right]x_j\hat{u}_i - 2\,[\Omega\boldsymbol{e}_3\times\hat{\boldsymbol{u}}]_i\\ &\quad - (\varepsilon\sin\tau)\,\hat{u}_2\delta_{i3} - \imath N_3^{-1}(\hat{p} + \boldsymbol{B}\cdot\hat{\boldsymbol{b}})k_i\\ &\quad + \hat{\theta}\delta_{i3} + \imath N_3^{-1}(\boldsymbol{k}\cdot\boldsymbol{B})\,\hat{b}_i, \end{aligned}$$

where the terms proportional to x_j must cancel (see, e.g., Refs. [10,12,33,34]) since the above equation must be valid for any $\|\boldsymbol{x}\|$, which is ensured when $(dk_j/dt) = -(\partial_{x_j}U_m)k_m$, or equivalently,

$$\dot{k}_1 = 0, \quad \dot{k}_2 = -(\varepsilon\sin\tau)\,k_3, \quad \dot{k}_3 = 0. \tag{15}$$

The time dependence of the wave vector allows the disturbance Fourier modes to be advected by the mean flow. The equation for the wave vector is called the *eikonal equation* since it is really closed to the one for disturbance flows as wave-packet following rays (see, e.g., Lebovitz and Lifschitz [35]).

Accordingly, system (12) becomes

$$\begin{aligned} \dot{\hat{u}}_i &= -2\,[\Omega\boldsymbol{e}_3\times\hat{\boldsymbol{u}}]_i - (\varepsilon\sin\tau)\,\hat{u}_2\delta_{i3} - \imath N_3^{-1}(\hat{p} + \boldsymbol{B}\cdot\hat{\boldsymbol{b}})k_i\\ &\quad + \hat{\theta}\delta_{i3} + \imath N_3^{-1}(\boldsymbol{k}\cdot\boldsymbol{B})\,\hat{b}_i,\\ \dot{\hat{\theta}} &= -(\varepsilon\cos\tau)\hat{u}_2 - \hat{u}_3,\\ \dot{\hat{b}}_i &= \imath N_3^{-1}(\boldsymbol{B}\cdot\boldsymbol{k})\,\hat{u}_i + (\varepsilon\sin\tau)\,\hat{b}_2. \end{aligned} \tag{16}$$

We also have the conditions $\boldsymbol{k}\cdot\hat{\boldsymbol{u}} = 0$ and $\boldsymbol{k}\cdot\hat{\boldsymbol{b}} = 0$ characterizing the fact that the inviscid perturbations $\boldsymbol{u}$ and $\boldsymbol{b}$ are divergence free.

Equation (15) can be easily solved to give

$$k_1 = K_1, \quad k_3 = K_3, \quad k_2 = (K_2 - \varepsilon k_3) + k_3\varepsilon\cos\tau, \tag{17}$$

where $\boldsymbol{K} = (K_1, K_2, K_3)$ is the wave vector at $\tau = 0$. As it can be remarked, the evolution of the wave vector $\boldsymbol{k}$ is similar to that of the magnetic field [see Eq. (10)] but with axes (x_2, x_3) interchanged so the inner product $\boldsymbol{B}\cdot\boldsymbol{k}$ is time independent, i.e.,

$$N_3\eta_k = \boldsymbol{B}\cdot\boldsymbol{k} = \boldsymbol{B}_0\cdot\boldsymbol{K} = B_{01}K_1 + B_{02}K_2 + B_{03}K_3. \quad (18)$$

On the other hand, the use of the relation $k_i\hat{u}_i = 0$ allows us to eliminate the pressure coefficient in the equation for $\hat{\boldsymbol{u}}$,

$$-\imath N_3^{-1}(\hat{p} + \boldsymbol{B}\cdot\hat{\boldsymbol{b}}) = k^{-2}k_3[(2\varepsilon\sin\tau)\,\hat{u}_2 - \hat{\theta}] + R_0^{-1}k^{-2}[k_2\hat{u}_1 - k_1\hat{u}_2], \quad (19)$$

where $R_0 = N_3/(2\Omega)$ is the Rossby number. Accordingly, system (17) can be rewritten as follows:

$$\begin{aligned}
\dot{\hat{u}}_1 &= R_0^{-1}\frac{k_1k_3}{k^2}\hat{u}_1 + \left[2\varepsilon\frac{k_1k_3}{k^2}\sin\tau + R_0^{-1}\frac{(k_2^2+k_3^2)}{k^2}\right]\hat{u}_2 + \imath\eta_k\hat{b}_1 - \frac{k_1k_3}{k^2}\hat{\theta},\\
\dot{\hat{u}}_2 &= -R_0^{-1}\frac{(k_1^2+k_3^2)}{k^2}\hat{u}_1 + \left[2\varepsilon\frac{k_2k_3}{k^2}\sin\tau - R_0^{-1}\frac{k_1k_2}{k^2}\right]\hat{u}_2 + \imath\eta_k\hat{b}_2 - \frac{k_2k_3}{k^2}\hat{\theta},\\
\dot{\hat{u}}_3 &= R_0^{-1}\frac{k_2k_3}{k^2}\hat{u}_1 - \left[\left(1 - 2\frac{k_3^2}{k^2}\right)(\varepsilon\sin\tau) + R_0^{-1}\frac{k_1k_3}{k^2}\right]\hat{u}_2 + \imath\eta_k\hat{b}_3 + \frac{(k_1^2+k_2^2)}{k^2}\hat{\theta}, \quad (20)\\
\dot{\hat{b}}_1 &= \imath\eta_k\hat{u}_1,\\
\dot{\hat{b}}_2 &= \imath\eta_k\hat{u}_2,\\
\dot{\hat{b}}_3 &= \imath\eta_k\hat{u}_3 + (\varepsilon\sin\tau)\,\hat{b}_2,\\
\dot{\hat{\theta}} &= -(\varepsilon\cos\tau)\,\hat{u}_2 - \hat{u}_3,
\end{aligned}$$

with $k_i\hat{u}_i = 0$ and $k_i\hat{b}_i = 0$. By setting $\varepsilon = 0$ in Eq. (20), one recovers the differential system governing the linear dynamic of magnetogravity Coriolis waves.

III. ANALYSIS OF THE MAGNETOGRAVITY SHEAR WAVES

In this section, we consider the stability of magnetogravity shear waves in a nonrotating frame, so that, $\Omega = 0$, or equivalently, $R_0 \to \infty$.

A. Preliminary analysis

The stability problem of magnetogravity waves is described by the following differential system deduced from (20):

$$\begin{aligned}
\dot{\hat{u}}_2 &= \left(2\varepsilon\frac{k_2k_3}{k^2}\sin\tau\right)\hat{u}_2 + \imath\eta_k\hat{b}_2 - \frac{k_2k_3}{k^2}\hat{\theta},\\
\dot{\hat{u}}_3 &= -\left(1 - 2\frac{k_3^2}{k^2}\right)(\varepsilon\sin\tau)\,\hat{u}_2 + \imath\eta_k\hat{b}_3 + \frac{(k_1^2+k_2^2)}{k^2}\hat{\theta},\\
\dot{\hat{b}}_2 &= \imath\eta_k\hat{u}_2,\\
\dot{\hat{b}}_3 &= \imath\eta_k\hat{u}_3 + (\varepsilon\sin\tau)\,\hat{b}_2,\\
\dot{\hat{\theta}} &= -(\varepsilon\cos\tau)\,\hat{u}_2 - \hat{u}_3. \quad (21)
\end{aligned}$$

In fact, when $k_1 \neq 0$, the conditions $k_i\hat{u}_i = 0$ and $k_i\hat{b}_i = 0$ allow us to find the two coefficients $\hat{u}_1$ and $\hat{b}_1$ once the above differential system has been solved. When $k_1 = 0$, the differential system (21) reduces to a three-dimensional differential system (see Sec. IV), while the evolution of the Fourier coefficients $\hat{u}_1$ and $\hat{b}_1$ is described by the following harmonic oscillator:

$$\dot{\hat{u}}_1 = \imath\eta_k\hat{b}_1, \quad \dot{\hat{b}}_1 = \imath\eta_k\hat{u}_1. \quad (22)$$

Therefore, the stability problem can be characterized by the analysis of the differential system (21) either $k_1 = 0$ or not.

On the other hand, the special case $k_3 = 0$, so that the wave vector $\boldsymbol{k} = (k_1, K_2, 0)$ is no longer time dependent, allows a simpler analytical treatment, as a forced oscillator. One easily shows that, in the presence of a horizontal magnetic field, there is a nonresonant forcing, while in the case of a vertical magnetic field or in the case without a magnetic field, the oscillator is forced by a resonant term. Indeed, when $k_3 = 0$, an alternative formulation of system (21) yields the following second-order differential equations,

$$\ddot{\hat{u}}_2 + \eta_k^2\hat{u}_2 = 0, \quad (23a)$$

$$\ddot{\hat{u}}_3 + \left(1 + \eta_k^2\right)\hat{u}_3 = -2\varepsilon\hat{u}_2\cos\tau, \quad (23b)$$

where

$$\eta_k = N_3^{-1}(B_{01}k_1 + B_{02}K_2). \quad (24)$$

Therefore, when $\eta_k \neq 0$, we have bounded solutions that characterize a nonresonance forcing,

$$\begin{aligned}
\hat{u}_2(\tau) &= A_1\cos(\eta_k\tau - \phi_1),\\
\hat{u}_3(\tau) &= A_2\cos\left(\sqrt{1+\eta_k^2}\,\tau - \phi_2\right) - \frac{\varepsilon A_1}{2\eta_k}\{\cos[(\eta_k+1)\tau - \phi_1] - \cos[(\eta_k - 1)\tau - \phi_1]\}. \quad (25)
\end{aligned}$$

Here, A_1, A_2, ϕ_1, and ϕ_2 are constants that are determined from the initial conditions. In contrast, when $\eta_k = 0$, so that $\boldsymbol{B}_0 = \boldsymbol{0}$ or $\boldsymbol{B}_0 = B_{03}\boldsymbol{e}_3$, the solution is not bounded (due to the secular term),

$$\begin{aligned}
\hat{u}_2 &= \text{const},\\
\hat{u}_3(\tau) &= A_2\cos(\tau - \phi_2) - \hat{u}_2\varepsilon\tau\sin\tau.
\end{aligned} \quad (26)$$

B. Reduced Floquet system

The coefficient $\hat{\pi}_m = \pi_m\exp(-\imath\boldsymbol{k}\cdot\boldsymbol{x})$ associated with the magnetic induction potential [see Eq. (13)] takes the form

$$N_3^{-2}\hat{\pi}_m = (\varepsilon\cos\tau)\,\hat{b}_2 + \hat{b}_3 + \imath\eta_k\hat{\theta} = \text{const}, \quad (27)$$

and constitutes a Lagrangian invariant for a nondiffusive fluid, as already indicated. Then, it is possible to express $\hat{\theta}$ in terms of $\hat{b}_2$,$\hat{b}_3$ and the constant,

$$\theta = \imath\eta_k^{-1}\left(N_3^{-2}\hat{\pi}_m + \hat{b}_2\varepsilon\cos\tau + \hat{b}_3\right),$$

reducing the number of dependent variables for the disturbance flow, and hence system (21) reduces to a four-dimensional inhomogeneous differential system. Without loss of generality from the point of view of a stability analysis, we consider that $\hat{\pi}_m$ is zero, so that

$$\hat{\theta} = \imath\eta_k^{-1}(\hat{b}_2\varepsilon\cos\tau + \hat{b}_3), \tag{28}$$

and hence we obtain a homogeneous differential (Floquet) system for $(\hat{u}_2,\hat{u}_3,\hat{b}_2,\hat{b}_3)$. In fact, one easily verifies that the stability problem is the same considering either $\hat{\pi}_m = 0$ or $\hat{\pi}_m = \text{const} \neq 0$. Therefore, by setting

$$(c_1, c_2, c_3, c_4) = \left(\hat{u}_2, \hat{u}_3, \imath\eta_k^{-1}\hat{b}_2, \imath\eta_k^{-1}\hat{b}_3\right), \tag{29}$$

the substitution of the form (28) into the system (21) gives rise to the following homogeneous four-dimensional system,

$$\dot{\boldsymbol{c}} = \mathbf{D}(\tau)\boldsymbol{\cdot}\boldsymbol{c}, \tag{30}$$

where

$$\mathbf{D} = \begin{pmatrix} 2\varepsilon\frac{k_2k_3}{k^2}\sin\tau & 0 & \eta_k^2 - \varepsilon\frac{k_2k_3}{k^2}\cos\tau & -\frac{k_2k_3}{k^2} \\ -\varepsilon\left(1-2\frac{k_3^2}{k^2}\right)\sin\tau & 0 & \varepsilon\frac{k_\perp^2}{k^2}\cos\tau & \eta_k^2 + \frac{k_\perp^2}{k^2} \\ -1 & 0 & 0 & 0 \\ 0 & -1 & \varepsilon\sin\tau & 0 \end{pmatrix}, \tag{31}$$

in which $k_\perp^2 = k_1^2 + k_2^2$ is the square of the horizontal wave number. The fact that the parameter $\eta_k = N_3^{-1}(\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{k})$ appearing in the expression of the matrix $\mathbf{D}(\tau)$ depends on the modulus of the initial wave vector signifies that the stability problem also depends on the wavelength of the perturbations.

