Generalized information entropies depending only on the probability distribution

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In the framework of superstatistics it has been shown that one can calculate the entropy of nonextensive statistical mechanics. We follow a similar procedure; we assume a $\Gamma(\chi^2)$ distribution depending on β that also depends on a parameter p_l. From it we calculate the Boltzmann factor and show that it is possible to obtain the information entropy $S = k\sum_{l=1}^{\Omega} s(p_l)$, where $s(p_l) = 1 - p_l^{p_l}$. By maximizing this information measure, p_l is calculated as function of βE_l and, at this stage of the procedure, p_l can be identified with the probability distribution. We show the validity of the saddle-point approximation and we also briefly discuss the generalization of one of the four Khinchin axioms. The modified axioms are then in accordance with the proposed entropy. As further possibilities, we also propose other entropies depending on p_l that resemble the Kaniakadis and two possible Sharma-Mittal entropies. By expanding in series all entropies in this work we have as a first term the Shannon entropy.

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I. INTRODUCTION

Since the seminal work by Shannon [1] about an information entropy to quantify predictability in a stochastic process, several other measures of information have been proposed in the literature [2]. By maximizing these information measures [3], their corresponding probability distributions can be calculated. Some of these generalized information and entropy measures and their potential physical applications have been discussed elsewhere [4].

Also, by considering nonequilibrium systems with a long-term stationary state that possess a spatiotemporally fluctuating intensive quantity, more general statistics have been formulated, known as superstatistics [5]. In these cases, the temperature is a fluctuating quantity among other various available intensive variables. The macroscopic system is considered as made up of many smaller cells that are temporarily in local equilibrium. Within each cell β, the inverse temperature, is approximately constant. Each cell is large enough to obey statistical mechanics but has a different β assigned to it, according to a general distribution $f(\beta)$; from it one can get an effective Boltzmann factor

$$B(E) = \int_0^\infty d\beta f(\beta) e^{-\beta E}, \qquad (1)$$

where E is the energy of a microstate associated with each of the considered cells. The ordinary Boltzmann factor is recovered for $f(\beta) = \delta(\beta - \beta_0)$. One can, however, consider other distributions for the temperature that will lead to their corresponding Boltzmann factors. The $\Gamma(\chi^2)$, log-normal, and F distributions were studied in this context as well as their corresponding Boltzmann factors. The analysis of these $B(E)$ showed that all these statistics present the same behavior for small variance of the fluctuations [5]. An extended discussion exists in the literature analyzing the possible viability of this kind of model to explain several physical phenomena [4,6].

In [7] a new formalism was developed to deduce entropies associated with Boltzmann factors $B(E)$. Following this procedure, the Boltzmann Gibbs entropy and the so-called nonextensive statistical mechanics entropy [8] S_q [corresponding to the Γ distribution (χ^2) and depending on a constant parameter q] were obtained. They expressed the entropy as $S = k\sum_{l=1}^{\Omega} s(p_l)$ in terms of a generic $s(x)$ and were able to calculate it, where x is at this stage a parameter and the sum over this parameter is defined as the entropy. Identifying this parameter as the probability distribution, it is possible to determine it by maximizing the appropriate information measure. For the log-normal and F distributions and other distributions it is not possible to get closed analytic expressions for their associated entropies and the calculations were performed numerically [7] utilizing the corresponding $B(E)$ in each case.

In a previous work by Obregón [9], the effect of assuming a generalized non-Boltzmann distribution depending only on a parameter p_l (i.e., the probability distribution after maximizing the information measure) was studied and applied for deriving the corresponding entropy of black holes, based on the Bekenstein-Hawking entropy model that identifies entropy with the black hole's horizon area. Since the Einstein equation can then be derived as an equation of state and gravity can be established as an entropic force, Obregón [9] derived corrections to Newton's law depending on the inverse power of the Planck's length $l_p = G\hbar/c^3$. This procedure is interesting in a general context, since it allows us to model modifications of a force law by considering alternate formulations of the entropy associated with the system, and vice versa.

