White-noise limit of nonwhite nonequilibrium processes

Stefano Bo[1] and Antonio Celani[2]

[1]*Physics Department and INFN, University of Turin, 10125 Turin, Italy*

[2]*Physics of Biological Systems, Institut Pasteur and CNRS UMR 3525, 28 rue du docteur Roux, 75015 Paris, France*

(Received 4 October 2013; published 31 December 2013)

The asymptotic behavior of a stochastic process subject to a colored noise is considered in the limit of vanishing correlation time of the noise. The interpretation of the multiplicative noise of the effective equation is investigated. The mathematically consistent formulation of the stochastic calculus for the limiting process is given. It differs in general from the Stratonovich one which is recovered when the colored noise obeys detailed balance or is a one-dimensional process.

DOI: 10.1103/PhysRevE.88.062150 PACS number(s): 05.40.−a, 05.10.Gg

I. INTRODUCTION

White noise is the cornerstone for the formal description and the understanding of most stochastic processes. This rather idealized setting can be seen to arise as the limit of a more realistic colored noise considered on timescales which are longer than its correlation time [1–4]. In general, the white noise of the resulting effective equation is a multiplicative one and it is therefore necessary to specify the prescription for its regularization (see Refs. [5–8] or others). It is known that if the colored noise is a one-dimensional Markov process the resulting equation follows the Stratonovich convention [1–4]. However, this is not necessarily true when the colored noise is multidimensional. In general there will be correction terms involving the autocorrelation matrix of the noise [4,9,10]. We will show here that in the multidimensional case the Stratonovich interpretation is guaranteed if the fast process admits an equilibrium solution. Conversely, if the fast noise reaches a nonequilibrium steady state we find that the required corrections to the Stratonovich regularization stem from the antisymmetric part of the autocorrelation function of the fast variables. The influence of the correlations of fast eliminated processes on the interpretation of the noise of the effective dynamics has been thoroughly investigated in a number of publications for richer systems involving several timescales. In Refs. [11–13] the authors studied the interplay between the correlation time of the noise and the timescales of the other processes. The authors of Ref. [14] considered the interpretation of the noise starting from deterministic systems. Here we focus on a minimal case in a setting close to the renowned example of Gardiner's book [2] where the only timescales are the fast one on which the correlation of the noise decays and the one of the effective process. Despite its simplicity the study case is rich enough to display the nontrivial deviation from the Stratonovich prescription which enables us to unveil the importance of the symmetries of the correlations of the fast process. We provide the mathematically consistent prescription for the regularization of the noise of the effective process which accounts for the multidimensional nonequilibrium effects and reduces to the Stratonovich one for the equilibrium or the one-dimensional case. In the main text we show an intuitive derivation of the effective process and in Appendices B and C we rigorously obtain the correct discretization by means of adiabatic elimination of the variables of the underlying fast random processes. We then study a simple example of a two-dimensional noise reaching a nonequilibrium steady state. The simple setting allows to explicitly derive the corrections to the Stratonovich prescription.

II. MAIN RESULTS

We are interested in the behavior of a process subject to some colored multidimensional noise in the limit of vanishing correlation time in analogy with the discussion in Ref. [2], i.e., considering the dynamics on timescales much longer than the correlation time of the noise. This problem can be formally addressed by studying the differential equation

$$\frac{dX_t^i}{dt} = \alpha^i(X_t) + \epsilon^{-1}\beta^{ia}(X_t)Y_t^a, \tag{1}$$

where the colored noise is represented by the fast process Y and obeys the Itō stochastic differential equation

$$dY_t^a = \epsilon^{-2}U^a(Y_t,t)dt + \epsilon^{-1}\sigma^{ab}(Y_t,t)\cdot dW_t^b \tag{2}$$

where $\cdot$ denotes the Itō product, $i,j = 1,\dots,n,a,b = 1,\dots,m$ and W_t^b is a Wiener process. Note that the choice of the prescription of the fast process is immaterial. The parameter ϵ sets the separation of timescales between the two processes. We require the fast process to have zero mean and to admit a stationary solution. At the steady state Y has a correlation

$$\langle Y_\tau^a Y_0^b\rangle = g^{ab}(\tau) = g^{ba}(-\tau), \tag{3}$$

where $\langle\cdots\rangle$ denotes the average over trajectories of Eq. (2) in the steady state. We remark that we are interested in investigating the formal aspects of this system in general and therefore consider generic drift and noise terms in Eq. (1). A consequence of this generality is that we will not couple the fluctuating term to the drift and will not address the issue of fluctuation-dissipation relations. Our focus will instead be on the time reversibility of the noise and its effects on the dynamics in the limit of vanishing correlation of the noise. Indeed, the fact that in multidimensional systems in general $g^{ab}(\tau) \neq g^{ab}(-\tau)$ is at the core of the results presented below. We consider the case in which the correlation decays with the characteristic time τ_c. Considering the scaling we can see that the limit $\epsilon \to 0$ corresponds to that of vanishing correlation time since we have that

$$\langle \epsilon^{-1}Y_{\epsilon^{-2}t}^a\epsilon^{-1}Y_0^b\rangle = \epsilon^{-2}g^{ab}(\epsilon^{-2}t), \tag{4}$$

which is a regularization of the Dirac delta function and where we see that $\tau_c \sim \epsilon^2$. Before moving forward it is worthwhile to make some considerations about the time integral of the correlation function of the noise:

$$\mu^{ab} = \int_0^\infty g^{ab}(\tau)d\tau, \tag{5}$$

which is a positive definite matrix and in general is not symmetric (see Appendix A). As every matrix it can be split in its symmetric and antisymmetric components $\mu^{ab} = \mu_S^{ab} + \mu_A^{ab}$. If the steady state of the noise is an equilibrium one it can be shown that the integral of the correlation function is symmetric and $\mu^{ab} = \mu_S^{ab}$ (see Appendix D). The same trivially holds if the process (and consequently the matrix) is one dimensional. Differently, the integral of the correlation between $-\infty$ and ∞ is always symmetric (see Appendix A):

$$\int_{-\infty}^{\infty} g^{ab}(\tau)d\tau = \mu^{ab} + \mu^{ba} = 2\mu_S^{ab} \equiv \nu^{ac}\nu^{bc}. \tag{6}$$