Though not always analytically solvable, the temporal behavior of $\boldsymbol{c}(\tau)$ may be characterized by using the standard Floquet theory (see Ref. [36]) since the matrix $\mathbf{D}$ is time periodic with period 2π. The general solution is a linear superposition of Floquet modes of the form $\boldsymbol{c}(\tau) = f(\tau)\exp(\sigma t)$, where $f(t)$ is periodic with period 2π. The Floquet exponent σ is determined by the requirement that $\exp(2\pi\sigma)$ is an eigenvalue of the Floquet multiplier matrix $\mathbf{M} = \Phi(2\pi)$, where $\Phi(\tau)$ is the fundamental matrix solution of Eq. (30) that reduces to identity at $\tau = 0$,

$$\dot{\Phi} = \mathbf{D}(\tau)\boldsymbol{\cdot}\Phi, \quad \Phi(0) = \mathbf{I_4},$$

where $\mathbf{I}_4$ is the 4×4 unit matrix. The fact that the determinant of the matrix Φ is a solution of the following first-order differential equation (see, e.g., Salhi and Cambon [37]),

$$\frac{d}{d\tau}\det\Phi = (\operatorname{trace}\mathbf{D})\det\Phi,$$

where

$$\operatorname{trace}\mathbf{D} = D_{ii} = 2\varepsilon\frac{k_2k_3}{k^2}\sin\tau = -2\frac{\dot{k}}{k}, \tag{32}$$

implies that

$$\det\Phi(\tau) = \frac{K^2}{k^2},$$

and hence $\det\mathbf{M}[=\det\Phi(2\pi)]$ is unity. It follows that the system (30) is conservative and possesses the property that if λ is an eigenvalue of $\mathbf{M}$, so also is its inverse λ^{-1} and its complex conjugate $\overline{\lambda}$ (see, e.g., Ref. [19]). The proof of this fact is the same as given by Lebovitz and Zweibel [18].[1] It follows that, in the stable case, the eigenvalues of $\mathbf{M}$ lie on the unit circle, while, if as parameters change (see below), an eigenvalue is at the onset of instability, it must have a multiplicity of two (or higher). Thus, a necessary condition for the onset of linear instability is a resonance where two Floquet multipliers coincide (see Ref. [18]).

[1]One can exactly follow the proof by Ref. [18] considering the matrix $\mathbf{R} = \operatorname{diag}(1,1,-1,-1)$ that verifies $\mathbf{RD}(-\tau) = -\mathbf{D}(\tau)\mathbf{R}$ and $\mathbf{R}^{-1}\mathbf{D}(\tau) = \overline{\mathbf{D}}(\tau)\mathbf{R}$.

C. Magnetogravity waves

When there is no shear, i.e., $\varepsilon = 0$, the buoyancy gradient in the transverse direction vanishes. If we put $\varepsilon = 0$, the coefficient matrix $\mathbf{D}_0$ (say) of Eq. (31) becomes constant,

$$\mathbf{D}_0 = \begin{pmatrix} 0 & 0 & \eta_k^2 & -\frac{K_2K_3}{K^2} \\ 0 & 0 & 0 & \left(\eta_k^2 + \frac{K_\perp^2}{K^2}\right) \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{33}$$

Its eigenvalues are

$$\sigma_1 = \imath\eta_k, \quad \sigma_2 = \imath\sqrt{\eta_k^2 + \frac{K_\perp^2}{K^2}}, \qquad \sigma_3 = -\imath\eta_k, \quad \sigma_4 = -\imath\sqrt{\eta_k^2 + \frac{K_\perp^2}{K^2}}. \tag{34}$$

They are distinct and nonzero as long as $\eta_k \neq 0$. Associated eigenvectors are reported in Appendix 2 for the sake of clarity. The frequencies (normalized by N_3) characterizing the magnetogravity (or Alfvén-Archimede) waves are

$$\omega_1 = \eta_k, \quad \omega_2 = \sqrt{\eta_k^2 + \frac{K_\perp^2}{K^2}}, \qquad \omega_3 = -\eta_k, \quad \omega_4 = -\sqrt{\eta_k^2 + \frac{K_\perp^2}{K^2}}. \tag{35}$$

The pair (ω_1,ω_3) characterizes the Alfvén waves, while the pair (ω_2,ω_4) characterizes the Alfvén-Archimede waves. Since there are two restoring forces acting on displaced fluid elements, there are two possible situations. The Lorentz and Archimede forces may act together, stiffening the system and producing the higher-frequency fast waves (i.e., the Alfvén-Archimede waves), or the Lorentz force acts as if it was alone to produce the lower-frequency slow waves (i.e., the Alfvén waves).

In the case of magnetocoriolis waves, the Coriolis and Lorentz forces act together, producing both fast and slow magnetocoriolis waves, as recently observed in the Taylor-Couette flow laboratory experiment by Nornberg *et al.* [38]. In that study, it has been demonstrated through a local stability analysis that with the addition of sufficient flow shear, the slow magnetocoriolis wave can be destabilized to produce the magnetorotational instability.

D. The resonant cases

Assume that $k_3 \neq 0$ and use the spherical coordinates (K,θ,φ) where

$$\mu \equiv \sin\theta = \frac{K_\perp}{K}, \quad \cos\varphi = \frac{k_1}{K_\perp}. \tag{36}$$

Then

$$k_1 = K\mu\cos\varphi, \quad K_2 = K\mu\sin\varphi, \quad k_3 = \pm K\sqrt{1-\mu^2}.$$

In order to use the asymptotic method by Lebovitz and Zweibel [18] (see Appendix), we assume that each eigenvalue σ_k of $\mathbf{D}_0$ is linear in μ. This can be achieved by considering that the initial magnetic field lies in the (x_1,x_2) plane, i.e., $\boldsymbol{B}_0 = B_{01}\boldsymbol{e}_1 + B_{02}\boldsymbol{e}_2$. The case where the initial magnetic field is vertical (i.e., $\boldsymbol{B}_0 = B_{03}\boldsymbol{e}_3$) is examined in Sec. IV.

Therefore, the expression of the four frequencies becomes

$$\omega_{1,3} = \pm\mu\eta_\varphi, \quad \omega_{2,4} = \pm\mu\sqrt{1+\eta_\varphi^2}, \tag{37}$$

in which the coefficient

$$\eta_\varphi = N_3^{-1}K\,(B_{01}\cos\varphi + B_{02}\sin\varphi) \tag{38}$$

is assumed to be positive or zero, without loss of generality.

As in the case of the elliptic instability [18,19], the destabilization of Floquet modes is a result of a special type of resonance between them (including resonances between their frequencies and the basic frequency, i.e., the buoyancy frequency for the problem considered here). For $\varepsilon = 0$, i.e., when the magnetogravity waves are not sheared, these resonances are defined by $\omega_i - \omega_j = n$, with $n \in \mathbb{N}$ and $i,j = 1,2,3,4$. As shown in Appendix A [see Eqs. (A17) and (A8)], at leading order in ε, instability associated with the parametric resonances can occur when $\omega_i - \omega_j = \pm 1$ $(i \neq j = 1,2,3,4)$. This is a consequence of the fact that the time-dependent elements of the matrix $\mathbf{D}_\varepsilon$ involved in the expansion of the matrix $\mathbf{D}$ in a Taylor series around $\varepsilon = 0$,

$$\mathbf{D}(\tau,\varepsilon,\mu,\eta_\varphi) = \mathbf{D}_0(\mu,\eta_\varphi) + \varepsilon\mathbf{D}_\varepsilon(\tau,\mu,\eta_\varphi) + \cdots,$$

behave as $\exp\pm(\imath N_3 t)$ [see Eq. (A8)]. Also, in the case of precessing sheared flow, instability associated with the parametric resonances can occur when $\omega_i - \omega_j = \pm 1$ since the time-dependent elements of $\mathbf{D}_\varepsilon$ behave as $\exp\pm(\imath\Omega_0 t)$, where Ω_0 is the solid body rotation rate (see Refs. [39,40]). In comparison, in the case of elliptical streamline flows, instability associated with the parametric resonances can occur when $\omega_i - \omega_j = \pm 2$ since the elements of $\mathbf{D}_\varepsilon$ behave as $\exp\pm(\imath 2\Omega_0 t)$ (see Refs. [14,18,19]).

Because the interchange $\varphi \to \varphi + \pi$ and $\theta \to \pi - \theta$ leads to the same set of frequencies $\{\omega_1,\omega_2,\omega_3,\omega_4\}$, we may assume without loss of generality that $0 \leqslant \varphi \leqslant \pi$ and $0 \leqslant \theta \leqslant \frac{\pi}{2}$, and hence we only need to consider the following three resonant cases:

$$\begin{aligned} \omega_1 - \omega_3 &= 2\eta_\varphi\mu = 1, \\ \omega_2 - \omega_3 &= \left(\eta_\varphi + \sqrt{1+\eta_\varphi^2}\right)\mu = 1, \\ \omega_2 - \omega_4 &= 2\mu\sqrt{1+\eta_\varphi^2} = 1. \end{aligned} \tag{39}$$

For each one of the above three cases we infer instability if $\mathrm{Re}\,\alpha \neq 0$, where α represents the roots of the following equation:

$$\alpha^2 - \left(\tilde{J}_{ii} + \tilde{J}_{jj} + \frac{2\pi\imath}{\mu}\nu(\omega_i+\omega_j)\right)\alpha + \begin{vmatrix} \tilde{J}_{ii} + \frac{2\pi\imath\nu\omega_i}{\mu} & \tilde{J}_{ij} \\ \tilde{J}_{ji} & \tilde{J}_{jj} + \frac{2\pi\imath\nu\omega_j}{\mu} \end{vmatrix} = 0. \tag{40}$$

where the elements $\tilde{J}_{ij}$ are defined by Eq. (A15). The coefficient ν characterizes the width of the unstable region at order $O(\varepsilon)$. In other words, the region in the (μ,ε) plane where instability occurs is typically a wedge with the apex at the point $(\mu,\varepsilon) = (\mu_0,0)$ and boundaries $\mu = \mu_0 + \nu_\pm\varepsilon$, where the slopes $\nu+$ and ν_- are to be found (see Fig. 1).

E. Case with $\omega_2 - \omega_4 = 1$: Hydrodynamic modes

When there is no magnetic field, so that $\eta_\varphi = 0$, both frequencies $\omega_{1,3}$ become zero, while the other two frequencies reduce to $\omega_2 = -\omega_4 = \mu$, characterizing purely gravity modes. They are modified by the presence of the magnetic field, $\omega_2 = -\omega_4 = \mu\sqrt{1+\eta_\varphi^2}$. Then, the fact that $\omega_2 - \omega_4 = 1$ implies $\omega_2 = -\omega_4 = \frac{1}{2}$, and hence

$$\mu = \mu_0 = \frac{1}{2\sqrt{1+\eta_\varphi^2}}. \tag{41}$$

Therefore, μ decreases as $|\eta_\varphi|$ increases, $\mu = 0.5$ for $\eta_\varphi = 0$, in agreement with the numerical results given in Ref. [30], and $\mu \to 0$ as $|\eta_\varphi| \to \infty$.