In this work, we review this entropy model and its possible extension to other systems besides black holes, studying a generalized distribution and its associated Boltzmann factor and entropy. Based on these results, we present other information entropies that also depend on p_l only. These models resemble the already well known $f(\beta)$, $B(E)$, and information entropies depending on a constant parameter q, whose possible applications and limitations, in relation with certain physical systems, have been discussed in the literature [4,6,8]. As we will discuss, the procedure given by Obregón [9] could be useful in the study of systems with a reduced number of microstates, such as systems under strong confinement, where the proposed entropy can be used to

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deduce corrections to van der Waals forces (vdW), that would also alter the corresponding equation of state. Modifications to van der Waals forces arise when we consider effects due to confinement and the effect of substrates. Sinanoglu and Pitzer [10] used third-order quantum perturbation theory to demonstrate that the standard Lennard-Jones model used to represent vdW forces in bulk phases is modified for adsorption of monolayers in solid substrates, with the presence of a new long-range repulsion originated from a quantum mechanical fluctuation that counterbalances the usual London attraction. This modification is the main mechanism that explains the changes on the phase diagram of an adsorbed substance (e.g., the critical temperature) with respect to its behavior in a bulk phase [11]. Similar effects have been proposed for the behavior of colloidal particles at the air-water interface [12] observed in experimental and simulated systems [13]. Considering the inverse problem, that is, the evaluation of pair-potential interactions based on the behavior of the entropy, suggests that corrections to the vdW forces observed in confined fluids also could be explained in terms of a nonextensive entropy as those proposed in this article.

In Sec. II, we first express [5] the Γ (χ^2) distribution and its corresponding Boltzmann factor in terms of the parameter p_l, and then the corresponding entropy [7] is obtained, as discussed in [9]. We will maximize this information entropy to identify our parameter and to get the corresponding $\{p_l\}$ probability distribution. In our model, the entropy resembles the one proposed in nonextensive statistical mechanics resulting also by assuming a (χ^2) distribution. In our case, however, it does not depend on a free constant q, but instead on a parameter p_l that as mentioned results in being the probability distribution. We will show that this entropy can be expanded in a series, whose first term corresponds to the Shannon entropy [1].

In Sec. III, we will consider other well-known information entropies [2] that can be generalized; the proposed entropies will not depend on the constant q, but on the probability p_l in each case. As examples we will consider the p_l-modified Kaniadakis and Sharma-Mittal entropies, which, as expected, will have also the Shannon entropy as a first term when expanded in series.

To complete this work we analyze two further aspects, the validity of the saddle-point approximation [14–16], and following [4,17,18] we also discuss a generalized version of the Khinchin axioms. As shown in [4], three of these axioms are kept and the fourth of them is replaced by a more general version similar to the one proposed in [17], leading to Tsallis entropy. We will study these two aspects only for the Boltzmann factor $B_{p_l}(E)$ and its associated entropy, arising from the proposed Γ distribution depending on p_l. For the other entropies we will not discuss these two features. However, for example, for these other entropies a similar procedure to that followed in [4,18] could be worked out in relation with an extension of the Khinchin axioms.

Section IV is dedicated to discussing the saddle-point approximation [16]. We will consider it in relation to the Boltzmann factor $B_{p_l}(E)$ arising from the p_l-Γ distribution. In Sec. V, we discuss how one can replace the fourth Khinchin axiom to get a set of axioms from which the entropy proposed here follows. As mentioned, this entropy is obtained from the Boltzmann factor $B_{p_l}(E)$ and this one from the Γ distribution. Section VI is devoted to discussion and outlook.

II. ENTROPY FROM THE BOLTZMANN FACTOR

We begin by assuming a Γ (or χ^2) inverse temperature β distribution depending on a parameter p_l, to be identified with the probability associated with the microscopic configuration of the system. We may write these parameter p_l-Γ distribution as

$$f_{p_l}(\beta) = \frac{1}{\beta_0 p_l \Gamma\left(\frac{1}{p_l}\right)} \left(\frac{\beta}{\beta_0}\frac{1}{p_l}\right)^{\frac{1-p_l}{p_l}} e^{-\beta/\beta_0 p_l}, \tag{2}$$

where β_0 is the average inverse temperature.