In an analogy to Ref. [2] we expect to find an effective equation of the kind

$$\frac{dX^i}{dt} = \alpha^i + \beta^{ia}\zeta_t^a, \tag{7}$$

where ζ_t^a is the white-noise limit of Y_t^a and has the covariance $\langle \zeta_t^a \zeta_{t'}^b \rangle = 2\mu_S^{ab}\delta(t-t')$. We intend to determine the correct interpretation of the product $\beta^{ia}\zeta_t^a$. We show below that there is a regularization such that it is possible to express the effective process as

$$dX_t^i = \alpha^i dt + \beta^{ib} \circlearrowleft d\xi_t^b, \tag{8}$$

where $d\xi_t^b \equiv \nu^{bc}dB^c$ and the whirl product above implies the discretization

$$X_{t+\Delta t}^i - X_t^i = \alpha^i(\hat{X}_t^i)\Delta t + \beta^{ia}(\hat{X}_t)\eta^a\sqrt{\Delta t} + O(\Delta t\sqrt{\Delta t}) \tag{9}$$

where η^a is a multivariate Gaussian with zero mean and covariance $\langle \eta^a\eta^b \rangle = 2\mu_S^{ab}$ and the point at which the functions are evaluated is implicitly defined by

$$\hat{X}_t^j = X_t^j + \tfrac{1}{2}\{\alpha^j(\hat{X}_t)\Delta t + \beta^{jd}(\hat{X}_t) \times [\delta^{cd} + (\mu_S^{-1})^{ce}(\mu_A)^{ed}]\eta^c\sqrt{\Delta t}\}. \tag{10}$$

Such a point can be seen to deviate from the midpoint because of the asymmetries of the integral of the correlation function induced by nonequilibrium:

$$\hat{X}_t^j = \frac{X_t^j + X_{t+\Delta t}^j}{2} + \frac{1}{2}\beta^{ja}(X_t)[(\mu_S^{-1})^{cd}(\mu_A)^{da}]\eta^c\sqrt{\Delta t} + \frac{1}{2}\beta^{ib}\frac{\partial\beta^{ja}}{\partial x^i}(X_t)\mu^{cb}[(\mu_S^{-1})^{cd}(\mu_A)^{da}]\Delta t. \tag{11}$$

When μ^{ab} is symmetric (which is the case when the fast process is at equilibrium or, as in Ref [2], is one dimensional) $\hat{X}_t^j = \frac{1}{2}(X_{t+\Delta t}^j + dX_t^j)$, i.e., the whirl regularization reduces to the Stratonovich midpoint one.

We are also interested in considering the limiting behavior of a functional of the trajectory described by Eq. (1):

$$\mathcal{J} = \int_t^{t'} h(X_\tau)d\tau + f^i(X_\tau)dX_\tau^i. \tag{12}$$

Notice that since X_t is a differentiable process, there is no ambiguity in the definition of the above functional, which is in fact an ordinary integral

$$\mathcal{J} = \int_t^{t'} [h(X_\tau) + \alpha^i(X_\tau)f^i(X_\tau) + \epsilon^{-1}f^i(X_\tau)\beta^{ia}(X_\tau)Y_\tau^a]d\tau. \tag{13}$$

In the limit of vanishing correlation time of the noise, we expect it to become a stochastic integral over the effective stochastic trajectories with an integration prescription to be specified. We find it to be

$$\lim_{\epsilon\to 0}\mathcal{J} = \int_{t'}^{t} h d\tau + f^i \circlearrowleft dX_\tau^i, \tag{14}$$

where the whirl product introduced in Eq. (9) is generalized and implies the discretization

$$\int_{t'}^{t} f^i \circlearrowleft dX_\tau^i \equiv \sum_{k=0}^{K-1} f^i(\hat{X}_{t'+k\Delta t})\left(X_{t'+(k+1)\Delta t}^i - X_{t'+k\Delta t}^i\right), \tag{15}$$

where $t' + K\Delta t = t$ and the evaluation point $\hat{X}_t$ is the one defined in Eq. (10).

III. HEURISTIC DERIVATION

We will give a rigorous derivation of the effective equation for the process by means of adiabatic elimination of the fast degrees of freedom in Appendix B. We report here a heuristic argument that clearly highlights the important role of the symmetries in the correlations of the fast process. Let us consider a discretized version of Eq. (1):

$$X_{t+n\tau}^i = X_t^i + \sum_{k=1}^{n}\alpha^i(X_{t+(k-1)\tau})\tau + \sum_{k=1}^{n}\beta^{ia}(X_{t+(k-1)\tau})Y_{t+(k-1)\tau}^a\tau, \tag{16}$$

where we should recall that $Y_t \sim \tau_c^{-1/2}$ and we do not need to explicitly write the bookkeeping parameter ϵ. We choose τ to be smaller than the correlation time of the noise ($\tau \ll \tau_c$). Then we consider a number of steps n such that the time $n\tau$ is much larger than the correlation time of the fast process τ_c but the functions $\alpha(X)$ and $\beta(X)$ have not yet varied considerably [i.e., $\tau_c \ll n\tau \ll (\partial_x\beta)^{-2}$ and $n\tau \ll (\partial_x\alpha)^{-1}$]. Given the large timescale separation this corresponds to the large range of n for which the autocorrelation of the fast process has decayed and the slow process has not moved much. Under these conditions we can Taylor expand the functions α and β and obtain

$$X_{t+n\tau}^i = X_t^i + n\tau\alpha^i(X_t) + \beta^{ia}(X_t)\sum_{k=1}^{n}Y_{t+(k-1)\tau}^a\tau + \frac{\partial\beta^{ia}(X_t)}{\partial x^j}\sum_{k=1}^{n}\left(X_{t+(k-1)\tau}^j - X_t^j\right)Y_{t+(k-1)\tau}^a\tau. \tag{17}$$

