

Boosting work characteristics and overall heat-engine performance via shortcuts to adiabaticity: Quantum and classical systems

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Under a general framework, shortcuts to adiabatic processes are shown to be possible in classical systems. We study the distribution function of the work done on a small system initially prepared at thermal equilibrium. We find that the work fluctuations can be significantly reduced via shortcuts to adiabatic processes. For example, in the classical case, probabilities of having very large or almost zero work values are suppressed. In the quantum case, negative work may be totally removed from the otherwise non-positive-definite work values. We also apply our findings to a micro Otto-cycle-based heat engine. It is shown that the use of shortcuts, which directly enhances the engine output power, can also increase the heat-engine efficiency substantially, in both quantum and classical regimes.

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I. INTRODUCTION

Shortcuts to adiabatic processes (STA) constitute a timely topic of broad interest [1–13], with several experimental realizations reported recently [7–9]. One important question is whether such STA are unique in quantum mechanics.

In this work we develop a simple and *general* framework for *classical* STA. Analogous to the classical adiabatic theorem, the involved Hamiltonian is assumed to be integrable [14]. A generic control field to achieve classical STA is then found. This is important because (i) classical STA may help design quantum STA (e.g., by quantizing the classical control field) and (ii) STA may be more general and robust than previously thought. Indeed, by applying our formalism to a parametric oscillator, the control Hamiltonian is precisely the classical limit of an early quantum result [15].

To make a different connection between STA and nonequilibrium statistical mechanics, we ask how STA impact the distribution function of the work done on a single system initially prepared in thermal equilibrium. We find that the work fluctuations can be significantly reduced. For example, both the long-tail part and the almost zero part of the work function in a classical example are substantially suppressed, leading to a faster convergence towards Jarzynski's equality. Remarkably, in the corresponding quantum case negative work values of the otherwise non-positive-definite quantum work may be removed *completely*. We then show how STA implemented in a prototypical micro heat engine [16,17] can increase engine efficiency and the power output at the same time.

II. SHORTCUTS TO ADIABATIC PROCESSES: FROM QUANTUM TO CLASSICAL SYSTEMS

Consider first STA for a quantum system (also called transitionless driving in Ref. [2]) for a quantum system with a nondegenerate Hamiltonian $\hat{H}_0[\lambda(t)]$, parametrized by $\lambda(t)$, with the n th instantaneous energy eigenstate given by $|n[\lambda(t)]\rangle$

possessing eigenenergy $E_n[\lambda(t)]$. The quantum adiabatic theorem states that if λ changes slowly enough, the system initially prepared on an eigenstate $|n[\lambda(0)]\rangle$ will continue to stay on the instantaneous energy eigenstate $|n[\lambda(t)]\rangle$, i.e.,

$$\hat{U}(t,0)|n[\lambda(0)]\rangle \approx e^{i\phi(t)}|n[\lambda(t)]\rangle, \quad (1)$$

where $\hat{U}(t,0)$ is the unitary time evolution operator and $\phi(t)$ includes a dynamical phase and a geometrical phase [18]. This solution clearly indicates the invariance of population on each instantaneous eigenstate. Such a type of time evolution can be accelerated by adding a control Hamiltonian $\hat{H}_C(t)$. In particular, by reverse engineering [1,2] one finds

$$\hat{H}_C(t) = i\hbar \frac{\partial U(t,0)}{\partial t} U^\dagger(t,0) - \hat{H}_0[\lambda(t)]. \quad (2)$$

This \hat{H}_C assists an adiabatic process associated with H_0 (the process is certainly nonadiabatic with respect to the full Hamiltonian $\hat{H}_0 + \hat{H}_C$), regardless of how fast $\lambda(t)$ varies. Note that $\hat{H}_C \propto \dot{\lambda}$ [1,2]. In the limit $\dot{\lambda} \rightarrow 0$, \hat{H}_C also approaches zero and the quantum adiabatic theorem is recovered.