Integration over β yields the generalized Boltzmann factor

$$B_{p_l}(E) = (1 + p_l\beta_0 E)^{-\frac{1}{p_l}}. \tag{3}$$

As shown in [5], this kind of expression can be expanded, for small $p_l\beta_0 E$, to get

$$B_{p_l}(E) = e^{-\beta_0 E}\left[1 + \tfrac{1}{2}p_l\beta_0^2 E^2 - \tfrac{1}{3}p_l^2\beta_0^3 E^3 + \cdots\right]. \tag{4}$$

The examples studied in [5] have been nicely addressed in [7] in order to deduce the entropies from their corresponding Boltzmann factors. Another possible way to reconstruct the entropy has been proposed in [19,20]; this approach provides other expressions. In [21] it has been shown that there exists a duality between these two procedures. We will restrict ourselves to the first proposal, where the Boltzmann-Gibbs entropy and the nonextensive statistical mechanics entropy can be obtained in a closed analytic form [7]. However, the entropies corresponding to the Boltzmann factors associated with other distributions like the log-normal or the F distributions cannot be obtained analytically and were calculated numerically. Following [7] and [9], we present the procedure to obtain the entropy corresponding to the $f(\beta)$ distribution Eq. (2) and to its associated generalized Boltzmann factor Eq. (3). We begin by defining the entropy $S = k\sum_{l=1}^{\Omega} s(p_l)$ in terms of a generic $s(p_l)$, where p_l can be considered at this moment an arbitrary parameter; for $s(x) = -x\ln x$ the Shannon entropy is recovered. As shown in [7] it is possible to express $s(x)$ and a generic internal energy $u(x)$ in terms of integrals on a function $E(y)$ that is obtained from the Boltzmann factor $B(E)$ of interest. By these means $s(x)$ and $u(x)$ can be written as

$$s(x) = \int_0^x dy \frac{\alpha + E(y)}{1 - E(y)/E^*} \tag{5}$$

and

$$u(x) = (1 + \alpha/E^*)\int_0^x \frac{dy}{1 - E(y)/E^*}, \tag{6}$$

where $E(y)$ is to be identified with the inverse function of $B_{p_l}(E)/\int_0^\infty dE' B_{p_l}(E')$. One selects first the $f(\beta)$ of interest; then $B(E)$ is calculated and the integral $\int_0^\infty B(E')dE'$ is performed. Inverting the axes of the variables, $E(y)$ for several superstatistics can be found [5], and from it E^*. In our case, the starting points are the distribution Eq. (2) and the Boltzmann factor Eq. (3). $E(y)$ and E^* are given by

$$E(y) = \frac{y^{-x} - 1}{x}, \tag{7}$$

$$E^* = -\frac{1}{x}. \tag{8}$$

A straightforward calculation gives for $u(x)$ and $s(x)$

$$u(x) = x^{x+1}, \tag{9}$$

$$s(x) = 1 - x^x, \tag{10}$$

where α has been determined by means of the condition $u(1) = 1$.

Expressions Eqs. (9) and (10) fulfill the expected conditions for the entropy and the energy $s(0) = 0, u(0) = 0$ and $u(1) = 1, s(1) = 0$. By these means the entropy results in

$$S = k \sum_{l=1}^{\Omega} \left(1 - p_l^{p_l}\right), \tag{11}$$

where k is the conventional constant and $\sum_{l=1}^{\Omega} p_l = 1$. The expansion of Eq. (11) gives

$$-\frac{S}{k} = \sum_{l=1}^{\Omega} \left[p_l \ln p_l + \frac{(p_l \ln p_l)^2}{2!} + \frac{(p_l \ln p_l)^3}{3!} + \cdots \right], \tag{12}$$

where the first term is the Shannon entropy.

Using this last expression, the corresponding functional including restrictions is given by

$$\Phi = \frac{S}{k} - \gamma \sum_{l=1}^{\Omega} p_l - \beta \sum_{l=1}^{\Omega} p_l^{p_l+1} E_l, \tag{13}$$

where the second restriction concerns the average value of the energy and γ and β are Lagrange parameters, and then by maximizing Φ, p_l is obtained as

$$1 + \ln p_l + \beta E_l (1 + p_l + p_l \ln p_l) = p_l^{-p_l}. \tag{14}$$

The dominant term in this expression corresponds to the Gibbs-Boltzmann prediction, $p_l = e^{-\beta_o E_l}$. In general, however, we cannot analytically express p_l as a function of βE_l. In Fig. 1, p_l is given as a function of the reduced energy βE_l. We notice that for relative large values of βE_l the usual values for p_l coincide with the ones given by Eq. (14). As expected, they coincide also for $p_l \approx 1$.