With a few considerations it is possible to express this equation as a stochastic differential equation (SDE) with multiplicative white noise since the first term in Y has a Gaussian limit. Indeed, variables with identical distributions that have exponentially decaying correlation functions and are summed for times longer than their correlation time satisfy the mixing conditions necessary to extend the central limit theorem to weakly correlated variables [15]. Therefore their sum is distributed as a Gaussian and we need only to specify its average and covariance. As introduced before, the process Y has zero average and so does its sum over consecutive time steps. The covariance matrix of the sum is

$$\left\langle \sum_{k=1}^{n}\sum_{l=1}^{n} Y^a_{t+(k-1)\tau} Y^b_{t+(l-1)\tau} \right\rangle \simeq \frac{n}{\tau}\int_{-\infty}^{\infty} \langle Y^a_s Y^b_0 \rangle ds = 2\frac{n}{\tau}\mu_S^{ab}, \tag{18}$$

where in moving from the sum to the integral we have exploited that $n\tau \gg \tau_c$ and for the last step we have used Eq. (6). The second sum term involves the correlation between Y and the process X. It can be taken as an average for the law of large numbers so that using iteratively the definition of the process and retaining only the lowest-order terms we have

$$\sum_{k=1}^{n}\sum_{l=1}^{k} \langle Y^b_{t+(l-1)\tau} Y^a_{t+(k-1)\tau} \rangle \simeq \frac{n}{\tau}\int_0^{\infty} \langle Y^a_s Y^b_0 \rangle ds \equiv \frac{n}{\tau}\mu^{ab}, \tag{19}$$

where it is important to note that the second sum (over l) runs only up to k. Finally, defining $\Delta t = n\tau$ we can write

$$X^i_{t+\Delta t} = X^i_t + \alpha^i(X_t)\Delta t + \beta^{jb}\frac{\partial \beta^{ia}}{\partial x^j}(X_t)\mu^{ab}\Delta t + \beta^{ia}(X_t)\eta^a\sqrt{\Delta t}, \tag{20}$$

where again η^a is a multivariate Gaussian with zero mean and covariance $\langle \eta^a \eta^b \rangle = 2\mu_S^{ab}$. This equation is the discrete version of

$$dX^i_t = \alpha^i dt + \beta^{jb}\frac{\partial \beta^{ia}}{\partial x^j}\mu^{ab}dt + \beta^{ia}\nu^{ab}\cdot dB^b_t, \tag{21}$$

where B^b_t is a Wiener process and $\cdot$ denotes the Itō product reflecting the fact that β^{ia} in Eq. (20) is evaluated at the initial point X_t. Our aim is to determine which interpretation of the product in Eq. (7) yields this equation. This clearly depends on the term $\beta^{jb}\frac{\partial \beta^{ia}}{\partial x^j}\mu^{ab}$, i.e., it is determined by the properties of the integral of the cross-correlation function μ^{ab} defined in Eq. (5). For instance, if the integral of the autocorrelation is symmetric $\mu^{ab} = \mu^{ba} = \mu_S^{ab}$ (which is the case when the fast process admits an equilibrium or is one dimensional as in Ref. [2]) we have that $\mu^{ab} = (1/2)\nu^{ac}\nu^{bc}$ and that Eq. (21) corresponds to the Stratonovich interpretation of Eq. (7). In fact, we can write it as

$$\mu^{ab} = \mu_S^{ab}, \quad dX^i = \alpha^i dt + \beta^{ia}\circ d\xi^a, \tag{22}$$

where $\circ$ denotes the midpoint Stratonovich product. It is important to note that Eq. (21) cannot be obtained by the Itō nor by the Hänggi–Klimontovich interpretation of the noise in Eq. (7). In fact, for Eq. (21) to correspond to the Itō interpretation of Eq. (7) we would have to require $\mu^{ab} = 0$ which would also imply that $\mu_S^{ab} = 0$ and the equation would not be a diffusive one. Similarly, for the Hänggi–Klimontovich one $\mu^{ab} = 2\mu_S^{ab}$ which again implies $\mu^{ab} = \mu_S^{ab} = 0$. For a generic μ^{ab} the correct discretization is the one given by the whirl product as in Eq. (8). Indeed, considering the discrete equation (9) and iteratively substituting the expression for $\hat{X}_t$ [Eq. (10)] we obtain Eq. (20).

IV. FUNCTIONALS OF THE TRAJECTORY

We now turn our attention to the limiting behavior of the functional defined in Eq. (12). We omit the heuristic discussion which would closely follow the one given for the study of the effective process. The complete derivation is contained in Appendix C and we list here the results. As for the effective equation for the dynamics the interpretation of the integral depends on the correlations of the eliminated fast process.

$$\lim_{\epsilon\to 0}\mathcal{J} = \int_{t'}^{t} h d\tau + f^i \circlearrowleft dX^i_\tau = \int_{t'}^{t}\left(h + \mu^{ab}\beta^{ia}\beta^{jb}\frac{\partial f^i}{\partial x^j}\right)d\tau + f^i\cdot dX^i_\tau \tag{23}$$

over effective trajectories as can be directly checked by substituting the definition of $\hat{X}_t$ given by Eq. (10) in Eq. (15). This is equivalent to the integral in Stratonovich form:

$$\int_{t'}^{t}\left[h + \frac{1}{2}\mu_A^{ab}\beta^{ia}\beta^{jb}\left(\frac{\partial f^i}{\partial x^j} - \frac{\partial f^j}{\partial x^i}\right)\right]d\tau + \int_{t'}^{t} f^i\circ dX^i_\tau. \tag{24}$$

From this expression it is again evident that when the fast process is at equilibrium, the correction due to the antisymmetric part of the correlation vanishes, and one recovers the Stratonovich prescription. It is interesting to note that when the integrand is a gradient (i.e., $f^i = -\partial U/\partial x^i$) the nonequilibrium correction vanishes and the stochastic integral is an exact differential, consistently with the original functional (12). However, the process itself [Eq. (8)] is still non-Stratonovich.