We now show that classical adiabatic processes also have shortcuts [12,19] and the physics is analogous to the above quantum picture. Though our treatment applies to arbitrary multidimensional integrable classical systems, for convenience we consider here a time-dependent classical Hamiltonian $H_0[p,q,\lambda(t)]$ with one degree of freedom [(p,q) are phase space variables]. To study classical adiabatic processes we further assume that $H_0[p,q,\lambda]$ at a fixed λ can be written as $\tilde{H}_0(I,\lambda)$: a function of the action variable I , but not a function of the angle variable θ [20]. For a general time-dependent $\lambda(t)$, the correct Hamiltonian $K_0[I,\theta,t]$ in the action-angle representation is obtained by use of a time-dependent type-II generating function $F_2[I,q,\lambda(t)]$ between (p,q) and (I,θ) [20], i.e.,

$$K_0[I,\theta,t] = \tilde{H}_0[I,\lambda] + \left[\left(\frac{\partial F_2(I,q,\lambda)}{\partial \lambda} \right)_{I,q} \dot{\lambda} \right]_{q=q(I,\theta,\lambda)}. \quad (3)$$

Classical adiabatic theorem then states that I is approximately a constant if λ is changing slowly.

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To realize classical STA, i.e., to have a constant I with fast changes in λ , we construct a control Hamiltonian $K_C[I, \theta, t]$ on top of $K_0[I, \theta, t]$. One obvious but exact solution is

$$K_C[I, \theta, t] = - \left[\left(\frac{\partial F_2(I, q, \lambda)}{\partial \lambda} \right)_{I, q} \dot{\lambda} \right]_{q=q(I, \theta, \lambda)}. \quad (4)$$

The total Hamiltonian then becomes

$$K = K_0 + K_C = \tilde{H}_0[I, \lambda], \quad (5)$$

which is θ independent and as such I is a constant of motion *exactly*. Note that $K_C \propto \dot{\lambda}$. In the limit $\dot{\lambda} \rightarrow 0$, the classical adiabatic theorem is recovered. The theoretical result here is consistent with the result of the above quantum STA also because I is an analog of the quantum number n .

III. CLASSICAL VERSUS QUANTUM WORK FLUCTUATIONS

For a small system under a control protocol, the work done on a system fluctuates. To study the impact of STA [which establishes a constant I (constant n) in classical (quantum) cases] on the work statistics, we introduce work functions below and advocate a useful scheme to connect STA with nonequilibrium statistical mechanics.

For a classical system $H_0[p, q, \lambda(t)]$ initially prepared at (p_0, q_0) , the inclusive work [21–23] during a period of $0 \leq t \leq \tau$ is given by

$$W_\tau = H_0[p(p_0, q_0, \tau), q(p_0, q_0, \tau), \lambda(\tau)] - H_0[p_0, q_0, \lambda(0)], \quad (6)$$

where $[p(p_0, q_0, t), q(p_0, q_0, t)]$ represents a classical trajectory emanating from (p_0, q_0) . The classical work function $P^c(W)$ is defined as

$$P^c(W) = \int \rho(p_0, q_0) \delta[W - W_\tau(p_0, q_0)] dp_0 dq_0, \quad (7)$$

where $\rho(p_0, q_0)$ describes the initial statistical ensemble, e.g., the Gibbs distribution corresponding to $H_0[\lambda(0)]$. To have a fair comparison with the bare cases without H_C , we propose a scheme in which $\dot{\lambda}(0) = \dot{\lambda}(\tau) = 0$ and hence

$$H_C(p, q, 0) = H_C(p, q, \tau) = 0. \quad (8)$$

That is, when evaluating the work, the control Hamiltonian H_C does not directly affect the calculation because H_C vanishes in the beginning and in the end. In contrast, the trajectories are affected by H_C and the work done to the system is now achieved by both a device achieving $\lambda(t)$ and by a field implementing H_C . Interestingly, under our control scheme, the Jarzynski equality [21, 23], i.e.,

$$\langle e^{-\beta W} \rangle = e^{-\beta \Delta F}, \quad (9)$$

still holds for the same ΔF even with H_C . Note that here $\langle \cdot \rangle$ represents the thermal average and ΔF is the free energy difference between a thermal equilibrium state of the final-state configuration (at the same temperature) and the initial thermal state.