Recall that the stability of the case without a magnetic field has been addressed in Salhi and Cambon [30], who determined numerically the boundaries of the unstable bands considering disturbances with $K_2 = 0$ [i.e., the initial wave vector lies in the (x_1,x_3) plane]. The following asymptotic analysis clearly shows that both the width and the maximal growth rate of the instability bands depend on the orientation of the initial wave vector.

The determination of α described by Eq. (40) needs the expression of $\tilde{J}_{22}, \tilde{J}_{44}, \tilde{J}_{24}$, and $\tilde{J}_{42}$ that is given in Appendix 4

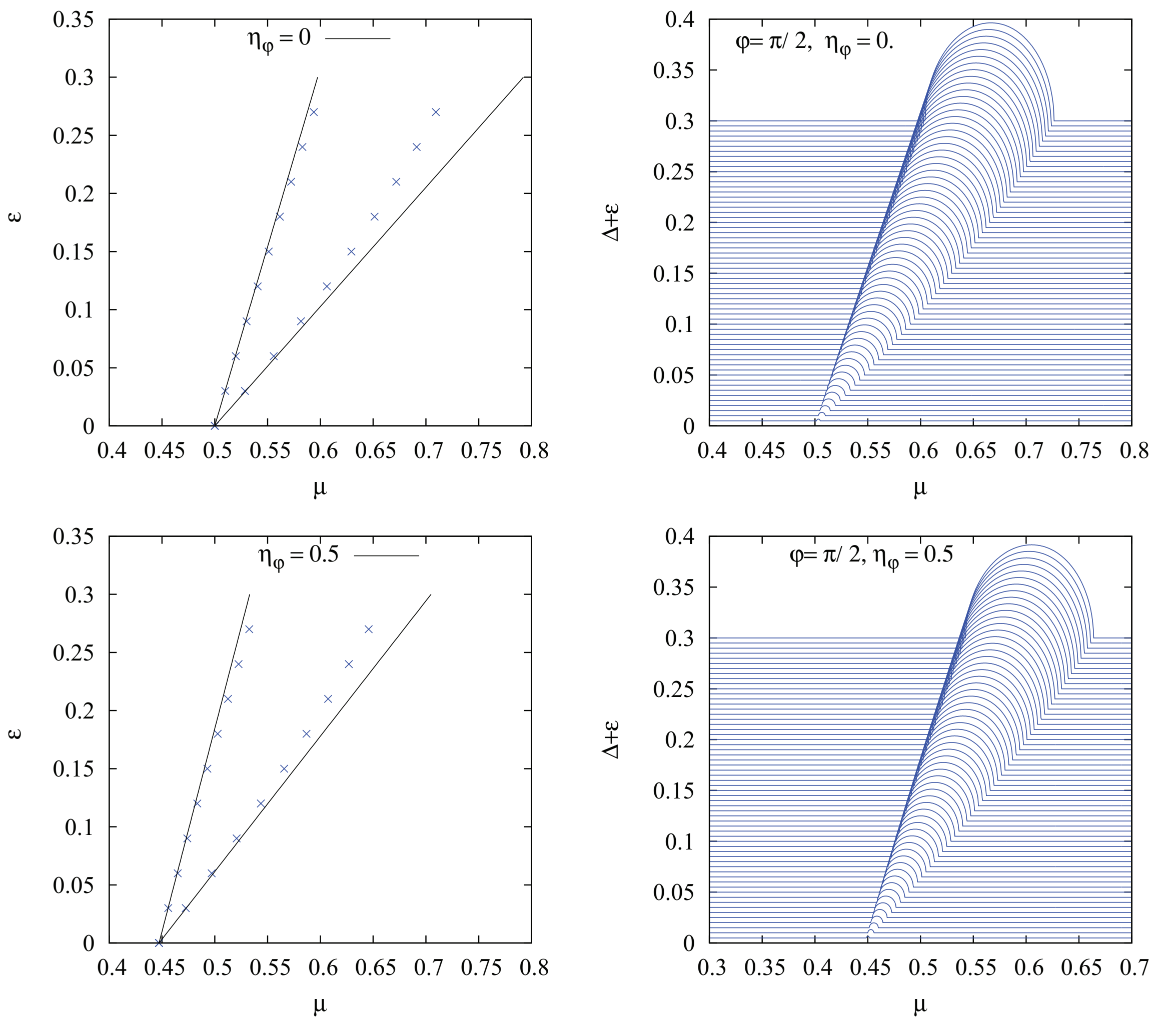

FIG. 1. (Color online) Hydrodynamic mode. The boundaries of the instability band in the (μ,ε) plane. The instability band emanates from the point $[\mu = 1/(2\sqrt{1+\eta_\varphi^2}),0]$. Top left panel: Pure hydrodynamic mode ($\varphi = \pi/2$ and $\eta_\varphi = 0$). Bottom left panel: The effect of a radial magnetic field with $\varphi = \pi/2$ and $\eta_\varphi = N_3^{-1} K B_{02} \sin\varphi = 0.50$ on the hydrodynamic mode. The analytical results given by Eq. (45) are represented by lines, while the numerical ones are represented by symbols. The right panels depict the numerical results giving, for fixed ε, the variation of the $\Delta + \varepsilon$ vs μ, where $\Delta = \mathrm{Re}(\alpha)$ is the growth rate.

[see Eqs. (A28) and (A29)],

$$\tilde{J}_{22} = -\tilde{J}_{44} = -2\pi\imath\,(\sin\varphi)\,\frac{(1-\eta^2)^{\frac{3}{2}}}{\sqrt{1+\eta_\varphi^2}},$$

$$\tilde{J}_{24}\tilde{J}_{42} = -\frac{\pi^2\mu^2(1-\mu^2)\sin^2\varphi}{4}\left[2 + \frac{(1-2\mu^2)}{\sqrt{1+\eta_\varphi^2}}\right]^2, \quad (42)$$

with $\tilde{J}_{24} = -\tilde{J}_{42}$. The substitution of these forms into Eq. (40) yields

$$\alpha^2 = \frac{\pi^2(1-\mu^2)\sin^2\varphi}{4\mu^2}\left[2\mu^2 + \frac{\mu(1-2\mu^2)}{\sqrt{1+\eta_\varphi^2}}\right]^2 - 4\pi^2\left[\nu\sqrt{1+\eta_\varphi^2} - \frac{(1-\mu^2)^{\frac{3}{2}}}{\sqrt{1+\eta_\varphi^2}}\sin\varphi\right]^2. \quad (43)$$

This has a maximum instability increment when

$$\nu = \frac{(1-\mu^2)^{\frac{3}{2}}}{\left(1+\eta_\varphi^2\right)^2}\sin\varphi,$$

and hence the substitution of the form (41) into the first term on the right-hand side of (43) leads to

$$\frac{\Delta_m}{\varepsilon} = \frac{(\mathrm{Re}\,\alpha)_{\max}}{2\pi} = \frac{3\sqrt{3}}{16}\,\frac{\left(1+\frac{4}{3}\eta_\varphi^2\right)^{\frac{3}{2}}}{\left(1+\eta_\varphi^2\right)^2}\sin\varphi. \quad (44)$$

In the pure hydrodynamic limit, so that $\eta_\varphi = 0$, this reduces to $\Delta_m/\varepsilon = (3\sqrt{3}/16)\sin\varphi$, and for a given $\sin\varphi \neq 0$, it decreases as η_φ increases, approaching zero as $\eta_\varphi \to \infty$.

The instability associated with the hydrodynamic modes has a bandwidth $(\nu_+ - \nu_-)\varepsilon$ that is, for given ε and η_φ, the length of the μ interval for which the unperturbed configuration is unstable (see Lebovitz and Zweibel [18]). Here, $\nu_\pm$ are the roots of the algebraic equation $\mathrm{Re}\,\alpha = 0$.

From Eq. (43), one derives the expression of $\nu_\pm$,

$$\nu_+ = 3\nu_- = \frac{9\sqrt{3}}{16}\frac{\left(1+\frac{4}{3}\eta_\varphi^2\right)^{\frac{3}{2}}}{\left(1+\eta_\varphi^2\right)^{\frac{5}{2}}}\sin\varphi, \tag{45}$$

Therefore, in the (μ,ε) plane, the hydrodynamic instability band is not symmetrical with respect to the axis $\varepsilon = 1/(2\sqrt{1+\eta_\varphi^2})$. For a given $\sin\varphi \neq 0$, the width of this instability band,

$$\delta = (\nu_+ - \nu_-)\,\varepsilon = \frac{3\sqrt{3}}{8}\varepsilon\frac{\left(1+\frac{4}{3}\eta_\varphi^2\right)^{\frac{3}{2}}}{\left(1+\eta_\varphi^2\right)^{\frac{5}{2}}}\sin\varphi,$$

decreases as η_φ increases, $\delta/\varepsilon = 3\sqrt{3}/8$ at $\eta_\varphi = 0$ (i.e., the pure hydrodynamic limit) and $\delta/\varepsilon \to 0$ as $\eta_\varphi \to \infty$.

Figure 1 shows the boundaries of the hydrodynamic subharmonic instability band in the (μ,ε) plane for $\varphi = \pi/2$. Both cases without a magnetic field (i.e., $\eta_\varphi = 0$) and cases with a radial magnetic field (i.e., $\eta_\varphi = N_3^{-1}KB_{02}\sin\varphi = 0.5$) have been considered. The numerical results reported in the figure were obtained by solving numerically the Floquet system (30) over one period $0 \leqslant \tau \leqslant 2\pi$ (using the fourth-order Runge-Kutta method) to compute $\mathbf{M}(2\pi)$ and to determine its eigenvalues (using the factorization for generalized eigenvalues, QZ, method) Λ_i $(i = 1,2,3,4)$, and hence the growth rate Δ_i,

$$\Delta_i = \frac{1}{2\pi}\log(\Lambda_i) \quad (i = 1,2,3,4).$$

As can be expected, for sufficiently small values of the parameter ε, there is an expected agreement, while for the large values of ε the asymptotic formula is less accurate in reproducing the numerical results (the relative error does not exceed 8%).

Under the same conditions as Fig. 1, Fig. 2 shows the numerical results giving the variation of Δ_m/ε vs ε. As can be seen, Δ_m/ε decreases as ε increases, but the decrease is more significant in the case without a magnetic field than in the case with the radial magnetic field. In the limit $\varepsilon \to 0^+$, the asymptotic formula [Eq. (44)] reproduces well the numerical results: $\Delta_m/\varepsilon \approx 0.3248$ for $\varphi = \pi/2$ and $\eta_\varphi = 0$, and $\Delta_m/\varepsilon = 0.32$ for $\varphi = \pi/2$ and $\eta_\varphi = 0.50$.

When the initial wave vector $\boldsymbol{K}$ lies in the (x_1,x_3) plane, so that $\sin\varphi = 0$, Eqs. (44) and (45) indicate that both Δ_m/ε and δ are zero. This signifies that, when $\varphi = 0$, there is no instability at the order $O(\varepsilon)$ emanating from the point $[\mu = 1/(2\sqrt{1+\eta_\varphi^2}),0]$. However, computations performed for $\varphi = 0$ and an azimuthal magnetic field with strength $\eta_\varphi = N_3^{-1}KB_{01}\cos\varphi = 0.5$ indicate that there exists a region of instability emanating from $\mu = 1/(2\sqrt{1+\eta_\varphi^2}) \approx 0.4472$ (see Fig. 3). This instability is of width $O(\varepsilon^n)$ with $n \geqslant 2$.

On the other hand, we note that, according to the present asymptotic analysis, there is no instability of order $O(\varepsilon)$ associated with the resonance $\omega_2 - \omega_4 = 2$. Indeed, in that case, both $\tilde{J}_{24}$ and $\tilde{J}_{42}$ are zero (see Appendix 4), while $\tilde{J}_{22}$ and $\tilde{J}_{44}$ are given by Eq. (42). Accordingly, the quadratic equation (40) reduces to

$$\alpha^2 = -4\pi^2\left[\nu\sqrt{1+\eta_\varphi^2} - \frac{(1-\mu^2)^{\frac{3}{2}}}{\sqrt{1+\eta_\varphi^2}}\sin\varphi\right]^2, \tag{46}$$

indicating that there is no instability [at the order $O(\varepsilon)$] associated with the resonance $\omega_2 - \omega_4 = 2$.