As we have shown, by choosing $f_{p_l}(\beta)$ Eq. (2), $B_{p_l}(E)$ Eq. (3) is obtained by integrating over β; by inverting the axes of the variable the inverse function $E(y)$ Eq. (7) and $E^\star$ Eq. (8) can be found. This procedure has allowed us to calculate $u(x)$ Eq. (9) and $s(x)$ Eq. (10) and consequently the entropy Eqs. (11) and (12). If we assume in $f_{p_l}(\beta)$ Eq. (2) the equiprobable condition, $p_l = \frac{1}{\Omega}$, then the corresponding distribution is given by

$$f_\Omega(\beta) = \frac{\Omega}{\beta_0 \Gamma(\Omega)} \left(\frac{\beta\Omega}{\beta_0}\right)^{\Omega-1} e^{-\frac{\beta\Omega}{\beta_o}}, \tag{15}$$

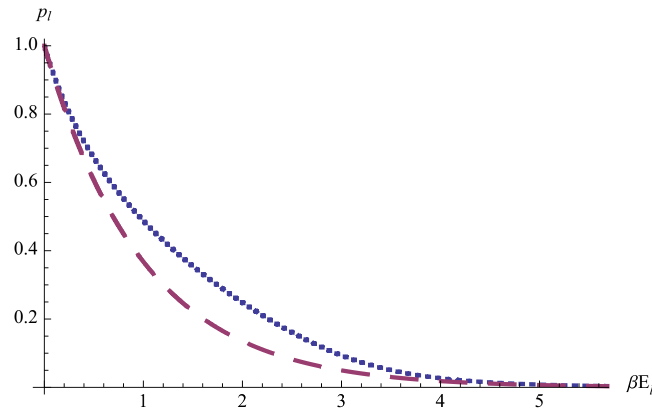

FIG. 1. (Color online) Comparison of the two probabilities. Blue dotted line corresponds to $p_l = f(\beta E_l)$, Eq. (14), and red dashed line to the standard one, $p_l = e^{-\beta E_l}$.

where the Boltzmann factor and the entropy are now

$$B_\Omega(E) = (1 + \beta_0 E/\Omega)^{-\Omega}, \tag{16}$$

$$S = k\Omega\left[1 - \frac{1}{\Omega^{\frac{1}{\Omega}}}\right], \tag{17}$$

or, in terms of the Boltzmann entropy, $S_B = k \ln \Omega$,

$$\frac{S}{k} = \frac{S_B}{k} - \frac{1}{2!} e^{-S_B/k} \left(\frac{S_B}{k}\right)^2 + \frac{1}{3!} e^{-\frac{2S_B}{k}} \left(\frac{S_B}{k}\right)^3 \cdots. \tag{18}$$

Figures 2 and 3 show the Boltzmann entropy $\frac{S_B}{k}$ and the entropy $\frac{S}{k}$ given by expression Eq. (17). As mentioned in the Introduction, it was shown in [9] in relation with the entropy of a black hole that if we associate its entropy, depending linearly on its area, with $\frac{S_B}{k}$ the standard entropy, then the entropy $\frac{S}{k}$ will be given as a complicated function of the area by means of Eqs. (17) and (18). This would imply a modification to the laws of Newton and to general relativity according with the possibility that gravity could be thought of as an equation of state [22], explained as an entropic force [23–25]. However,