V. TWO-DIMENSIONAL NONEQUILIBRIUM STEADY STATE NOISE

In this section we give a simple example where all details are worked out explicitly. In order to have an asymmetric μ^{ab} we need to consider a nonequilibrium fast process in more than one dimension. Let us choose as a simple example the case in which the fast process is given by the two-dimensional motion of a particle in a parabolic well with an additional nonpotential rotational force (see, e.g., Ref. [16]):

$$dY^a = \epsilon^{-2}\frac{1}{\gamma}F^{ab}Y^b dt + \epsilon^{-1}\sqrt{2D}dW^a_t, \tag{25}$$

where $a,b = 1,2$,

$$F = \begin{pmatrix} -k & \omega \\ -\omega & -k \end{pmatrix}, \tag{26}$$

and $D = T/\gamma$. The corresponding forward Fokker–Planck equation reads

$$\frac{\partial p(y^1,y^2)}{\partial t} = \frac{k}{\gamma}\frac{\partial}{\partial y^1}y^1 p - \frac{\omega}{\gamma}y^2\frac{\partial}{\partial y^1}p + D\frac{\partial^2}{\partial y^1\partial y^1}p + \frac{k}{\gamma}\frac{\partial}{\partial y^2}y^2 p + \frac{\omega}{\gamma}y^1\frac{\partial}{\partial y^2}p + D\frac{\partial^2}{\partial y^2\partial y^2}p. \tag{27}$$

By casting these equations in polar coordinates (r,ϕ) it is immediate to find a solution of the kind $p_s \propto \exp[-kr^2/(2T)]$. This is a stationary nonequilibrium solution with a vanishing flux in the radial direction and an angular one $J_\phi = -\frac{\omega}{\gamma}p_s$. We can now proceed to the evaluation of the autocorrelation matrix. By multiplying the Fokker–Planck equation by y^a and y'^b and integrating by parts we obtain (for $t \geqslant t'$)

$$d_t\left\langle Y_t^a Y_{t'}^b\right\rangle = \frac{1}{\gamma}F^{ac}\left\langle Y_t^c Y_{t'}^b\right\rangle dt, \tag{28}$$

and consequently the correlation function for $t \geqslant 0$ then reads

$$g^{ab}(t) = \left\langle Y_t^a Y_0^b\right\rangle = [e^{\frac{1}{\gamma}Ft}]^{ac}\left\langle Y_0^c Y_0^b\right\rangle. \tag{29}$$

Integrating it we obtain

$$\mu^{ab} = \int_0^\infty \left\langle Y_t^a Y_0^b\right\rangle dt = -\gamma F^{-1ac}\left\langle Y_0^c Y_0^b\right\rangle = \frac{\gamma T}{k^2+\omega^2}\begin{pmatrix} 1 & \omega/k \\ -\omega/k & 1\end{pmatrix}, \tag{30}$$

where we have exploited the fact that $\langle Y_0^c Y_0^b\rangle = \delta^{cb}\frac{T}{k}$. The off-diagonal terms of μ are antisymmetric so that if we sum μ to its transpose we obtain

$$2\mu_S^{ab} = \mu^{ab} + \mu^{ba} = \frac{2\gamma T}{k^2+\omega^2}\delta^{ab}. \tag{31}$$

The asymmetry of the matrix μ implies that the effective equation (20) does not correspond to the Stratonovich interpretation of the SDE. In fact, even in the case in which X is a one-dimensional process we get the effective equation

$$dX_t = \alpha dt + \frac{\gamma T}{k^2+\omega^2}\frac{\omega}{k}\left(\beta^2\frac{\partial\beta^1}{\partial x} - \beta^1\frac{\partial\beta^2}{\partial x}\right)dt + \sqrt{\frac{2\gamma T}{k^2+\omega^2}}\left(\beta^1\circ dB^1 + \beta^2\circ dB^2\right), \tag{32}$$

where B^1, B^2 are independent Wiener processes. In general, this equation does not correspond to the SDE (7) with a Stratonovich interpretation of the noise (unless $\beta^2\frac{\partial\beta^1}{\partial x} = \beta^1\frac{\partial\beta^2}{\partial x}$). It is the nonequilibrium rotatory term of the fast process ω/k which originates the deviation from the Stratonovich equation.

If we now consider the functional (12) for a two-dimensional process in X we find that the limiting integral becomes

$$\int_{t'}^t\left[h + \frac{\gamma T}{k^2+\omega^2}\frac{\omega}{k}(\beta^{11}\beta^{22} - \beta^{12}\beta^{21})\left(\frac{\partial f^1}{\partial x^2} - \frac{\partial f^2}{\partial x^1}\right)\right]d\tau + \int_{t'}^t f^i\circ dX_\tau^i. \tag{33}$$

where again the correction to the Stratonovich prescription is due to the nonequilibrium term of the noise ω/k. As discussed for the general case if $\frac{\partial f^1}{\partial x^2} = \frac{\partial f^2}{\partial x^1}$ (i.e., f^i is a gradient) the correction vanishes. However, the process X is still non-Stratonovich displaying corrections which are the two-dimensional equivalent of the ones in Eq. (32).