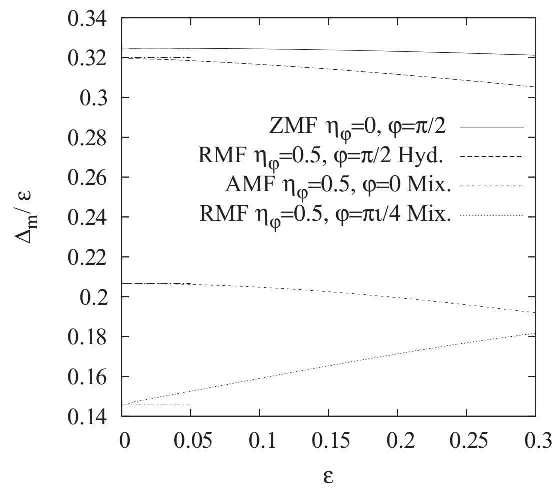

FIG. 2. Numerical results giving the variation of Δ_m/ε vs ε for the instability associated with the hydrodynamic mode (Hyd.) or mixed mode (Mix.). ZMF: Zero magnetic field. RMF or AMF: Radial or azimuthal magnetic field. In the limit $\varepsilon \to 0^+$, the numerical results agree with the asymptotic results [see Eqs. (44) and (51)] that are represented by horizontal segments (at $\Delta_m/\varepsilon \approx 0.3248, 0.32, 0.2067, 0.1461$).

F. Case with $\omega_2 - \omega_3 = 1$: Mixed modes

Without a magnetic field, so that $\eta_\varphi = 0$, the frequency $\omega_2 = \mu\sqrt{1+\eta_\varphi^2}$ reduces to $\omega_2 = \mu$, while the frequency $\omega_3 = -\mu\eta_\varphi$ vanishes. Therefore these modes consist of a hydrodynamic mode and a purely magnetic mode. When $\omega_2 - \omega_3 = 1$, the third relation in Eq. (39) implies that

$$\mu = \mu_0 = \frac{1}{\sqrt{1+\eta_\varphi^2}+\eta_\varphi} = \sqrt{1+\eta_\varphi^2} - \eta_\varphi. \tag{47}$$

Therefore μ changes from 1 at $\eta_\varphi = 0$ to 0 as $\eta_\varphi \to \infty$.

The evaluation of the elements $\tilde{J}_{22}$, $\tilde{J}_{33}$, $\tilde{J}_{23}$ and $\tilde{J}_{32}$ is reported in Appendix [see Eqs. (A27), (A28), (A36), and (A37)]. The result is

$$\begin{aligned}\tilde{J}_{22} &= -2\pi\imath\,(\sin\varphi)\frac{(1-\eta^2)^{\frac{3}{2}}}{\sqrt{1+\eta_\varphi^2}},\\ \tilde{J}_{23} &= -\pi\eta_\varphi\mu,\\ \tilde{J}_{32} &= -\frac{\pi\mu\cos^2\varphi}{\sqrt{1+\eta_\varphi^2}},\end{aligned} \tag{48}$$

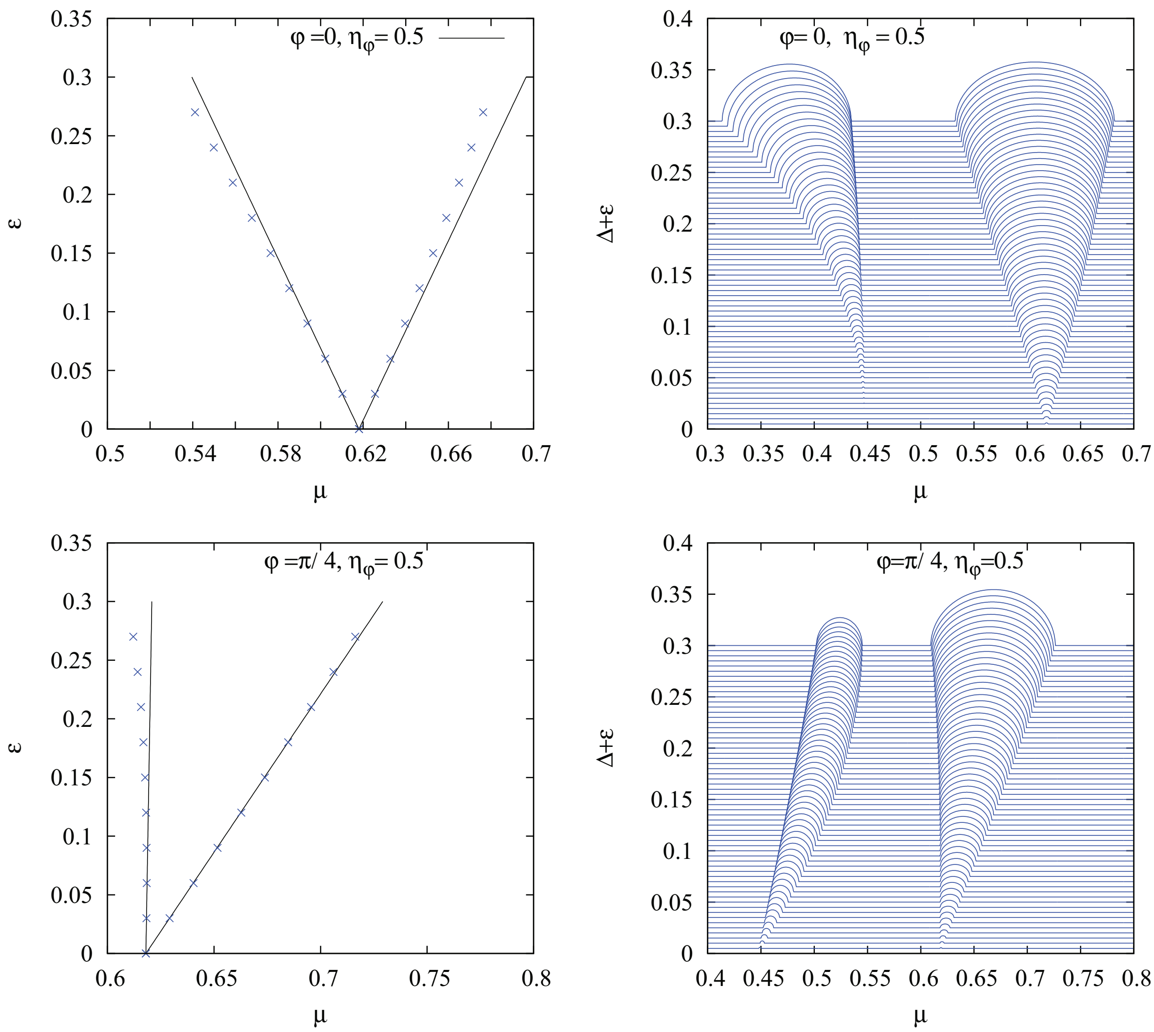

FIG. 3. (Color online) Mixed mode. The boundaries of the instability band in the (μ,ε) plane. For $\eta_\varphi = 0.5$, the instability emanates from the point $(\mu = \sqrt{1+\eta_\varphi^2} - \eta_\varphi \approx 0.62,0)$. Top left panel: Case of an azimuthal magnetic field with $\varphi = 0$ and $\eta_\varphi = N_3^{-1} K B_{01} \cos\varphi = 0.5$. Bottom left panel: Case of a radial magnetic field with $\varphi = \pi/4$ and $\eta_\varphi = N_3^{-1} K B_{02} \sin\varphi = 0.5$. The analytical results given by Eq. (52) are represented by lines while the numerical results are represented by symbols. The right panels show the numerical results giving, for fixed ε, the variation of the $\Delta + \varepsilon$ vs μ. The instability regions emanating from the point $(\mu = 0.447,0)$ are associated with the hydrodynamic mode. The one appearing in the bottom right panel is of order $O(\varepsilon)$, while the one appearing in the top right panel is of order $O(\varepsilon^2)$ and it cannot be captured by the present asymptotic analysis.

while $\tilde{J}_{33} = 0$, as indicated previously. Substituting the above forms into Eq. (40) and remarking that

$$1 - \mu^2 = 2\eta_\varphi \mu, \quad \mu^2 - 4\eta_\varphi \sqrt{1+\eta_\varphi^2} = \mu^{-2},$$

we find

$$\alpha = \imath\pi(\mu\nu - \mu^{-1}\nu_m) \pm \sqrt{D}, \tag{49}$$

where

$$D = \pi^2 \left[\frac{\mu^2 \eta_\varphi \cos^2\varphi}{\sqrt{1+\eta_\varphi^2}} - \mu^{-2}(\nu - \nu_m)^2 \right], \qquad \nu_m = \frac{\mu(1-\mu^2)^{\frac{3}{2}}}{\sqrt{1+\eta_\varphi^2}} \sin\varphi. \tag{50}$$

Therefore, instability prevails if and only if $D > 0$. In that case, the instability can indeed occur and has its maximal growth rate when $\nu = \nu_m$. Then

$$\frac{\Delta_m}{\varepsilon} \equiv \frac{(\mathrm{Re}\,\alpha)_{\max}}{2\pi} = \frac{\left(\sqrt{1+\eta_\varphi^2} - \eta_\varphi\right)\sqrt{\eta_\varphi}}{2\left(1+\eta_\varphi^2\right)^{\frac{1}{4}}} |\cos\varphi|. \tag{51}$$

The variation of Δ_m vs η_φ is maximal $\Delta_m/\varepsilon = 0.2071$ at $\eta_\varphi = 0.4551$. For this reason, the value $\eta_\varphi = 0.5$ has been chosen for the illustration of both the analytical and numerical results. At the limit $\varepsilon \to 0^+$, the numerical results for Δ_m/ε agree well with the asymptotic formula (51) characterizing the maximal growth rate for the subharmonic instability associated with the mixed modes (see Fig. 2).

The upper and lower edges of the band of subharmonic instability associated with the mixed modes are expressed by the formula $\mu = \mu_0 \pm \nu_\pm \varepsilon$, where $\nu_\pm$ may be determined from

Eq. (50) by requiring D to vanish (a consequence of the fact that $\mathrm{Re}\,\alpha = 0$). We find

$$\nu_\pm = \left[\frac{\mu(1-\mu^2)^{\frac{3}{2}}}{\sqrt{1+\eta_\varphi^2}}\sin\varphi\right] \pm \left[\frac{\mu^2\sqrt{\eta_\varphi}}{(1+\eta_\varphi^2)^{\frac{1}{4}}}\cos\varphi\right], \quad (52)$$

and hence the bandwidth is

$$\frac{\delta}{\varepsilon} = \nu_+ - \nu_- = \frac{\left(\sqrt{1+\eta_\varphi^2}-\eta_\varphi\right)}{\left(\sqrt{1+\eta_\varphi^2}+\eta_\varphi\right)}\frac{\sqrt{\eta_\varphi}}{\left(1+\eta_\varphi^2\right)^{\frac{1}{4}}}|\cos\varphi|.$$

For fixed $\cos\varphi \neq 0$, both Δ_m/ε and δ/ε are zero at $\eta_\varphi = 0$, increase for $0 < \eta_\varphi < (\eta_\varphi)_m$, reach the maximal value at $(\eta_\varphi)_m$, and decrease for $(\eta_\varphi)_m < \eta_\varphi$, approaching zero as $\eta_\varphi \to \infty$. Here, $(\eta_\varphi)_m \approx 0.455$ when considering the variation of Δ_m/ε vs η_φ and $(\eta_\varphi)_m \approx 0.243$ when considering the variation of δ/ε vs η_φ.

Figure 3 shows the boundaries of the subharmonic instability band associated with the mixed mode in the (ε,μ) plane for an azimuthal magnetic field with $\varphi = 0$ and $\eta_\varphi = 0.5$, and a radial magnetic field with $\varphi = \pi/4$ and $\eta_\varphi = 0.5$. As can be seen, there is an expected agreement between the asymptotic formula [see Eq. (52)] and the numerical results.