A comparison between classical and quantum work fluctuations will be stimulating. To that end we adopt the two-time measurement definition of quantum work [24], which is

known as $W = E_m^\tau - E_n^0$, with E_m and E_n being the energy values upon energy measurement. The quantum work function $P^q(W)$ is then given by

$$P^q(W) = \sum_{m, n} P_n^0 P_{n \rightarrow m}^\tau \delta[W - (E_m^\tau - E_n^0)], \quad (10)$$

where P_n^0 is the initial probability on $|n[\lambda(0)]\rangle$ and $P_{n \rightarrow m}^\tau$ is the transition probability between $|n[\lambda(0)]\rangle$ and $|m[\lambda(\tau)]\rangle$. Adopting the same scheme as in classical cases, i.e., $\dot{\lambda}(0) = \dot{\lambda}(\tau) = 0$, we have

$$\hat{H}_C(0) = \hat{H}_C(\tau) = 0. \quad (11)$$

IV. APPLICATIONS TO A PARAMETRIC OSCILLATOR

A parametric oscillator is feasible for experimental investigations [16, 17]. We are thus motivated to consider a parametric oscillator with a time-dependent angular frequency $\omega(t)$. In the quantum version, the Hamiltonian is

$$\hat{H}_0(t) = \frac{\hat{p}^2}{2m} + \frac{m}{2} \omega^2(t) \hat{q}^2. \quad (12)$$

The required $\hat{H}_C(t)$ used to realize quantum transitionless driving is found to be [15]

$$\hat{H}_C(t) = -\frac{\dot{\omega}}{4\omega} (\hat{q} \hat{p} + \hat{p} \hat{q}), \quad (13)$$

which is proportional to $\dot{\omega}$. Such a control Hamiltonian may be realized by considering a magnetic field whose vector potential \hat{A} is proportional to \hat{q} (this also effectively changes ω). In the classical domain, the classical Hamiltonian is

$$H_0[p, q, \omega(t)] = \frac{p^2}{2m} + \frac{m}{2} \omega^2(t) q^2. \quad (14)$$

Using our general result above one finds

$$K_C[I, \theta, t] = -\frac{\dot{\omega}}{\omega} I \sin \theta \cos \theta. \quad (15)$$

Detailed calculations in Appendix A [19] show

$$H_C[p, q, t] \equiv K_C[I(p, q, t), \theta(p, q, t), t] = -\frac{\dot{\omega}}{2\omega} pq, \quad (16)$$

which is precisely the classical limit of the quantum \hat{H}_C .

For work fluctuations we consider the classical case first. Because $\tilde{H}_0[I, \omega(t)] = \omega(t)I$ and we have set $\dot{\omega}(0) = \dot{\omega}(\tau) = 0$, we find $K[I, 0] = \omega(0)I$ and $K[I, \tau] = \omega(\tau)I$, with I being an exact constant of motion under H_C . The work expression then reduces to

$$W_\tau = H_0[p(p_0, q_0, \tau), q(p_0, q_0, \tau), \omega(\tau)] - H_0[p_0, q_0, \omega(0)] = K[I, \omega(\tau)] - K[I, \omega(0)] = \Delta\omega I, \quad (17)$$

with $\Delta\omega \equiv \omega_f - \omega_i$ taken to be positive. For simplicity we also define $\omega_f \equiv \omega(\tau)$ and $\omega_i \equiv \omega(0)$. Assuming that the system is initially prepared in a canonical distribution with inverse temperature β , i.e.,

$$\rho(I, \theta) = \frac{\beta \omega_i}{2\pi} e^{-\beta \omega_i I}, \quad (18)$$

we obtain

$$P_{\text{ad}}^c(W) = \frac{\omega_i \beta}{\Delta\omega} \exp\left(-\frac{\omega_i \beta W}{\Delta\omega}\right) \Theta(W), \quad (19)$$

with Θ being the unit step function. Because this result is independent of the details of $\dot{\omega}$, it applies to conventional adiabatic cases as well. That is, the work function for STA remains identical to that in conventional adiabatic processes [25].

It is necessary and interesting to compare $P_{\text{ad}}^c(W)$ with the work function $P_{\text{nonad}}^c(W)$ of a process with the same time dependence of $\omega(t)$ but without H_C . For the bare protocol we obtain

$$P_{\text{nonad}}^c(W) = \frac{1}{\sqrt{\mu_+\mu_-}} \exp\left[-\frac{\mu_+ + \mu_-}{2\mu_+\mu_-}W\right] \times I_0\left[\frac{\mu_+ - \mu_-}{2\mu_+\mu_-}W\right] \Theta(W), \quad (20)$$

where $I_\alpha(x)$ is the modified Bessel function of the first kind with parameter α . Here we have further assumed a (positive-valued) angular frequency $\omega(t)$ increasing monotonically during the protocol, with a simple proof of the positive definiteness of W detailed in Appendix C. The constant factors μ_\pm should be determined by specific realizations of $\omega(t)$. In particular, in the sudden-change limit (SL) of $\tau \rightarrow 0$, $P_{\text{nonad}}^c(W)$ reduces to

$$P_{\text{SL}}^c(W) = \sqrt{\frac{1}{\pi W}} \sqrt{\frac{\beta\omega_i^2}{(\omega_f^2 - \omega_i^2)}} \exp\left[-\frac{\beta\omega_i^2}{\omega_f^2 - \omega_i^2}W\right] \Theta(W). \quad (21)$$