VI. CONCLUSIONS AND DISCUSSION

In the present contribution we have investigated the white-noise limit of systems driven by a colored process. The extension of this classic limit to higher dimensions introduces a richer scenario. The discretization choice for the resulting multiplicative noise is no longer the Stratonovich one and differences in the specific process modeling the colored noise affect the final result. We have provided the expression for the correct discretization of the process and derived explicitly the deviation from the Stratonovich one. We have chosen a simple starting point in order to allow a transparent interpretation of the results. By doing so we have been able to trace the corrections back to the antisymmetric part of the time integral of the autocorrelation matrix of the noise. The most relevant result of our present work is that the correct discretization crucially depends on the features of the eliminated fast process: if it admits an equilibrium solution the resulting process will follow the Stratonovich prescription otherwise it will display additional fast-process dependent corrections. Finally, it is worth pointing out that the choice of our starting point is not the most general one. We have in fact restricted consideration to the case in which the fast process is independent of the slow one and is a Markov process on its own. This would not be the case if the drift coefficient or the diffusion matrix in Eq. (2) depended on the slow process such as, for example, for Langevin Kramers dynamics in the limit of vanishing inertia with a space-dependent temperature or friction (see, e.g., Refs. [17–20]).

APPENDIX A: PROPERTIES OF INTEGRAL OF CORRELATION MATRIX

The symmetry of the integral can be readily checked by splitting it

$$G^{ab} = \int_{-\infty}^\infty g^{ab}(\tau)d\tau = \int_{-\infty}^\infty\left\langle Y_\tau^a Y_0^b\right\rangle d\tau = \int_{-\infty}^0\left\langle Y_\tau^a Y_0^b\right\rangle d\tau + \int_0^\infty\left\langle Y_\tau^a Y_0^b\right\rangle d\tau,$$

and exploiting the stationarity of the process which ensures $\langle Y_\tau^a Y_0^b\rangle = \langle Y_0^a Y_{-\tau}^b\rangle$ together with a simple change of variable:

$$G^{ab} = \int_0^\infty\left\langle Y_0^a Y_\tau^b\right\rangle d\tau + \int_{-\infty}^0\left\langle Y_0^a Y_\tau^b\right\rangle d\tau = \int_{-\infty}^\infty\left\langle Y_0^a Y_\tau^b\right\rangle d\tau = G^{ba}. \tag{A1}$$

Similarly one can prove that

$$G^{ab} = \mu^{ab} + \mu^{ba} = 2\mu_S^{ab}. \tag{A2}$$

Concerning the positivity we can exploit the Wiener–Kinchin theorem which states that

$$\langle \tilde{y}^a(k)\tilde{y}^{b*}(q)\rangle = \frac{1}{2\pi}\delta(k-q)\int_{-\infty}^{\infty} d\tau \left\langle Y^a_\tau Y^b_0\right\rangle e^{ik\tau}, \quad \text{(A3)}$$

where $\tilde{f}(k)$ denotes the Fourier transform. By setting $q = k = 0$ and multiplying to the left and right by any vector v the positivity is proven. Also μ can be shown to be positive definite by multiplying left and right by any vector v^a and v^b and applying the Wiener–Kinchin theorem to the resulting scalar.

APPENDIX B: EFFECTIVE PROCESS BY ADIABATIC ELIMINATION

This appendix contains the derivation of the effective equation for X_t by means of adiabatic elimination of the fast degrees of freedom (see Ref. [4] for a detailed exposition of such method). The Kolmogorov equations for the propagator of the joint process X_t [Eq. (1)] and Y_t [Eq. (2)], $p(x,Y,t|x',Y',t')$, read

$$\partial_t p = L^\dagger p, \quad \partial_{t'} p = -L' p, \quad \text{(B1)}$$

where the generator of the diffusion process is

$$L = \epsilon^{-2}\underbrace{\left(U^a\frac{\partial}{\partial y^a} + \frac{1}{2}V^{ab}\frac{\partial^2}{\partial y^a\partial y^b}\right)}_{L_1} + \epsilon^{-1}\underbrace{b(x)^{ia}y^a\frac{\partial}{\partial x^i}}_{L_2} + \underbrace{a^i(x)\frac{\partial}{\partial x^i}}_{L_3}, \quad \text{(B2)}$$

with

$$V^{ab} = \sigma^{ac}\sigma^{cb}. \quad \text{(B3)}$$

In order to solve this problem by means of multiple-scale asymptotic methods let us expand the solution as

$$p = p^{(0)} + \epsilon p^{(1)} + \epsilon^2 p^{(2)} + \cdots \quad \text{(B4)}$$

and introduce the very fast and the fast time variables $\tau = \epsilon^{-2}t$ and $\tilde{t} = \epsilon^{-1}t$. At order ϵ^{-2} the backward equation reads

$$\left(\frac{\partial}{\partial\tau} + L_1\right)p^{(0)} = 0. \quad \text{(B5)}$$

We are considering the case in which the fast dynamics relaxes on very fast timescales to a stationary solution

$$L_1^\dagger w(y,t) = 0. \quad \text{(B6)}$$

This amounts to saying that the spectrum of $L_1^\dagger$ has a top eigenvalue $E_0 = 0$ and a finite gap ($E_1 < 0$). (If $L_1^\dagger w = wL_1$, then detailed balance holds and the solution is an equilibrium one.) This implies that $p^{(0)}$ is in the kernel of L_1, i.e., does not depend on y,

$$p^{(0)} = \rho(x,t,\tilde{t}), \quad \text{(B7)}$$

after a very fast transient.