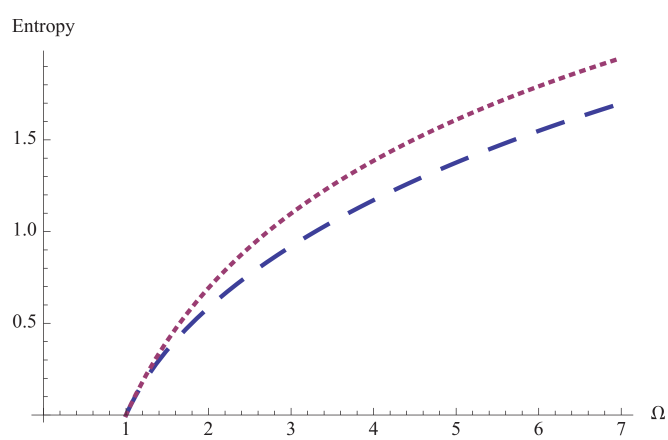

FIG. 2. (Color online) Entropies as function of Ω (small). Blue dashed and red dotted lines correspond to $\frac{S}{k}$, Eq. (17), and $\frac{S_\beta}{k}$, respectively ($p_l = 1/\Omega$ equipartition).

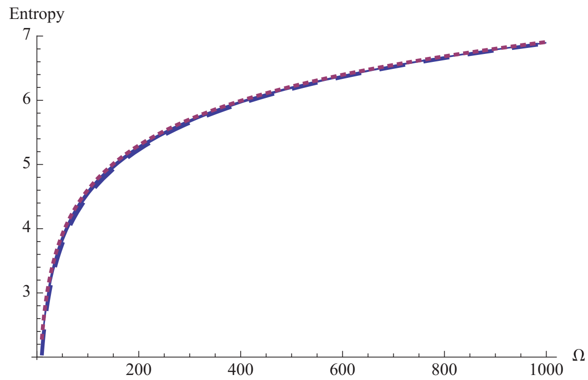

FIG. 3. (Color online) Entropies as function of Ω (large). Blue dashed and red dotted lines correspond to $\frac{S}{k}$, Eq. (17), and $\frac{S_\beta}{k}$, respectively ($p_l = 1/\Omega$ equipartition).

for most gravitational systems one expects a large Ω, namely large $\frac{S_B}{k}$, and this will not greatly differ from $\frac{S}{k}$, Eq. (18) and Fig. 3.

We notice that in the range of low values of Ω the entropies $\frac{S_\beta}{k}$ and $\frac{S}{k}$ differ. Instead, for large Ω the two entropies essentially coincide. Since Ω is basically a measure of the phase space volume, what we are finding is that for systems with reduced number of microstates the entropies become different, whereas for the opposite case they will be essentially identical. Then, we could conclude from here that the model derived for the particular case of black holes in [9] could be useful when we have a clear indication of restriction of available states, like by strong confinement of fluids or low temperatures.

The entropy derived in [9] and whose main properties have been studied here, obtaining Eqs. (11) and (17), has as an important feature to be independent of any arbitrary constant parameter, and to depend only on the probability distribution p_l Eq. (14) associated with the microscopic configuration of the system. Its expansion provides as a first term the Shannon entropy Eq. (12) and correspondingly the Boltzmann entropy Eq. (18). This entropy corresponds to the Γ distribution Eqs. (2) and (15).

III. OTHER GENERALIZED INFORMATION ENTROPIES IN TERMS OF p_l

More general measures of information than the Shannon entropy have been proposed in the literature [2]. Maximizing these entropies subject to suitable constraints allow us to obtain associated probability distributions. In [4,6,8] several of these entropies have been reviewed and their potential physical applications discussed.

Similar entropies depending only on p_l can be proposed. As examples let us consider modified Kaniadakis and Sharma-Mittal entropies. The Kaniadakis entropy is defined by the expression

$$S_\kappa = -k \sum_l^{\Omega} \frac{p_l^{1+\kappa} - p_l^{1-\kappa}}{2\kappa}. \tag{19}$$

This is an entropy which reduces to the original Shannon entropy for $\kappa = 0$ [4]. Inspired by this Kaniadakis entropy, we propose to consider the following generalized entropy,

$$S = -k \sum_l^{\Omega} \frac{p_l^{p_l} - p_l^{-p_l}}{2}; \tag{20}$$

the two terms in this expression can be expanded in a similar manner as in Eqs. (11) and (12) to get

$$-\frac{S}{k} = \sum_l^{\Omega} \left[p_l \ln p_l + \frac{(p_l \ln p_l)^3}{3!} + \cdots \right]. \tag{21}$$

The first term corresponds to the Shannon entropy. It is interesting to notice that the expansion Eq. (21) of the entropy Eq. (20) differs from the expansion Eq. (12) of the entropy Eq. (11) corresponding to the Γ distribution Eq. (2) and that we have analyzed in more detail in the previous sections; in Eq. (21) only the "odd" terms in the expansion arise.