Finally, we now show that there is no instability associated with the resonant case with $\omega_1 - \omega_3 = 2\mu\eta_\varphi = 1$, so that $\mu = \mu_0 = 1/(2\eta_\varphi)$. This resonant case characterizes the purely magnetic modes since they vanish when there is no a magnetic field. Because $\tilde{J}_{11} = \tilde{J}_{13} = \tilde{J}_{33} = 0$ [see Eqs. (A27) and (A40) in Appendix 4], the quadratic equation (40) reduces to

$$\alpha^2 = -4\pi^2\nu^2\eta_\varphi^2,$$

signifying that $\mathrm{Re}\,\alpha = 0$, and hence there is no instability associated with this resonance.

IV. SIMILARITIES BETWEEN MAGNETOGRAVITY AND GRAVITY-CORIOLIS WAVES

In this section, we attempt to point out similarities between the response of the magnetogravity waves and the gravity-Coriolis waves to a vertical periodic shear considering axisymmetric disturbances (i.e., those corresponding to an infinite wavelength in the azimuthal direction, $k_1 = 0$, or equivalently, the $m = 0$ mode). Both the magnetic field and the Coriolis force are vertical.

A. Hill's equation

1. Magnetogravity waves

As indicated at the end of Sec. II B, when $k_1 = 0$, the time evolution of the coefficients $\hat{u}_1$ and $\hat{b}_1$ is described by the autonomous differential system (22) for which there is no instability. Because, at $k_1 = 0$, the solenoidal condition implies

$$c_2(\tau) = -\frac{k_2}{k_3}c_1(\tau), \quad c_4(\tau) = -\frac{k_2}{k_3}c_3(\tau),$$

provided $k_3 \neq 0$, the system (30) reduces to a two-dimensional one,

$$\dot{c}_1 = \left(2\varepsilon\frac{k_2k_3}{k^2}\sin\tau\right)c_1 + \left(\eta_k^2 + \frac{k_2^2}{k^2} - \varepsilon\frac{k_2k_3}{k^2}\sin\tau\right)c_3, \quad (53)$$
$$\dot{c}_3 = -c_1,$$

where $k^2 = k_2^2(\tau) + k_3^2$ and

$$\eta_k = N_3^{-1}B_{03}k_3 \quad (54)$$

or $\eta_k = N_3^{-1}(B_{02}K_2 + B_{03}k_3)$ if one considers arbitrary orientation of the magnetic field. From system (53), we deduce the following second-order differential equation,

$$\ddot{c}_3 - \left(2\varepsilon\frac{k_2k_3}{k^2}\sin\tau\right)\dot{c}_3 + \left(\eta_k^2 + \frac{k_2^2}{k^2} - \varepsilon\frac{k_2k_3}{k^2}\cos\tau\right)c_3 = 0, \quad (55)$$

and by setting $\psi = [k(\tau)/K]c_3(\tau)$, we transform it as follows:

$$\ddot{\psi} + \underbrace{\left[\eta_k^2 + \frac{k_2^2}{k^2} - \varepsilon^2\frac{k_3^4}{k^4}\sin^2\tau\right]}_{V(\tau)}\psi = 0, \quad (56)$$

The latter equation has the form of a Hill equation (see Ref. [41]) since the potential $V(\tau)$ is periodic.

2. Gravity-Coriolis waves

The stability problem of the gravity-Coriolis shear waves is obtained by setting $\eta_k = 0$ (i.e., there is no magnetic field) in system (20),

$$\dot{\hat{u}}_1 = R_0^{-1}\frac{k_1k_3}{k^2}\hat{u}_1 + \left[2\varepsilon\frac{k_1k_3}{k^2}\sin\tau + R_0^{-1}\frac{\left(k_2^2+k_3^2\right)}{k^2}\right]\hat{u}_2 - \frac{k_1k_3}{k^2}\hat{\theta},$$
$$\dot{\hat{u}}_2 = -R_0^{-1}\frac{\left(k_1^2+k_3^2\right)}{k^2}\hat{u}_1 + \left[2\varepsilon\frac{k_2k_3}{k^2}\sin\tau - R_0^{-1}\frac{k_1k_2}{k^2}\right]\hat{u}_2 - \frac{k_2k_3}{k^2}\hat{\theta},$$
$$\dot{\hat{u}}_3 = R_0^{-1}\frac{k_2k_3}{k^2}\hat{u}_1 - \left[\left(1-2\frac{k_3^2}{k^2}\right)(\varepsilon\sin\tau) + R_0^{-1}\frac{k_1k_3}{k^2}\right]\hat{u}_2 + \frac{\left(k_1^2+k_2^2\right)}{k^2}\hat{\theta},$$
$$\dot{\hat{\theta}} = -(\varepsilon\cos\tau)\hat{u}_2 - \hat{u}_3, \quad (57)$$

with $k_i\hat{u}_i = 0$. As indicated at the beginning of Sec. IV, we consider axisymmetric disturbances (i.e., $k_1 = 0$), so that $k_2\hat{u}_2 + k_3\hat{u}_3 = 0$. Accordingly, system (57) reduces to

$$\dot{\hat{u}}_1 = R_0^{-1}\hat{u}_2,$$
$$\dot{\hat{u}}_2 = -R_0^{-1}\frac{k_3^2}{k^2}\hat{u}_1 - (2\varepsilon\sin\tau)\frac{k_2k_3}{k^2}\hat{u}_2 - \frac{k_2k_3}{k^2}\hat{\theta},$$
$$\dot{\hat{\theta}} = -\left[(\varepsilon\cos\tau) - \frac{k_2}{k_3}\right]\hat{u}_2. \quad (58)$$

In view of Eq. (32),

$$-(2\varepsilon\sin\tau)\frac{k_2k_3}{k^2}=2\frac{k\dot{k}}{k^2},$$

we set

$$c_1=\hat{u}_1,\quad c_2=\frac{k^2}{K^2}\hat{u}_2,\quad c_3=\hat{\theta}.$$

Then the latter differential system can be rewritten as

$$\begin{aligned}\dot{c}_1&=R_0^{-1}\frac{K^2}{k^2}c_2,\\ \dot{c}_2&=-R_0^{-1}\frac{k_2^2}{K^2}c_1-\frac{k_3k_2}{K^2}c_3,\\ \dot{c}_3&=\frac{(K_2-\varepsilon k_3)}{k_3}\frac{K^2}{k^2}c_2,\end{aligned}\tag{59}$$

from which one easily deduce the following constant of motion,

$$\frac{\hat{\pi}_\theta}{\imath N_3^2}=\frac{2\Omega}{N_3}k_3c_3-(K_2-\varepsilon k_3)\,c_1=\text{const.}\tag{60}$$

The latter relation is a consequence of the fact that the potential vorticity

$$\pi_\theta=(\nabla\theta)\cdot(\nabla\times\boldsymbol{u}+2\boldsymbol{\Omega})$$

is a Lagrangian invariant for a nondiffusive fluid (see Ref. [42]). As in Sec. II, without loss of generality from the point of view of a stability analysis, we take $\pi_\theta=0$. Accordingly, we deduce from system (59) the following second-order differential equation for c_3,

$$\begin{aligned}\ddot{c}_3&-\left(2\varepsilon\frac{k_2k_3}{k^2}\sin\tau\right)\dot{c}_3\\ &+\left(R_0^{-2}\frac{k_3^2}{k^2}+\frac{k_2^2}{k^2}-\varepsilon\frac{k_2k_3}{k^2}\cos\tau\right)c_3=0,\end{aligned}\tag{61}$$

where R_0 is the Rossby number

$$R_0=\frac{N_3}{2\Omega}.\tag{62}$$

By setting $\eta_k=R_0^{-1}(k_3/k)$ and $\psi=[k(\tau)/K]c_3(\tau)$, Eq. (61) can be transformed to the Hill equation (56).

3. *Stratified accretion disks with vertical periodic shear*

Zaqarashvili *et al.* [2] suggested that, in galactic disks, a periodic shear along the rotational axis can be generated by the spiral density waves propagating in these disks. In that study, the response of the magnetoacoustic waves to a vertical periodic shear has been addressed.

Here we briefly consider a vertically stratified nonmagnetized accretion disk under a vertical periodic shear and we use the shearing sheet approximation (see, e.g., Refs. [23,34]) with a coordinate system that is standard for shear flow that differs from the shearing box (SB) convention such that $x_2=x_{\rm SB}$ (radial direction), $x_1=-y_{\rm SB}$ (azimuthal direction), and $x_3=z$ (vertical direction). It should be remarked that the shearing sheet approximation contains most of the physics that is relevant to phenomena occurring on scales of an order smaller than the disk thickness [34]. In this coordinate system, the base flow takes the form

$$\begin{aligned}&\boldsymbol{U}=\mathbf{S}\cdot\boldsymbol{x},\quad \mathbf{S}=\begin{pmatrix}0&S&0\\0&0&0\\0&\varepsilon N_3\sin\tau&0\end{pmatrix},\\ &\boldsymbol{W}=\nabla\times\boldsymbol{U}+2\boldsymbol{\Omega}=(\varepsilon N_3\sin\tau,0,-S+2\Omega)\,,\\ &\Theta=\left(N_3^2\varepsilon\cos\tau\right)x_2+N_3^2x_3,\end{aligned}\tag{63}$$

where S and Ω are constants. The flow of Eq. (63) is an exact solution of the Boussinesq-Euler equations.

Accordingly, we show that axisymmetric plane waves (superimposed to the latter base flow) are also governed by Eq. (56) provided

$$\eta_k=\frac{\omega}{N_3}\frac{k_3}{k},\quad \omega^2=2\Omega\,(2\Omega-S)\,,\tag{64}$$

where ω is the epicyclic frequency, and $\omega=\Omega$ for a Keplerian disk. The stability of the base flow (63) will be addressed in more detail in a subsequent paper.

B. Mathieu's equation

When the parameter ε is sufficiently small $\varepsilon\ll1$, Eq. (56) may be expressed, within a good approximation, by a truncated power series in ε,

$$\begin{aligned}k_2^2(\tau)&=K_2^2-2\varepsilon K_2k_3\,(1-\cos\tau)+O(\varepsilon^2),\\ k^{-2}(\tau)&=K^{-2}[1+2\varepsilon K^{-2}K_2k_3(1-\cos\tau)]+O(\varepsilon^2).\end{aligned}$$

If the terms of $O(\varepsilon^2)$ are neglected, this reduces to Mathieu's equation,

$$\ddot{\psi}+\left[\eta_k^2+\frac{K_2^2}{\left(K_2^2+k_3^2\right)}+2\varepsilon\frac{K_2k_3^3}{\left(K_2^2+k_3^2\right)^2}(\cos\tau-1)\right]\psi=0.\tag{65}$$

The solutions of the Mathieu equation are generally bounded except in the vicinity of resonances defined by

$$\omega_n^2=\eta_k^2+\frac{K_2^2}{\left(K_2^2+k_3^2\right)}=\frac{n^2}{4}\quad(n=1,2,3,4,\ldots),\tag{66}$$

where the solutions are exponentially growing with a growth rate of order ε^n (see, e.g., Ref. [43]).

For convenience, we introduce the polar coordinates $(\sqrt{K_2^2+k_3^2},\vartheta)$ in the (x_2,x_3) plane such that $\vartheta=\arctan(K_3/k_2)$, and we set

$$\begin{aligned}\mu&=\cos\vartheta=K_2\left(K_2^2+k_3^2\right)^{-\frac{1}{2}},\\ \eta_0&=N_3^{-1}KB_{03}\quad\text{or}\quad\eta_0=2\Omega N_3^{-1}.\end{aligned}\tag{67}$$

Then, for the vertical magnetic field or the vertical Coriolis force, one has $\eta_k^2=\eta_0^2(1-\mu^2)$, and hence Eq. (66) with $n=1$ can be rewritten as

$$\left(\tfrac{1}{2}-\mu\right)\left(\tfrac{1}{2}+\mu\right)=\eta_0^2\,(1-\mu)\,(1+\mu)\,.\tag{68}$$

Because the latter equation is invariant under the interchange $\mu\to-\mu$, we only consider $0\leqslant\mu\leqslant1$. In the (μ,ε) plane, the point lying in the μ axis from which emanates the subharmonic

($n = 1$) instability (if attainable) is characterized by

$$0 \leqslant \mu(\eta_0) \leqslant \tfrac{1}{2},$$

with the maximal growth rate

$$\Delta_m = \varepsilon\mu(1-\mu^2)^{\frac{3}{2}}. \tag{69}$$

The function $\Delta_m(\mu)$ is maximal at $\mu = \frac{1}{2}$ and takes a zero value at $\mu = 0$. When $\mu = 0$, Eq. (68) reduces to $4\eta_0^2 = 1$. This signifies that the subharmonic instability region is attainable only if

$$k_3 < \frac{N_3}{2B_{03}} \quad \text{or equivalently} \quad L_3 > \frac{N_3}{2B_{03}} \tag{70}$$

when considering the magnetogravity shear waves or if $R_0 > 2$ when considering the gravity-Coriolis shear waves. Here, L_3 ($k_3L_3 \sim 1$) is a characteristic vertical length scale.