At order ϵ^{-1} one has

$$\left(\frac{\partial}{\partial\tau} + L_1\right)p^{(1)} = -\left(\frac{\partial}{\partial\tilde{t}} + L_2\right)p^{(0)}. \quad \text{(B8)}$$

After relaxation, for L_1 to be invertible, one has to impose that the right-hand side is orthogonal to the null space of $L_1^\dagger$ which is generated by the stationary w. Therefore, multiplying by w and integrating over y one obtains the solvability condition

$$\frac{\partial\rho}{\partial\tilde{t}} = -\beta^{ia}\overline{y^a}\frac{\partial\rho}{\partial x^i}, \quad \text{(B9)}$$

where $\overline{\cdots} = \int dy \cdots w$ denotes the average over the stationary density of the fast variables and we have made use of $\int dyw = 1$ and the explicit expression of L_2. Since we are considering the case in which $\overline{y^a} = 0$ we have that $\partial\rho/\partial\tilde{t} = 0$ and $p^{(0)}$ does not depend on the fast timescale $\tilde{t}$. The formal solution then reads

$$p^{(1)} = -b^{ia}(x)\frac{\partial\rho(x,t)}{\partial x^i}L_1^{-1}y^a + \text{zero modes of } L_1. \quad \text{(B10)}$$

The equation at order ϵ^0 is

$$\left(\frac{\partial}{\partial\tau} + L_1\right)p^{(2)} = -\left(\frac{\partial}{\partial\tilde{t}} + L_2\right)p^{(1)} - \left(\frac{\partial}{\partial t} + L_3\right)p^{(0)}. \quad \text{(B11)}$$

The solvability condition becomes an effective backward Kolmogorov equation on slow timescales t

$$\frac{\partial\rho}{\partial t} + \alpha^i\frac{\partial\rho}{\partial x^i} + \overline{y^b\left(-L_1^{-1}y^a\right)}\beta^{jb}\frac{\partial}{\partial x^j}\left(\beta^{ia}\frac{\partial\rho}{\partial x^i}\right) = 0. \quad \text{(B12)}$$

The quantity $\overline{y^b(-L_1^{-1}y^a)}$ can be shown to coincide with the integral of the correlation function

$$\mu^{ab} = \overline{y^b\left(-L_1^{-1}y^a\right)} = \int_0^\infty \left\langle Y^a_t Y^b_0\right\rangle dt, \quad \text{(B13)}$$

which is in general not symmetric (see Appendix D). The effective equation can then be rewritten as

$$\frac{\partial\rho}{\partial t} + \left(\alpha^i + \mu^{ab}\beta^{jb}\frac{\partial}{\partial x^j}\beta^{ia}\right)\frac{\partial\rho}{\partial x^i} + \mu^{ab}\beta^{jb}\beta^{ia}\frac{\partial^2\rho}{\partial x^j\partial x^i} = 0. \quad \text{(B14)}$$

With the results of Appendix A it is possible to see that

$$\mu^{ab} + \mu^{ba} = \int_{-\infty}^{\infty} d\tau\left\langle Y^a_\tau Y^b_0\right\rangle = \int_{-\infty}^{\infty} g^{ab}d\tau = G^{ab}, \quad \text{(B15)}$$

which is symmetric. The diffusion term $\mu^{ab}\beta^{jb}\beta^{ia}\frac{\partial^2\rho}{\partial x^j\partial x^i}$ is also symmetric but because of the interchangeable role of $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial x^j}$ which allows to express it as

$$\tfrac{1}{2}d^{ij} \equiv \beta^{jb}\beta^{ia}\mu^{ab}_S. \quad \text{(B16)}$$

In fact,

$$\begin{aligned}\mu^{ab}\beta^{jb}\beta^{ia}\frac{\partial^2\rho}{\partial x^j\partial x^i} &= \mu^{ab}\beta^{ib}\beta^{ja}\frac{\partial^2\rho}{\partial x^j\partial x^i}\\ &= \frac{1}{2}\mu^{ab}(\beta^{jb}\beta^{ia} + \beta^{ib}\beta^{ja})\frac{\partial^2\rho}{\partial x^j\partial x^i}\\ &= \frac{1}{2}\beta^{jb}\beta^{ia}(\mu^{ab} + \mu^{ba})\frac{\partial^2\rho}{\partial x^j\partial x^i}\\ &= \beta^{jb}\beta^{ia}\mu^{ab}_S\frac{\partial^2\rho}{\partial x^j\partial x^i}. \end{aligned}\quad \text{(B17)}$$

With these identifications it is possible to write Eq. (B14) as

$$\frac{\partial \rho}{\partial t}+\left(\alpha^{i}+\mu^{ab}\beta^{jb}\frac{\partial}{\partial x^{j}}\beta^{ia}\right)\frac{\partial \rho}{\partial x^{i}}+\mu_{S}^{ab}\beta^{jb}\beta^{ia}\frac{\partial^{2}\rho}{\partial x^{j}\partial x^{i}}=0, \tag{B18}$$

which is the Kolmogorov backward equation associated to the SDE of Eq. (21) in the main text. It is possible to rearrange the terms of equation (B18) to obtain

$$\begin{aligned}0=&\frac{\partial \rho}{\partial t}+\left(\alpha^{i}+\frac{1}{2}(\mu^{ab}-\mu^{ba})\beta^{jb}\frac{\partial}{\partial x^{j}}\beta^{ia}\right)\frac{\partial \rho}{\partial x^{i}}\\&-\mu_{S}^{ab}\left(\beta^{ia}\frac{\partial}{\partial x^{j}}\beta^{jb}\right)\frac{\partial \rho}{\partial x^{i}}+\frac{\partial}{\partial x^{j}}\left(\mu_{S}^{ab}\beta^{jb}\beta^{ia}\frac{\partial \rho}{\partial x^{i}}\right),\end{aligned} \tag{B19}$$

from which we can see the form that Eq. (21) takes when considering the kinetic Hänggi–Klimontovich interpretation of the noise:

$$\begin{aligned}dX_{t}^{i}=&\alpha^{i}dt+\frac{1}{2}(\mu^{ab}-\mu^{ba})\beta^{jb}\frac{\partial\beta^{ia}}{\partial x^{j}}dt\\&-\mu_{S}^{ab}\beta^{ia}\frac{\partial\beta^{jb}}{\partial x^{j}}dt+\beta^{ia}\nu^{ab}\star dB_{t}^{b},\end{aligned} \tag{B20}$$

where $\star$ denotes the Hänggi–Klimontovich product. In the equilibrium case $\mu^{ab}=\mu^{ba}$ (see Appendix D) and taking the square root $G^{ab}=\mu^{ab}+\mu^{ba}=\nu^{ac}\nu^{bc}$ and defining $\hat{\beta}^{jc}=\nu^{bc}\beta^{jb}$ one obtains the Stratonovich generator

$$\frac{\partial \rho}{\partial t}+\alpha^{i}\frac{\partial \rho}{\partial x^{i}}+\frac{1}{2}\hat{\beta}^{jc}\frac{\partial}{\partial x^{j}}\left(\hat{\beta}^{ic}\frac{\partial \rho}{\partial x^{i}}\right)=0\ \text{(equilibrium)}. \tag{B21}$$

APPENDIX C: ADIABATIC ELIMINATION FOR A FUNCTIONAL OF TRAJECTORIES

The generating function the functional $\mathcal{J}$ defined in Eq. (12) reads

$$G_{s}=\langle e^{-s\mathcal{J}}\rangle \tag{C1}$$

and obeys the backward Feynman–Kac equation

$$\partial_{t}G+LG=s(h+f^{i}\alpha^{i}+\epsilon^{-1}f^{i}\beta^{ia}y^{a})G. \tag{C2}$$

In order to solve this problem by means of multiple-scale asymptotic methods let us expand the solution as

$$G=G^{(0)}+\epsilon G^{(1)}+\epsilon^{2}G^{(2)}+\cdots. \tag{C3}$$

At order ϵ^{-2} the backward equation reads

$$\left(\frac{\partial}{\partial\tau}+L_{1}\right)G^{(0)}=0, \tag{C4}$$

with solution

$$G^{(0)}=q\left(x,t,\tilde{t}\right), \tag{C5}$$

after a very fast transient.

At order ϵ^{-1} one has

$$\left(\frac{\partial}{\partial\tau}+L_{1}\right)G^{(1)}=-\left(\frac{\partial}{\partial\tilde{t}}+L_{2}-sf^{i}\beta^{ia}y^{a}\right)G^{(0)}. \tag{C6}$$

After relaxation, for L_1 to be invertible, one has to impose that the right-hand side is orthogonal to the null space of $L_1^{\dagger}$ which is generated by the stationary w.

Therefore, multiplying by w and integrating over y one obtains the solvability condition

$$\frac{\partial q}{\partial\tilde{t}}+\beta^{ia}\overline{y^{a}}\frac{\partial q}{\partial x^{i}}=s\beta^{ia}f^{i}\overline{y^{a}}q, \tag{C7}$$

where we have made use of $\int dyw=1$ and the explicit expression of L_2. Recalling that $\overline{y^{a}}=0$ as required for the proper behavior of the propagator one concludes that $\partial q/\partial\tilde{t}=0$ and $G^{(0)}$ does not depend on the fast timescale $\tilde{t}$. The formal solution reads

$$G^{(1)}=-b^{ia}\frac{\partial q}{\partial x^{i}}L_{1}^{-1}y^{a}+s\beta^{ia}f^{i}qL_{1}^{-1}y^{a}. \tag{C8}$$

The equation at order ϵ^{0} then is

$$\begin{aligned}\left(\frac{\partial}{\partial\tau}+L_{1}\right)G^{(2)}=&s(h+\alpha^{i}f^{i})G^{(0)}+sf^{i}\beta^{ia}y^{a}G^{(1)}\\&-\left(\frac{\partial}{\partial\tilde{t}}+L_{2}\right)G^{(1)}-\left(\frac{\partial}{\partial t}+L_{3}\right)G^{(0)}.\end{aligned} \tag{C9}$$

The solvability condition becomes an effective backward Kolmogorov equation on slow timescales t:

$$\begin{aligned}&\frac{\partial q}{\partial t}+\alpha^{i}\frac{\partial q}{\partial x^{i}}+\mu^{ab}\beta^{jb}\frac{\partial}{\partial x^{j}}\left(\beta^{ia}\frac{\partial q}{\partial x^{i}}\right)\\&=s(h+\alpha^{i}f^{i})q+s\mu^{ab}\beta^{jb}\frac{\partial}{\partial x^{j}}(\beta^{ia}f^{i}q)\\&\quad+s\mu^{ab}\beta^{ia}\beta^{jb}f^{j}\frac{\partial q}{\partial x^{i}}-s^{2}\mu^{ab}\beta^{ia}\beta^{jb}f^{i}f^{j}q.\end{aligned} \tag{C10}$$

This corresponds to the Feynman–Kac equation for the generating function of the functional

$$\lim_{\epsilon\to0}\mathcal{J}=\int_{t'}^{t}\left(h+\mu^{ab}\beta^{ia}\beta^{jb}\frac{\partial f^{i}}{\partial x^{j}}\right)d\tau+f^{i}\cdot dX_{\tau}^{i} \tag{C11}$$

over effective trajectories. Recasting the integral in Stratonovich form one has

$$\begin{aligned}\lim_{\epsilon\to0}\mathcal{J}=&\int_{t'}^{t}\left[h+\frac{1}{4}(\mu^{ab}-\mu^{ba})\beta^{ia}\beta^{jb}\left(\frac{\partial f^{i}}{\partial x^{j}}-\frac{\partial f^{j}}{\partial x^{i}}\right)\right]d\tau\\&+\int_{t'}^{t}f^{i}\circ dX_{\tau}^{i}.\end{aligned} \tag{C12}$$