We consider now the Sharma-Mittal entropies; these are two constant parameter families of entropic forms. They can be written as

$$S_{\kappa,r} = -k \sum_l^{\Omega} p_l^r \left(\frac{p_l^\kappa - p_l^{-\kappa}}{2\kappa} \right). \tag{22}$$

We now assume that κ and r are functions only of p_l; we get the entropy Eq. (11) for $-2\kappa = p_l$ and $r = \frac{p_l}{2} + 1$ and the entropy Eq. (20) is obtained for $r = 1$ and $\kappa = p_l$. These Sharma-Mittal entropies can be generalized in several manners as functions of the probability p_l by means of other different assumptions. Two of them correspond to the entropies given in Eqs. (11) and (20). Other entropies considered, for example in [2], can also be generalized as functions of p_l. As examples we have analyzed the p_l-dependent generalized Kaniadakis and two Sharma-Mittal entropies that reduce to the entropies Eqs. (11) and (20).

IV. THE SADDLE-POINT APPROXIMATION, THE p_l-Γ DISTRIBUTION CASE

We have already obtained the low-energy asymptotics Eq. (4) for the Boltzmann factor Eq. (3) arising from the Γ distribution Eq. (2). This approximation represents the leading order correction to ordinary statistical mechanics systems with temperature fluctuations for small values of the energy E. The zeroth-order approximation to this Boltzmann factor corresponds, as expected, to the Boltzmann statistics $B(E) \sim e^{-\beta_0 E}$ with inverse temperature β_0.

To find the high-energy asymptotics of $B(E)$ we follow [16] where the fact is used that Eq. (1) has the form of a Laplace integral for $E \to \infty$. In this limit, the integral can be approximated by its largest integrand. This is the essence of the saddle-point approximation, namely the Laplace method. The conditions of applicability of this approximation method are basically the conditions that one assumes regarding the shape of $f(\beta)$ and its differentiability. By putting $B_{p_l}(E)$ in the form

$$B_{p_l}(E) = \int_0^\infty e^{-\beta E + \ln f(\beta)} d\beta, \tag{23}$$

one attempts to find the unique value of β which maximizes the exponential function

$$\Psi(\beta,E) = -\beta E + \ln f(\beta), \tag{24}$$

for any large enough energy value E. The value of β maximizing $\Psi(\beta,E)$ for large fixed E is denoted β_E. Having assumed $f(\beta)$ to be unimodal ensures the uniqueness of β_E. Since $f(\beta)$ is unimodal, $\ln f(\beta)$ must be a concave function of β. In this manner the maximum of $\Psi(\beta,E)$ can only be obtained at the single point β_E. It is such that

$$E = [\ln f_{p_l}(\beta)]' = \frac{f'_{p_l}(\beta)}{f_{p_l}(\beta)}. \tag{25}$$

In this way we get β_E and in the limit $E \to \infty$

$$B_{p_l}(E) \sim e^{\Psi(\beta_E,E)} = f_{p_l}(\beta_E)e^{-\beta_E E}. \tag{26}$$

This saddle-point or Laplace approximation can be improved by using a Gaussian approximation of the integrand in Eq. (23). The refined high-energy asymptotics results in

$$B_{p_l}(E) \sim \frac{f_{p_l}(\beta_E)e^{-\beta_E E}}{\sqrt{-[\ln f_{p_l}(\beta_E)]''}}. \tag{27}$$