Interestingly, $P_{\text{SL}}^c(W)$ diverges at $W = 0$, whereas $P_{\text{ad}}^c(W)$ is always finite. Equally interesting, since we assume $\omega_f > \omega_i$ (without loss of generality), we have

$$\frac{\omega_i^2}{\omega_f^2 - \omega_i^2} < \frac{\omega_i}{2\Delta\omega}. \quad (22)$$

Thus the exponential decay rate of $P_{\text{SL}}^c(W)$, which is $\frac{\beta\omega_i^2}{\omega_f^2 - \omega_i^2}$, is less than half of the exponential decay rate of $P_{\text{ad}}^c(W)$ for STA. Two main impacts of STA on the work function are hence clear: The probabilities for both very small and large work values are strongly suppressed.

To illustrate and corroborate our analysis we present in Fig. 1 numerical results of the work function. The results agree with our theoretical calculations. Indeed, compared with a bare protocol of the same duration, the STA case (i) suppresses the long tail of the work distribution and (ii) also significantly decreases the weights of almost zero work. Quantitatively, in terms of mean work $\langle W \rangle$ and the standard deviation of work $\sigma(W)$,

$$\langle W \rangle_{\text{STA}} = \sigma(W)_{\text{STA}} = \frac{\Delta\omega}{\omega_i\beta} \quad (23)$$

for STA, whereas for a nonadiabatic process in the sudden-change limit we find

$$\langle W \rangle_{\text{SL}} = \frac{\omega_f^2 - \omega_i^2}{2\beta\omega_i^2} \quad (24)$$

and a larger work variance

$$\sigma(W)_{\text{SL}} = \frac{1}{\sqrt{2}} \left(\frac{\omega_f}{\omega_i} + 1 \right) \sigma(W)_{\text{STA}}. \quad (25)$$

Certainly, suppressing the probabilities for small and large W can be more significant than what is manifested in $\sigma(W)$. To stress this point, we display in Fig. 2 a typical result converging towards Jarzynski's equality with a limited number

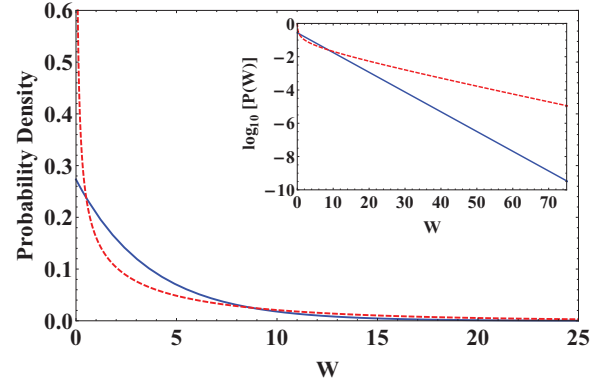


FIG. 1. (Color online) Work function for a parametric oscillator with $\beta = 0.2$, $\omega_i = 10$, and $\omega_f = 10\sqrt{3}$, with all variables in scaled and hence dimensionless units. The solid (blue) line is for STA, with $\omega_i\tau = 0.001$ and $\omega(t)$ chosen to be $\omega(t) = \omega_i\sqrt{(a^2 + 1)/2 - [(a^2 - 1)/2]\cos[\pi(t/\tau)]}$, with $a = \omega_f/\omega_i = \sqrt{3}$. The dashed (red) line is for a bare process of the same duration. The same results are shown in the inset using a semilogarithmic plot.

of classical simulation trajectories. It is seen that $\langle e^{-\beta W} \rangle$ in either the bare protocol or the STA converges towards the same theoretical value, but the case of STA does converge faster (statistically). This is an intriguing consequence of work fluctuation suppression. Indeed, the dissipated work, i.e., $\langle W \rangle - \Delta F$, is also suppressed via STA. Shortcuts to adiabatic processes hence mimic the so-termed escorted free energy simulations put forth in Ref. [26]. Note, however, that here we use a strictly Hamiltonian control term $H_C(t)$ that not only renders the validity of the Jarzynski equality but (in contrast to Ref. [26]) leaves the expression for physical work unchanged as well.