In the (μ,ε) plane, the subharmonic instability (if attainable) emanates from the point (μ_0,0) such that

$$\mu = \mu_0 = \frac{1}{2}\frac{\sqrt{1-4\eta_0^2}}{\sqrt{1-\eta_0^2}}. \tag{71}$$

The substitution of the latter form into (69) allows us to express Δ_m as a function of η_0,

$$\frac{\Delta_m}{\varepsilon} = \frac{3\sqrt{3}}{16}\frac{\left(1-4\eta_0^2\right)^{\frac{1}{2}}}{\left(1-\eta_0^2\right)^2}. \tag{72}$$

Therefore, we may conclude that, when the strength of the magnetic field k_3B_{03} (or the Coriolis parameter 2Ω) exceeds $N_3/2$, the subharmonic instability is inhibited.

On the other hand, we remark that when the initial magnetic field is radial, so that

$$\eta_k = N_3^{-1}K_2B_{02} = \underbrace{\left(N_3^{-1}B_{02}\sqrt{K_2^2+k_3^2}\right)}_{\eta_0}\mu,$$

we deduce from Eq. (66) the point (μ_0,0) of the (μ,ε) plane from which emanates the subharmonic instability,

$$\mu = \mu_0 = \frac{1}{2}\frac{1}{\sqrt{1+\eta_0^2}},$$

with the maximal growth rate

$$\frac{\Delta_m}{\varepsilon} = \frac{3\sqrt{3}}{16}\frac{\left(1+\frac{4}{3}\eta_0^2\right)^{\frac{3}{2}}}{\left(1+\eta_0^2\right)^2}.$$

These expressions can also be recovered by setting $\varphi = \pi/2$ in relations (41) and (44) obtained by the asymptotic method of Lebovitz and Zweibell.

V. CONCLUDING REMARKS

In a nondiffusive unbounded (vertically) stratified fluid, gravity waves propagate with frequency $\omega_g = N_3K_\perp/K$, where $K_\perp$ is the horizontal wave number and N_3 is the Brunt-Väisälä frequency. When the fluid is a conductor and in the presence of a uniform magnetic field, $\sqrt{\rho_0\chi_0}\boldsymbol{B}_0$, there are slow magnetogravity waves propagating with the Alfvén frequency $\omega_a = \boldsymbol{B}_0\cdot\boldsymbol{K}$, and fast magnetogravity waves propagating with the Alfvén-Archimede frequency $\omega_{ag} = \sqrt{\omega_a^2+\omega_g^2}$. Because the Boussinesq approximation filters out the high-frequency acoustic waves, then $\omega_{ag} \ll c_sK$, where c_s is the sound speed. On the other hand, when $\omega_a \ll \omega_g$, one has $\omega_{ag} = \omega_g[1+\omega_a^2/(2\omega_g^2)+\cdots] \approx \omega_g$. This can occur at large scales (i.e., $K \ll 1$), while at small scales (i.e., $K \gg 1$) one has $\omega_{ag} \approx \omega_a$.

We have performed a linear stability analysis in terms of the base-flow-advected Fourier mode of the case where these waves are excited by a time-periodic shear along the vertical (x_3) axis and varying linearly with the radial (x_2) coordinate, and having a frequency $\omega_f = N_3$ and amplitude ε. This idealized base flow serves as a local model to study the angular moment transport in more complex flows arising in astrophysical systems (galactic disks, solar atmosphere, etc.).

Owing to the fact that the magnetic induction potential is a Lagrangian invariant, it has been shown that planes waves are governed by a four-dimensional (MHD) Floquet system. The method by Lebovitz and Zweibel [18] has been adopted to analyze the stability of the Floquet system at the order $O(\varepsilon)$. It has been shown that some modes can display a local three-dimensional instability.

Without a magnetic field, the resonant modes of the gravity waves that are candidates to generate a subharmonic instability when the vertical periodic shear is present are the modes for which $\omega_g = \omega_f/2$, or equivalently, $\mu = K_\perp/K = 1/2$. The occurrence of subharmonic instability depends on the orientation of the horizontal wave vector $\boldsymbol{K}_\perp$ since the maximal growth rate Δ_m is found to be $\Delta_m/\varepsilon = (3\sqrt{3}/16)|K_2/K_\perp|$. This implies that Δ_m/ε is maximal for axisymmetric plane-wave disturbances (i.e., $K_2/K_\perp = \pm 1$) and vanishes for the $K_2 = 0$ mode. Note that computations indicate that the resonant mode $\mu = 1/2$ induces an instability even if $K_2 = 0$. Such an instability is at the order $O(\varepsilon^n)$ ($n \geqslant 2$) since it is not captured by the present asymptotic analysis which is valid at the order $O(\varepsilon)$. It should be instructive to mention here that, in the case of unbounded flow with elliptical streamlines, it is found that $\Delta_m/\varepsilon = 9/16$ in which ε is the departure of streamlines of the unperturbed flow from axial symmetry (see Ref. [13]), while for precessing sheared flows, it is found that $\Delta_m/\varepsilon = 5\sqrt{15}/32$ in which ε is the precessing parameter (see Refs. [40,44,45]).

In the presence of a uniform horizontal magnetic field, the resonant modes that can induce subharmonic instabilities are the modes for which $\omega_{ag} = \omega_f/2$ (i.e., hydrodynamic modes) or ($\omega_{ag}+\omega_a) = \omega_f$ (i.e., mixed modes), while the resonant modes for which $\omega_a = \omega_f/2$ (i.e., magnetic modes) do not induce any instability. The magnetic field exerts a stabilizing effect by decreasing the values of the maximal growth rate of the subharmonic instability associated with the hydrodynamic modes [see Eq. (44)], as in magnetoelliptic instability studies (see Refs. [18,19]). It is also found that the subharmonic instability associated with the mixed modes vanishes for axisymmetric disturbances (i.e., $K_2/K_\perp = \pm 1$), while the subharmonic instability associated with the hydrodynamic modes vanishes for $K_2 = 0$, as already indicated. For $\varepsilon \leqslant 0.25$, the asymptotic analysis results giving the boundaries of the subharmonic instability are in agreement with computations.

Due to the complexity of an analytical treatment of the subharmonic instabilities that can arise in the magnetogravity-Coriolis waves under the vertical time-periodic shear, we have addressed separately the effect of a vertical magnetic field and the effect of a vertical Coriolis force on the gravity shear waves considering axisymmetric disturbances. It has been shown that, for both cases, plane waves are governed by a Hill equation, and, when ε is sufficiently small, the subharmonic instability band is determined by a Mathieu equation. When the Coriolis parameter 2Ω (or the magnetic strength $B_0 K$) exceeds half of the vertical Brunt-Väisälä frequency, N_3 the subharmonic instability vanishes.

APPENDIX: LEBOVITZ AND ZWEIBEL'S METHOD

Here we report the calculations leading to the quadratic equation (40) from which one can characterize the stability of the three subharmonic resonances: the hydrodynamic modes ($\omega_2 - \omega_4 = 1$), the mixed modes ($\omega_2 - \omega_3 = 1$), and the magnetic modes ($\omega_1 - \omega_3 = 1$). These calculations are similar to the asymptotic calculations (for $\varepsilon \ll 1$) done by Lebovitz and Zweibel [18] in their study of magnetoelliptic instabilities (see also Refs. [19,39]). For the elliptical flow case, ε represents the departure of the streamlines of the basic flow from axial symmetry, as indicated previously. The asymptotic analysis of the Floquet system (30) for sufficiently small ε proceeds in two steps: finding the Floquet multiplier matrix $\mathbf{M}$ and calculating its eigenvalues. Let us also recall that $\mathbf{M} = \Phi(2\pi,\varepsilon,\mu_0,\eta_\varphi,\varphi)$, where $\Phi(\tau,\varepsilon,\mu_0,\eta_\varphi,\varphi)$ is the fundamental solution of the Floquet system (30), μ_0 and φ are defined by Eq. (36), while η_φ is given by Eq. (38). In the following, both the parameters η_φ and φ will be held fixed so, to simplify the notation, we suppress the dependence of the matrices on these two parameters.

1. Construction of the Floquet multiplier matrix M at the first order of ε

We expand $\mathbf{M}$ in a Taylor series around $\varepsilon = 0$ and $\mu = \mu_0$, for some wedge apex $(\mu_0,0)$ in the (μ,ε) plane,

$$\mathbf{M}(\varepsilon,\mu) = \mathbf{M}(0,\mu_0) + \varepsilon \mathbf{M}_\varepsilon(0,\mu_0) + (\mu - \mu_0)\,\mathbf{M}_\mu(0,\mu_0) + O[\varepsilon^2, (\mu-\mu_0)^2], \tag{A1}$$

and we assume that

$$\mu(\varepsilon) = \mu_0 + \varepsilon\nu + O(\varepsilon^2), \tag{A2}$$

which signifies that the direction of propagation of a mode also changes when it becomes unstable, but we are including only the thickest wedges, whose thickness is of order ε in the present analysis (see also Ref. [19]). Here,

$$\mathbf{M}_\varepsilon = \frac{\partial}{\partial\varepsilon}\mathbf{M}(0,\mu_0), \quad \mathbf{M}_\mu = \frac{\partial}{\partial\mu}\mathbf{M}(0,\mu_0).$$

Accordingly, Eq. (A1) is rewritten as

$$\mathbf{M}(\varepsilon,\mu) = \mathbf{M}_0 + \varepsilon\mathbf{M}_1 + O(\varepsilon^2), \tag{A3}$$

where

$$\mathbf{M}_0 = \mathbf{M}(0,\mu_0), \quad \mathbf{M}_1 = \mathbf{M}_\varepsilon(0,\mu_0) + \nu\mathbf{M}_\mu(0,\mu_0). \tag{A4}$$

The determination of the matrices $\mathbf{M}_0$ and $\mathbf{M}_1$ requires the expansion of the matrix $\mathbf{D}(\tau)$ of the Floquet system (30) in a Taylor series around $\varepsilon = 0$ at fixed μ,

$$\mathbf{D}(\tau,\varepsilon,\mu) = \mathbf{D}_0(\mu) + \varepsilon\mathbf{D}_\varepsilon(\tau,\mu) + O(\varepsilon^2), \tag{A5}$$

where $\mathbf{D}_0$ does not depend on the dimensionless time τ,

$$\mathbf{D}_0 = \begin{pmatrix} 0 & 0 & \mu^2\eta_\varphi^2 & -\mu\sqrt{1-\mu^2}\sin\varphi \\ 0 & 0 & 0 & \mu^2\left(1+\eta_\varphi^2\right) \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tag{A6}$$

and its eigenvalues are of the form

$$\sigma_{1,3} = \imath\omega_{1,3} = \pm\imath\mu\eta_\varphi, \quad \sigma_{2,4} = \imath\omega_{2,4} = \pm\imath\mu\sqrt{1+\eta_\varphi^2}. \tag{A7}$$