The approximations of $B(E)$ Eqs. (26) and (27) show that the mixture of Boltzmann statistics defining $B(E)$ reduces at high energy E to a particular Boltzmann statistics, like in the equilibrium situation, but now this Boltzmann statistics is a function of β_E which depends on E, the energy considered and determined by $f_{p_l}(\beta)$ Eq. (25). The long-term stationary behavior of the nonequilibrium system considered for high values of E is dominated by the equilibrium behavior of a subset of cells having an inverse temperature close to β_E. We now consider the asymptotic behavior of $B_{p_l}(E)$ Eq. (1) for $E \to \infty$ and $f_{p_l}(\beta)$ given by Eq. (2). We first solve Eq. (25) to find β_E for this case. One gets

$$\beta_E = \frac{(1-p_l)\beta_0}{E p_l \beta_0 + 1}, \tag{28}$$

as expected [16] as $E \to \infty$, $\beta_E \to 0$. We want now to calculate $B_{p_l}(E)$ Eq. (26) for this β_E. This can be expressed as

$$B_{p_l}(E) = \frac{1}{\beta_0 p_l \Gamma\left(\frac{1}{p_l}\right)} e^{-\frac{1-p_l}{p_l}[\ln\frac{p_l}{1-p_l} + \ln(E p_l \beta_0 + 1) + 1]}. \tag{29}$$

For $E \to \infty$ and a certain value of p_l,

$$B_{p_l}(E) \sim e^{-\frac{1-p_l}{p_l}\ln E} \sim E^{1-\frac{1}{p_l}}. \tag{30}$$

The more refined approximation Eq. (27) can be obtained by dividing Eq. (30) by $\sqrt{-[\ln f_{p_l}(\beta_E)]''}$; this in the high-energy limit is proportional to E. In this way

$$B_{p_l}(E) \sim E^{-1/p_l}. \tag{31}$$

In this more accurate calculation, we get, asymptotically, a decaying power law for the effective Boltzmann factor. As mentioned in [16] power-law superstatistics seem to be physically relevant for several physical systems, among others defect turbulence [26] and cosmic-ray statistics [27].

V. AN APPROPRIATE INFORMATION MEASURE

The well-established four Khinchin axioms are extensively discussed and presented in [4]. As known, the celebrated Shannon entropy $S = -k\sum_l^{\Omega} p_l \ln p_l$ satisfies all these axioms. It has been however argued in the literature that the fourth of these axioms is not an obvious property [4,17,18]. We will concentrate our discussion on it. This fourth axiom deals with the composition of two systems I and II (not necessarily independent). We denote the probabilities of the first system as p_i^I, those of the second system as p_j^{II}. The joint system is described by the joint probabilities $p_{ij}^{I,II} = p_i^I p^{II}(j|i)$, where $p^{II}(j|i)$ is the conditional probability of event j in system II under the condition that event i has already occurred in system I. The conditional information of system II formed with the conditional probabilities $p^{II}(j|i)$ is denoted by $I(\{p^{II}(j|i)\})$, under the condition that system I is in the state i. The fourth axiom states that the conditional information is related by

$$I\left(\left\{p_{ij}^{I,II}\right\}\right) = I\left(\left\{p_i^I\right\}\right) + \sum_i p_i^I I(\{p^{II}(j|i)\}). \tag{32}$$

This axiom postulates that the information measure should be independent of the way the information is collected. We can collect the information in II, assuming a given event i in system I, and then sum the result over all possible events i in the system I, weighting with the probabilities p_i^I. If the two systems are independent the probability of the two systems factorizes $p_{ij}^{I,II} = p_i^I p_j^{II}$. Only in this case Eq. (32) reduces to

$$I\left(\left\{p_{ij}^{I,II}\right\}\right) = I\left(\left\{p_i^I\right\}\right) + I\left(\left\{p_j^{II}\right\}\right), \tag{33}$$

the rule of additivity of information for independent systems. From a physical point of view this axiom Eq. (32) is not an obvious property. Should the information be considered as independent from the way we collect it? In complex systems, the order in which the information is collected can be very relevant. This has led to the replacement of the fourth Khinchin axiom by something more general. In particular in [17] it was shown that the Tsallis entropy follows uniquely by replacing only the fourth axiom Eq. (32) by the more general version

$$S_q^{I,II} = S_q^I + S_q^{II|I} - (q-1)S_q^I S_q^{II|I}. \tag{34}$$

The meaning of this new axiom is that if we collect information from two subsystems, the total information should be the sum of the information collected from system I and the conditional information from system II, plus a correction term. *A priori* this correction term can be anything. One restricts the possible assumptions to

$$S^{I,II} = S^I + S^{II|I} + g(S^I, S^{II|I}), \tag{35}$$

where g is some function. One of the simplest forms is of the kind given by Eq. (34). We may as well formulate another axioms which then would lead to other possible information measures. This is the case, for example, in [18] where a set of axioms has been assumed that leads to the Sharma-Mittal entropy.