In terms of a kinetic Hänggi–Klimontovich integral one would find

$$\lim_{\epsilon\to0}\mathcal{J}=\int_{t'}^{t}\left(h-\mu^{ba}\beta^{ia}\beta^{jb}\frac{\partial f^{i}}{\partial x^{j}}\right)d\tau+f^{i}\star dX_{\tau}^{i}. \tag{C13}$$

APPENDIX D: EQUILIBRIUM OF FAST PROCESSES AND SYMMETRIES IN μ^{ab}

In general, for any f such that $\overline{f}=0$ one has

$$\begin{aligned}(-L_1^{-1}f)(z') &= \int dz \int_{-\infty}^{t} dt' W(z,t|z',t') f(z)\\ &= \int dz \int_{-\infty}^{t} dt' \sum_E \phi_E(z')\phi_E^+(z) e^{-E(t-t')} f(z)\\ &= \sum_{E>0} E^{-1}\phi_E(z') \int dz \phi_E^+(z) f(z). \qquad \text{(D1)}\end{aligned}$$

Above, W is the propagator of the fast process generated by L_1 (with frozen-in slow variables), i.e.,

$$\left(\frac{\partial}{\partial t'} + L_1'\right) W(z,t|z',t') = 0, \qquad \text{(D2)}$$

$$\left(\frac{\partial}{\partial t} - L_1^\dagger\right) W(z,t|z',t') = 0, \qquad \text{(D3)}$$

and its biorthogonal decomposition is

$$W(z,t|z',t') = \sum_E \phi_E(z')\phi_E^+(z) e^{-E(t-t')}, \qquad \text{(D4)}$$

$$L_1\phi_E = -E\phi_E, \quad L_1^\dagger \phi_E^+ = -E\phi_E^+. \qquad \text{(D5)}$$

With this property we can express the Green–Kubo–Taylor formula for the diffusion matrix of the effective equation

$$\begin{aligned}\overline{y^b(-L_1^{-1})y^a} &= \int d\mathbf{y}' d\mathbf{y} \int_{-\infty}^{t} dt' w(\mathbf{y}') y'^b W(\mathbf{y},t|\mathbf{y}',t') y^a\\ &= \int_{-\infty}^{t} dt' \left\langle Y_t^a Y_{t'}^b \right\rangle, \qquad \text{(D6)}\end{aligned}$$

which, given that the process is stationary, can be written as

$$\overline{y^b(-L_1^{-1})y^a} = \int_0^\infty d\tau \langle Y_\tau^a Y_0^b\rangle = \mu^{ab}. \qquad \text{(D7)}$$

Since in general

$$\overline{y^b(-L_1^{-1})y^a} \neq \overline{y^a(-L_1^{-1})y^b}, \qquad \text{(D8)}$$

μ^{ab} may not be symmetric.

If the steady state of the fast variable is an equilibrium one then has

$$\phi_E^+ = w_{eq}\phi_E \qquad \text{(D9)}$$

by detailed balance and we can invert the order of the terms in Eq. (D7) since for any f and g with $\overline{f}=\overline{g}=0$,

$$\overline{g\,(-L_1^{-1}f)} = \sum_{E>0} E^{-1}\overline{f\phi_E}\;\overline{g\phi_E} = \overline{f\,(-L_1^{-1}g)}. \qquad \text{(D10)}$$

This implies that μ^{ab} is symmetric and consequently the effective equation is to be interpreted with the Stratonovich convention.

[1] E. Wong and M. Zakai, Ann. Math. Stat. **36**, 1560 (1965).
[2] C. W. Gardiner, *Handbook of Stochastic Methods* (Springer, Berlin, 1997).
[3] P. Hänggi and P. Jung, Adv. Chem. Phys. **89**, 239 (1995).
[4] G. A. Pavliotis and A. M. Stuart, *Multiscale Methods: Averaging and Homogenization* (Springer, New York, 2008), Vol. 53.
[5] K. Itô, Proc. Imp. Acad. **20**, 519 (1944).
[6] R. L. Stratonovich, *Conditional Markov Processes and their Application to the Theory of Optimal Control* (American Elsevier Pub. Co., New York, 1968).
[7] P. Hänggi and H. Thomas, Phys. Rep. **88**, 207 (1982).
[8] Yu. L. Klimontovich, Phys. A (Amsterdam, Neth.) **163**, 515 (1990).
[9] H. J. Kushner and H. Huang, SIAM J. Appl. Math. **40**, 528 (1981).
[10] H. J. Kushner and H. Huang, Stochastics **14**, 85 (1985).
[11] R. Graham and A. Schenzle, Phys. Rev. A **26**, 1676 (1982).
[12] R. Kupferman, G. A. Pavliotis, and A. M. Stuart, Phys. Rev. E **70**, 036120 (2004).
[13] G. A. Pavliotis and A. M. Stuart, Multiscale Model. Simul. **4**, 1 (2005).
[14] D. Givon and R. Kupferman, Phys. A (Amsterdam, Neth.) **335**, 385 (2004).
[15] P. Billingsley, *Probability and Measure* (John Wiley & Sons, New York, 1995).
[16] V. Y. Chernyak, M. Chertkov, and C. Jarzynski, J. Stat. Mech.: Theory Exp. (2006) P08001.
[17] J. M. Sancho, M. San Miguel, and D. Dürr, J. Stat. Phys. **28**, 291 (1982).
[18] M. Matsuo and S.-i. Sasa, Phys. A (Amsterdam, Neth.) **276**, 188 (2000).
[19] A. W. C. Lau and T. C. Lubensky, Phys. Rev. E **76**, 011123 (2007).
[20] M. Yang and M. Ripoll, Phys. Rev. E **87**, 062110 (2013).