We assume now $\mu \neq 0$ and $\eta_\varphi \neq 0$, and then the eigenvalues are distinct and nonzero. Recall that the first two correspond to magnetic modes since they are zero when $\eta_\varphi = 0$, while the second two refer to hydrodynamic modes since they reduce to the eigenvalues of the purely hydrodynamic case in the limit $\eta_\varphi = 0$. Regarding the matrix $\mathbf{D}_\varepsilon$, which represents the derivative of the matrix $\mathbf{D}$ with respect to the parameter ε, one can easily work it out from Eq. (31). We find that nonzero elements are of the form

$$\begin{aligned}
(D_\varepsilon)_{11} &= -\imath\mu\sqrt{1-\mu^2}(\sin\varphi)[\exp(\imath\tau) - \exp(\imath\tau)], \\
(D_\varepsilon)_{13} &= -\frac{1}{2}\mu\sqrt{1-\mu^2}(\sin\varphi)[\exp(\imath\tau) + \exp(\imath\tau)], \\
(D_\varepsilon)_{14} &= (\mu^2 - 1)(1-\mu^2)[\exp(\imath\tau) + \exp(\imath\tau) - 2], \\
(D_\varepsilon)_{21} &= -\imath 2(1-2\mu^2)[\exp(\imath\tau) - \exp(\imath\tau)], \\
(D_\varepsilon)_{23} &= \frac{1}{2}\mu^2[\exp(\imath\tau) + \exp(\imath\tau)], \\
(D_\varepsilon)_{24} &= \mu(1-\mu^2)^{\frac{3}{2}}(\sin\varphi)[\exp(\imath\tau) + \exp(\imath\tau) - 2], \\
(D_\varepsilon)_{43} &= -\frac{\imath}{2}[\exp(\imath\tau) - \exp(\imath\tau)].
\end{aligned} \tag{A8}$$

From the matrices $\mathbf{D}_0$ and $\mathbf{D}_\varepsilon$ we may now construct the matrices $\mathbf{M}_0$ and $\mathbf{M}_\varepsilon$ needed in Eq. (A4). Indeed, for fixed τ in $[0,2\pi]$ and μ, we expand the fundamental matrix Φ around $\varepsilon = 0$,

$$\Phi(\tau,\varepsilon,\mu) = \Phi_0(\tau,\mu) + \varepsilon\Phi_1(\tau,\mu) + O(\varepsilon^2), \quad \Phi_1(0,\mu) = \mathbf{0},$$

and we substitute the latter form into Eq. (30), or equivalently, $\dot{\Phi} = \mathbf{D}\cdot\Phi$, we obtain the following system:

$$\begin{aligned}
\dot{\Phi}_0 &= \mathbf{D}_0\cdot\Phi_0, \\
\dot{\Phi}_1 &= \mathbf{D}_0\cdot\Phi_1 + \mathbf{D}_\varepsilon\cdot\Phi_0.
\end{aligned} \tag{A9}$$

The integration of the first relation yields

$$\Phi_0(\tau,\mu) = \exp(\tau \mathbf{D}_0), \quad \mathbf{M}_0(\mu) = \exp(2\pi \mathbf{D}_0), \tag{A10}$$

and hence the use of the variation of constants formula allows us to determine the solution for Φ_1,

$$\begin{aligned} \mathbf{M}_\varepsilon(\mu) &= \Phi_1(2\pi,\mu) = \mathbf{M}_0(\mu)\mathbf{J}(\mu), \\ \mathbf{J}(\mu) &= \int_0^{2\pi} \Phi_0^{-1}(s,\mu)\mathbf{D}_\varepsilon(s,\mu)\Phi_0(s,\mu)ds. \end{aligned} \tag{A11}$$

We next proceed to simplify this expression by working in the base, diagonalizing the matrix $\mathbf{D}_0$ since the characteristic polynomial of the Floquet multiplier matrix $\mathbf{M}$ is the same in any coordinate system.

2. Calculations in the base diagonalizing $\mathbf{D}_0$

Because the eigenvalues $\{\sigma_\ell\}$ $(\ell = 1,2,3,4)$ of the matrix $\mathbf{D}_0$ are all distinct [provided $\eta_\varphi \neq 0$ and $\mu \neq 0$, see Eq. (A7)], the eigenvectors are linearly independent and the matrix $\mathbf{T}(\mu)$ formed from their columns diagonalizes $\mathbf{D}_0$,

$$\begin{aligned} \tilde{\mathbf{D}}_0 &= \mathbf{T}^{-1}\mathbf{D}_0\mathbf{T} = \mathrm{diag}\,(\sigma_1,\sigma_2,\sigma_3,\sigma_4) \\ &= (\imath\,\mu)\,\mathrm{diag}\Big(\eta_\varphi,\sqrt{1+\eta_\varphi^2},-\eta_\varphi,-\sqrt{1+\eta_\varphi^2}\Big), \end{aligned} \tag{A12}$$

where the columns of $\mathbf{T}$ are the eigenvectors of $\mathbf{D}_0$ expressed in the old base, i.e.,

$$\mathbf{T} = \begin{pmatrix} \imath\mu\eta_\varphi & \mu\sqrt{1-\mu^2}(\sin\varphi) & -\imath\mu\eta_\varphi & \mu\sqrt{1-\mu^2}(\sin\varphi) \\ 0 & -\mu^2 & 0 & -\mu^2 \\ -1 & \imath\frac{\sqrt{1-\mu^2}(\sin\varphi)}{\sqrt{1+\eta_\varphi^2}} & -1 & -\imath\frac{\sqrt{1-\mu^2}(\sin\varphi)}{\sqrt{1+\eta_\varphi^2}} \\ 0 & -\frac{\imath\mu}{\sqrt{1+\eta_\varphi^2}} & 0 & \frac{\imath\mu}{\sqrt{1+\eta_\varphi^2}} \end{pmatrix} \tag{A13}$$

and

$$\mathbf{T}^{-1} = \frac{1}{2}\begin{pmatrix} -\frac{\imath}{\mu\eta_\varphi} & -\frac{\imath(\sin\varphi)\sqrt{1-\mu^2}}{\mu^2\eta_\varphi} & -1 & -\frac{(\sin\varphi)\sqrt{1-\mu^2}}{\mu} \\ 0 & -\frac{1}{\mu^2} & 0 & \frac{\imath\sqrt{1+\eta_\varphi^2}}{\mu} \\ \frac{\imath}{\mu\eta_\varphi} & \frac{\imath(\sin\varphi)\sqrt{1-\mu^2}}{\mu^2\eta_\varphi} & -1 & -\frac{(\sin\varphi)\sqrt{1-\mu^2}}{\mu} \\ 0 & -\frac{1}{\mu^2} & 0 & -\frac{\imath\sqrt{1+\eta_\varphi^2}}{\mu} \end{pmatrix}. \tag{A14}$$

Therefore, in place of Eq. (A11) we now obtain

$$\begin{aligned} \tilde{\mathbf{M}}_\varepsilon(\mu) &= \tilde{\Phi}_1(2\pi,\mu) = \tilde{\mathbf{M}}_0(\mu)\tilde{\mathbf{J}}(\mu), \\ \tilde{\mathbf{J}}(\mu) &= \int_0^{2\pi} \tilde{\Phi}_0^{-1}(s,\mu)\tilde{\mathbf{D}}_\varepsilon(s,\mu)\tilde{\Phi}_0(s,\mu)ds, \end{aligned} \tag{A15}$$

where the tilde ($\sim$) indicates that the matrix is expressed in the base diagonalizing $\mathbf{D}_0$. Because the eigenvalues $\{\sigma_\ell\}$ are distinct, the matrices $\tilde{\Phi}_0 = \exp(\tau\tilde{\mathbf{D}})$ and $\tilde{\mathbf{M}}_0 = \exp(2\pi\tilde{\mathbf{D}})$ are diagonal,

$$\begin{aligned} \tilde{\Phi}_0 &= \mathrm{diag}[\exp(\sigma_1\tau), \exp(\sigma_2\tau), \exp(\sigma_3\tau), \exp(\sigma_4\tau)], \\ \tilde{\mathbf{M}}_0 &= \mathrm{diag}\,(\lambda_1,\lambda_2,\lambda_3,\lambda_4)\,, \end{aligned} \tag{A16}$$

where

$$\lambda_\ell = \exp(2\pi\sigma_\ell) = \exp(2\pi\imath\omega_\ell) \quad (\ell = 1,2,3,4).$$

Therefore, the expression of the matrix $\tilde{\mathbf{J}}$ in Eq. (A15) becomes

$$\tilde{J}_{ij} = T_{im}^{-1}T_{nj}\underbrace{\int_0^{2\pi} e^{\imath(\omega_j-\omega_i)\tau}(\mathbf{D}_\varepsilon)_{mn}(\tau)d\tau}_{H_{mn}^{(ij)}}. \tag{A17}$$

In Appendix 4 we calculate the elements of $\tilde{\mathbf{J}}$ which we need for the stability analysis of the three resonant cases: hydrodynamic modes, mixed modes, and magnetic modes.

To complete the calculation of the matrix $\tilde{\mathbf{M}}_1$,

$$\tilde{\mathbf{M}}_1 = \tilde{\mathbf{M}}_\varepsilon + \nu\tilde{\mathbf{M}}_\mu = \tilde{\mathbf{M}}_0\cdot\tilde{\mathbf{J}} + \nu\tilde{\mathbf{M}}_\mu,$$

we need only the derivative of $\tilde{\mathbf{M}}_0$ with respect to μ,

$$\begin{aligned} \tilde{\mathbf{M}}_\mu &= \frac{\partial}{\partial\mu}\tilde{\mathbf{M}}_0(\mu) = \mathrm{diag}\left(\frac{\partial\lambda_\ell}{\partial\mu}\right) \\ &= \mathrm{diag}\left(2\pi\lambda_\ell\frac{\partial\sigma_\ell}{\partial\mu}\right) \quad (\ell = 1,2,3,4), \end{aligned}$$

since $\lambda_\ell = \exp(2\pi\sigma_\ell)$ as indicated previously.

When the initial magnetic field is horizontal, each eigenvalue σ_ℓ of the matrix $\mathbf{D}_0$ is linear in μ [see Eq. (37)], so that

$$\frac{\partial\sigma_\ell}{\partial\mu} = \frac{\sigma_\ell}{\mu},$$

and hence

$$\tilde{\mathbf{M}}_\mu(0,\mu_0) = 2\pi\,\mathrm{diag}\left(\frac{\sigma_\ell}{\mu}\lambda_\ell\right). \tag{A18}$$

It should be noted that, when the initial magnetic field has a nonzero vertical component, the asymptotic calculations for three-dimensional disturbances become complicated. For this purpose, the stability analysis of the case of a vertical initial magnetic field has been performed considering only axisymmetric disturbances (see Sec. IV).

Finally, from Eqs. (A16)–(A18) we deduce the expression of the maxtrix $\tilde{\mathbf{M}}_1$,

$$\tilde{\mathbf{M}}_1 = \tilde{\mathbf{M}}_0\tilde{\mathbf{J}} + \nu\tilde{\mathbf{M}}_\mu, \tag{A19}$$

$$(\tilde{\mathbf{M}}_1)_{ij} = [\exp(2\pi\sigma_i)]\left[\tilde{\mathbf{J}}_{ij} + \left(\frac{2\pi\nu}{\mu}\sigma_i\right)\delta_{ij}\right].$$

For the derivation of the quadratic equation (40) which allows us to analyze the stability of three resonant cases, we further analyze the characteristic polynomial of the matrix $\mathbf{M}$.

3. Characteristic polynomial

The aim of this section is to analyze the characteristic polynomial of the matrix $\mathbf{M}$ leading to the quadratic equation (40) for the first-order correction to the growth rate. Because the calculation is the same as in Lebovitz and Zweibel [18] (see also Refs. [19,39]), we will briefly report the necessary steps allowing us to derive Eq. (40).