The entropies Eqs. (11) and (17) associated with the Γ distribution Eqs. (2) and (15) are composable. Suppose the two systems I and II are not independent. In this case one can still write the joint probability p_{ij} as a product of p_i and the condition probability $p(j|i)$; the probability of event j under the condition that event i has already occurred is $p_{ij} = p(j|i)p_i$.

Then the conditional entropy associated with system II, under the condition that system I is in state i, is ($k = 1$)

$$S^{II|i}_{p_{j|i}} = 1 - \sum_j p_{(j|i)}^{p^{(j|i)}}. \tag{36}$$

One can then verify the condition

$$S^I_{p_i} + \sum_i p_i^{Ip_i^I} S^{II|i}_{p(j|i)} = S^{I,II}_{p_{ij}}. \tag{37}$$

This relation is similar to the original fourth axiom, Eq. (32); one has however the probability with an exponent that is the probability itself. We weigh now the events in system I with $p_i^{p_i}$ instead of p_i. Hence, the p_l-dependent information considered in Eq. (11) is not independent of the way it is collected for the various subsystems; namely the information associated with Eq. (37) will not be additive [Eq. (33)], for the case of two independent systems for which, as mentioned, $p_{ij}^{I,II} = p_i^I p_j^{II}$.

VI. DISCUSSION AND OUTLOOK

The distribution Eq. (2), Boltzmann factor Eqs. (3) and (4), and entropies Eqs. (11), (17), (20), and (22) proposed in this work depend on a parameter p_l. By maximizing the functional Eq. (13) using the entropy Eq. (11), p_l is identified with the microscope probability distribution and its dependence on βE_l has been obtained Eq. (14). Moreover, the generalized information entropies proposed in this work Eqs. (11), (20), and (22) can be expanded to get as a first term the Shannon entropy Eqs. (12) and (21).

We have analyzed the difference between the entropy derived in this work and the Shannon expression, and we have found that they differ for low values of Ω. In the context of condensed matter, this effect implies that both approaches will differ for systems with reduced number of microscopical states, like in systems under strong confinement. We can expect that for small amounts of substances confined within pores, the entropy model of Ref. [9] will give noticeable differences with respect to the standard Gibbs-Boltzmann statistics. We expect that modifications to the van der Waals equation and forces in this case will be present and a route to derive them could be from the entropy of the system, as done in [9] for the gravitational force. The proposed entropy Eqs. (11), (17), and (18) defines now a nonextensive statistical mechanics and consequently a modified thermodynamics. As mentioned, in the Introduction considering an inverse procedure, one would get from this entropy an associated equation of state and pair-potential interactions that would alter the van der Waals forces. As discussed, it is known that by considering effects due to confinement and the effect of substrates one gets modifications to these forces [10–13].

We have analyzed the saddle-point approximation for the p_l-Γ distribution and have got, in the high-energy limit, an asymptotically decaying power law for the effective Boltzmann factor Eq. (31); this is the physically expected appropriate behavior [16]. We have also shown how the fourth Khinchin axiom should be modified so that the associated entropy results in Eq. (11). In Fig. 1 we have compared the probability distribution arising from this entropy with the standard one. In Figs. 2 and 3 we have compared (for $p_l = \frac{l}{\Omega}$) the entropy Eq. (17) with Boltzmann entropy. As shown, the entropies Eqs. (11), (20), and (22) resemble the well-known nonextensive statistical mechanics entropy, the Kaniadakis and two Sharma-Mittal entropies correspondingly (other well-known q entropies can also be generalized in terms of p_l). For the entropy depending on a constant parameter q, there exists an extensive literature on their physical reach; their relation with the experiments is under discussion [4,6,8]. Also, other theoretical developments have been proposed [19–21]. We will study, in further work, some of the p_l-dependent entropies proposed here in connection with these aspects.

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