Let us now turn to the quantum work fluctuations with the initial quantum state at thermal equilibrium. Using Eq. (10) one finds that in STA or normal adiabatic processes

$$P_{\text{ad}}^q(W) = \sum_{n=0}^{\infty} P_n \delta[W - \hbar(\omega_f - \omega_i)(n + 1/2)], \quad (26)$$

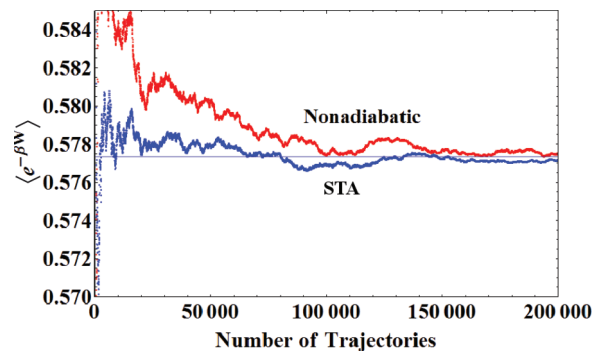


FIG. 2. (Color online) Numerical average value of $\langle e^{-\beta W} \rangle$ vs the number of classical trajectories used, with $\omega_i = 10$, $\beta = 0.2$, and $\omega_i\tau = 0.001$. The upper red (lower blue) line is for a bare nonadiabatic process (STA). The horizontal thin line indicates the theoretical value $1/\sqrt{3}$. The protocol for $\omega(t)$ is the same as in Fig. 1.

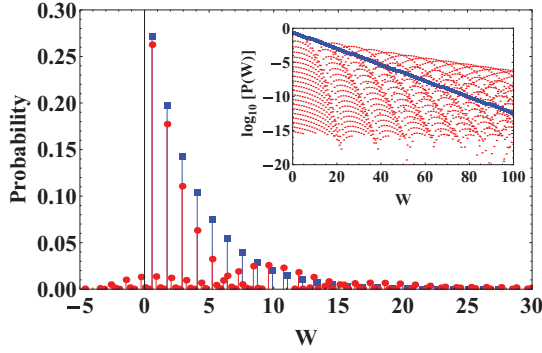


FIG. 3. (Color online) Quantum work function from direct simulations, with $2\pi\hbar = 1$, $\beta = 0.2$, and $\omega(t)$ the same as in Fig. 2. The red dots (blue squares) denote $P^q(W)$ of a bare nonadiabatic process (STA) with $\tau\omega_i = 0.001$. For clarity only data points with $P^q(W) \geq 0.0002$ are plotted. The inset displays all data points on a semilogarithmic plot using a wider range of W , with the thick line for STA.

with

$$P_n = (1 - e^{-\beta\hbar\omega_i})e^{-n\beta\hbar\omega_i}. \quad (27)$$

The discrete sum is due to quantization. For general nonadiabatic processes without \hat{H}_C , analytical but rather complicated work functions are available [27–29]. Here we perform direct numerical investigations with a specific realization of $\omega(t)$ considered in Figs. 1 and 2.

Our results in Fig. 3 show that the long-tail distribution of quantum work is also strongly suppressed in STA, with the degree of suppression in good correspondence with our classical results. Quantitatively, for the parameters in Fig. 3, the work variance $\sigma(W) = 3.1$ [$\sigma(W) = 8.7$] in the STA (bare nonadiabatic) case. There are two other inspiring aspects. First, in the classical protocol (STA or a bare process) and in the quantum STA, W is strictly positive definite with monotonically increasing $\omega(t)$. In distinct contrast, however, in a bare quantum process (i.e., without \hat{H}_C), work W attains an appreciable probability of becoming negative. Second, while W for STA assumes equally spaced discretized values only, in the bare processes work W assumes a rich variety of different discrete values. It is also checked that via the quantum Jarzynski equality the bare protocol and the STA indeed yield the same quantum ΔF .

V. ENHANCEMENT IN EFFICIENCY AND POWER OF A PROTOTYPICAL MICRO HEAT ENGINE

We now consider an Otto-cycle-based heat engine using a parametric oscillator as its working medium [16,17]. This involves ω -changing strokes without a reservoir (steps 1 and 3) and two relaxation processes with reservoirs at two different inverse temperatures β_1 and β_2 (steps 2 and 4). For the details of the four steps, see, e.g., Ref. [17].