We denote by $p(\varepsilon,\lambda) = |\tilde{M} - \lambda\mathbf{I}_4|$ the characteristic of the Floquet multiplier matrix $\mathbf{M}$ and by Λ_ℓ ($\ell = 1,2,3,4$) its roots. We expand $p(\varepsilon,\lambda)$ in a perturbative series around ε to second order in ε,

$$p(\varepsilon,\lambda) = p_0(\lambda) + \varepsilon p_1(\lambda) + \varepsilon^2 p_2(\lambda) + O(\varepsilon^3), \tag{A20}$$

where $p_0(\lambda) = \Pi_{i=1}^4(\lambda_i - \lambda) = \Pi_{i=1}^4\left[\exp(2\pi\sigma_i) - \lambda\right]$ is the characteristic polynomial of $\tilde{\mathbf{M}}_0$,

$$p_1(\lambda) = \left[\frac{d}{d\varepsilon}\left|\tilde{\mathbf{M}} - \lambda\mathbf{I}\right|\right]_{\varepsilon=0} \quad \text{and}$$

$$p_2(\lambda) = \left[\frac{d^2}{d\varepsilon^2}\left|\tilde{\mathbf{M}} - \lambda\mathbf{I}\right|\right]_{\varepsilon=0}.$$

The condition for destabilization is that there be double (or higher) roots of $p(\varepsilon,\lambda)$. For instance, we restrict the consideration to the case of double roots. Then the Puiseux expansion takes the form (see Hille [46])

$$\Lambda_1 = \lambda_1 + \varepsilon^{\frac{1}{2}}\beta_{\frac{1}{2}} + \varepsilon\beta_1 + O(\varepsilon^{\frac{3}{2}}), \tag{A21}$$

where, for definiteness, we have assumed $\lambda_1 = \lambda_2$. Because $p(\lambda_1) = 0$, the coefficient $\beta_{1/2}$ is zero [see Eq. (A19) in Ref. [18],

$$\beta_{\frac{1}{2}}^2 = -2\frac{p_1(\lambda_1)}{p_0''(\lambda_1)} = 0,$$

where $'$ denotes the derivative with respect to λ, while the coefficient β_1 is found by solving the quadratic equation [see Eq. (32) and Appendix A 3 in Ref. [18]),

$$a_0\beta_1^2 + a_1\beta_1 + a_2 = 0, \tag{A22}$$

with

$$\begin{aligned} a_0 &= \frac{1}{2}p_0''(\lambda_1) = (\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1), \\ a_1 &= p_1'(\lambda_1) = -[(\tilde{\mathbf{M}}_1)_{11} + (\tilde{\mathbf{M}}_1)_{22}]a_0, \\ a_2 &= p_2(\lambda_1) = \begin{vmatrix} (\tilde{\mathbf{M}}_1)_{11} & (\tilde{\mathbf{M}}_1)_{12} \\ (\tilde{\mathbf{M}}_1)_{21} & (\tilde{\mathbf{M}}_1)_{22} \end{vmatrix} a_0. \end{aligned} \tag{A23}$$

By setting $\alpha = \beta_1/\lambda_1$, Eq. (A21) can be rewritten as

$$\begin{aligned} \frac{\Lambda_1}{\lambda_1} &= 1 + \varepsilon\alpha + O(\varepsilon^{\frac{3}{2}}), \\ \alpha &= \frac{1}{2a_0}\left[a_1 \pm \sqrt{a_1^2 - 4a_0a_2}\right], \end{aligned} \tag{A24}$$

or equivalently,

$$\alpha^2 - \left(\tilde{J}_{11} + \tilde{J}_{22} + \frac{2\pi\imath}{\mu}\nu(\omega_1 + \omega_2)\right)\alpha + \begin{vmatrix} \tilde{J}_{11} + \frac{2\pi\imath\nu\omega_1}{\mu} & \tilde{J}_{12} \\ \tilde{J}_{21} & \tilde{J}_{22} + \frac{2\pi\imath\nu\omega_2}{\mu} \end{vmatrix} = 0, \tag{A25}$$

where Eqs. (A19) and (A23) have been used. Consequently, at the first order of ε, there is instability if $\mathrm{Re}\,\alpha \neq 0$. For the cases where $\lambda_1 = \lambda_3$, $\lambda_1 = \lambda_4$, and $\lambda_3 = \lambda_4$, the coefficients a_0, a_1, and a_2 are calculated in a similar manner, and hence one obtains Eq. (40).

4. Calculating some elements of $\tilde{\mathbf{J}}$

The aim of this section is to calculate the elements of the matrix $\tilde{\mathbf{J}}$ which we need for the analysis of the quadratic equation (40) in each one of the three resonant cases: $\omega_2 - \omega_4 = 1$ (hydrodynamic modes), $\omega_2 - \omega_3 = 1$ (mixed modes), and $\omega_1 - \omega_3 = 1$ (magnetic modes). For the sake of clarity, we report here the expression of $\tilde{\mathbf{J}}$ given by Eq. (A17),

$$\tilde{J}_{ij} = T_{im}^{-1}T_{nj}\underbrace{\int_0^{2\pi} e^{\imath(\omega_j - \omega_i)\tau}(\mathbf{D}_\varepsilon)_{mn}(\tau)d\tau}_{H_{mn}^{(ij)}}.$$

We now proceed to calculate the diagonal elements $\tilde{J}_{jj}$. For these elements the exponential factors in the integrand reduce to unity, and hence $H_{mn}^{(jj)} = \int_0^{2\pi}(D_\varepsilon)_{mn}d\tau$. Moreover, with the aid of Eq. (A8) giving the expression of the elements $(D_\varepsilon)_{ij}$, we deduce that only the elements $H_{14}^{(jj)}$ and $H_{24}^{(jj)}$ are nonzero,

$$H_{14}^{(jj)} = -(2\mu^2 - 1)(1 - \mu^2), \quad H_{24}^{(jj)} = -(2\mu^2 - 1)^{3/2}\sin\varphi,$$

so that

$$\tilde{J}_{jj} = \left(T_{j1}^{-1}H_{14}^{(jj)} + T_{j2}^{-1}H_{24}^{(jj)}\right)T_{4j}. \tag{A26}$$

By using for the elements T_{ij} and T_{ij}^{-1} the expression given above in Eqs. (A13) and (A14), we find

$$\tilde{J}_{11} = \tilde{J}_{33} = 0 \tag{A27}$$

and

$$\tilde{J}_{44} = -\tilde{J}_{22} = 2\pi\imath(\sin\varphi)\frac{(1 - \mu^2)^{\frac{3}{2}}}{\sqrt{1 + \eta_\varphi^2}}. \tag{A28}$$

Before proceeding to evaluate the off-diagonal elements of $\tilde{\mathbf{J}}$, let us recall that the destabilization of the Floquet system (30) occurs through a resonance between at least two eigenvalues of the matrix $\mathbf{M}$, and as in Refs. [18,19,39], only the case of double multiplicity of the eigenvalues is considered here. Because the eigenvalues of the matrix $\mathbf{M}_0$ are of the form $\lambda_\ell = \exp(2\pi\sigma_\ell)$, $\ell = 1,2,3,4$, resonance occurs for $\sigma_i - \sigma_j = \imath(\omega_i - \omega_j) = \imath n$, where n is a nonzero integer.

It is clear from Eqs. (A8) and (A17) that for the resonant cases with $\omega_i - \omega_j = \pm 2, \pm 3, \pm 4, \ldots$, all the off-diagonal elements of $\tilde{\mathbf{J}}$ are zero. In comparison, the resonant cases with $\omega_i - \omega_j = \pm 1$ contribute off-diagonal terms to leading order in ε.

For the case with $\omega_2 - \omega_4 = 1$ (hydrodynamic modes) we need $\tilde{J}_{24}$ and $\tilde{J}_{42}$,

$$\begin{aligned}\tilde{J}_{24} &= T_{22}^{-1}\big(H_{21}^{(24)}T_{14} + H_{23}^{(24)}T_{34} + H_{24}^{(24)}T_{44}\big) + T_{24}^{-1}H_{43}^{(24)}T_{34},\\ \tilde{J}_{42} &= T_{42}^{-1}\big(H_{21}^{(42)}T_{12} + H_{23}^{(42)}T_{32} + H_{24}^{(42)}T_{42}\big) + T_{44}^{-1}H_{43}^{(42)}T_{32},\end{aligned} \tag{A29}$$

since $T_{21}^{-1} = T_{41}^{-1} = 0$, where

$$\begin{aligned}H_{21}^{(24)} &= -H_{21}^{(42)} = -\imath\pi(1-2\mu^2),\\ H_{23}^{(24)} &= H_{23}^{(42)} = \pi\mu^2,\\ H_{24}^{(24)} &= H_{24}^{(42)} = 2\pi\mu(1-\mu^2)^{\frac{3}{2}}(\sin\varphi),\\ H_{43}^{(24)} &= -H_{43}^{(42)} = -\imath\pi.\end{aligned} \tag{A30}$$

With the aid of (A14) and (A13) giving the expression of $\mathbf{T}$ and $\mathbf{T}^{-1}$, respectively, we obtain

$$\tilde{J}_{42} = -\tilde{J}_{24} = \imath\frac{\pi\mu\sqrt{1-\mu^2}}{2}\left[2 + \frac{(1-2\mu^2)}{\sqrt{1+\eta_\varphi^2}}\right]\sin\varphi. \tag{A31}$$

For the resonant case with $\omega_2 - \omega_3 = 1$ (mixed modes), we need $\tilde{J}_{23}$ and $\tilde{J}_{32}$,

$$\tilde{J}_{23} = T_{22}^{-1}\big(H_{21}^{(23)}T_{13} + H_{23}^{(23)}T_{33}\big) + T_{23}^{-1}H_{43}^{(23)}T_{33}, \tag{A32}$$

where

$$\begin{aligned}H_{21}^{(23)} &= -\imath\pi(1-2\mu^2),\\ H_{23}^{(23)} &= \pi\mu^2,\\ H_{43}^{(23)} &= -\imath\pi,\end{aligned} \tag{A33}$$

and

$$\begin{aligned}\tilde{J}_{32} = T_{31}^{-1}&\big(H_{11}^{(32)}T_{12} + H_{13}^{(32)}T_{32} + H_{14}^{(32)}T_{42}\big)\\ &+ T_{32}^{-1}\big(H_{21}^{(32)}T_{12} + H_{23}^{(32)}T_{22} + H_{24}^{(32)}T_{42}\big)\\ &+ T_{34}^{-1}H_{43}^{(32)}T_{32},\end{aligned} \tag{A34}$$

where

$$\begin{aligned}H_{11}^{(32)} &= 2\imath\pi\mu(1-2\mu^2)\sin\varphi,\\ H_{13}^{(32)} &= -\pi\mu\sqrt{1-\mu^2}\sin\varphi,\\ H_{14}^{(32)} &= -\pi(1-\mu^2)(1-2\mu^2\sin^2\varphi),\\ H_{21}^{(32)} &= \imath\pi(1-2\mu^2),\\ H_{23}^{(32)} &= \pi\mu^2,\\ H_{24}^{(32)} &= 2\pi\mu(1-\mu^2)^{\frac{3}{2}}\sin\varphi,\\ H_{43}^{(32)} &= \imath\pi.\end{aligned} \tag{A35}$$

By the use of (A13) and (A14) and after lengthy calculations, we find

$$\tilde{J}_{23} = \frac{\pi}{2\mu}\Big(\mu + \eta_\varphi(1-2\mu^2) - \sqrt{1+\eta_\varphi^2}\Big) = -\pi\eta_\varphi\mu, \tag{A36}$$

$$\tilde{J}_{32} = -\frac{\pi\mu}{\sqrt{1+\eta_\varphi^2}}\cos^2\varphi. \tag{A37}$$

As for the resonant case $\omega_1 - \omega_3 = 1$ (magnetic modes), we show that the off-diagonal element $\tilde{J}_{13}$ is zero, so that $\tilde{J}_{13}\tilde{J}_{31} = 0$, and hence it not necessary to determine $\tilde{J}_{31}$:

$$\begin{aligned}\tilde{J}_{13} = T_{11}^{-1}&\big(H_{11}^{(13)}T_{13} + H_{13}^{(13)}T_{33}\big)\\ &+ T_{12}^{-1}\big(H_{21}^{(13)}T_{13} + H_{23}^{(13)}T_{33}\big) + T_{14}^{-1}H_{43}^{(13)}T_{33},\end{aligned} \tag{A38}$$

where

$$\begin{aligned}H_{11}^{(13)} &= -H_{11}^{(31)} = -2\pi\imath(\sin\varphi)\mu\sqrt{1-\mu^2},\\ H_{13}^{(13)} &= H_{13}^{(31)} = -\pi(\sin\varphi)\mu\sqrt{1-\mu^2},\\ H_{21}^{(13)} &= -H_{21}^{(42)} = -\pi\imath(1-2\mu^2),\\ H_{23}^{(13)} &= H_{23}^{(13)} = \pi\mu^2,\\ H_{43}^{(13)} &= -H_{43}^{(31)} = -\pi\imath.\end{aligned} \tag{A39}$$

Accordingly, the use of Eqs. (A13) and (A14) allows us to deduce that

$$\tilde{J}_{13} = 0. \tag{A40}$$

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