It is convenient to assume steps 1 and 3 to be conventional (quasi)adiabatic or extremely nonadiabatic (sudden-change limit) processes. With β_1 , β_2 , ω_i , and the duration of each

cycle step fixed, ω_f can be optimized to maximize the net work output, with the corresponding engine efficiency being the efficiency at maximum power [30], denoted as η . For parameters in the classical regime, η obtained using (quasi)adiabatic steps 1 and 3, which is

$$\eta_{\text{ad}} = 1 - \sqrt{\beta_2/\beta_1}, \quad (28)$$

is higher than (by more than two times)

$$\eta_{\text{nonad}} = \frac{1 - \sqrt{\beta_2/\beta_1}}{2 + \sqrt{\beta_2/\beta_1}} \quad (29)$$

which is obtained using steps 1 and 3 in their sudden-change limit [16,17]. Nevertheless, conventional (quasi)adiabatic processes lead to a very long cycle time, which yields a low engine power. In contrast, rapid steps 1 and 3 can generate higher power output but with the drawback of yielding a low engine efficiency η_{nonad} .

The balance between power and efficiency is of vast interest to heat-engine designs [10,16]. Here we apply classical STA to the above-described Otto cycle in the classical domain. That is, we replace steps 1 and 3 by two classical STA. In principle, steps 1 and 3 can now be almost instantaneous, but the work function remains identical to that obtained in conventional adiabatic processes. With the cycle duration now being bounded only by the relaxation time scales associated with steps 2 and 4, the power of the heat engine is also drastically enhanced. We stress that the engine efficiency remains η_{ad} simply because the work functions of steps 1 and 3 are still adiabatic work functions. A heat engine based on STA can hence exploit *both* advantages of fast strokes (high power) and adiabatic processes (high efficiency).

For the heat engine operating in the quantum regime, we consider quantum STA for steps 1 and 3. Though the engine power is enhanced, the expression of quantum work output per cycle remains the same as in Ref. [17]. The efficiencies with optimized ω_f are shown in Fig. 4. For a large β_1 such as $\beta_1 = 10$, the obtained efficiency results can be far from the two classical curves (solid lines) describing η_{ad} and η_{nonad} . This case hence represents an engine operating in the deep quantum domain. Interestingly, there the heat-engine

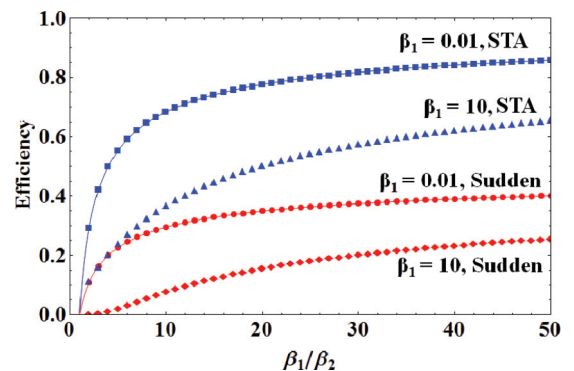


FIG. 4. (Color online) Efficiency at maximized work output as a function of β_1/β_2 for a prototypical quantum heat engine [16,17], with $\omega_i = 10$, $2\pi\hbar = 1$, and $\beta_1 = 10$ or 0.01. The two solid lines describe the classical results η_{ad} and η_{nonad} given in the text. Note the efficiency increase by STA.

efficiency associated with STA is still more than twice that obtained in the sudden-change limit. The enhancement factor is even much larger if β_2 approaches β_1 .

VI. CONCLUSION

Shortcuts to adiabatic processes are shown to have simple classical counterparts. They can substantially suppress work fluctuations in rapid processes.

Work values, though classically positive definite in a protocol, may still be negative in the quantum domain. However, quantum STA may completely suppress the negative work values. Shortcuts to adiabatic processes can also enhance the efficiency and at the same time the power of a micro heat engine, in both classical and quantum regimes. Finally, we compare our work with two related and independent studies [11,12]. Reference [11] studied the use of STA (in a quantum framework) in a heat-engine model via a tailored time dependence of the frequency of the parametric oscillator setup [5]. Though using a different control scheme, the main conclusion drawn in Ref. [11] echoes ours in the heat-engine application. Reference [12] also studied the classical analog of quantum STA, but using concepts and techniques different from this work.

APPENDIX A: \hat{H}_C FOR A QUANTUM PARAMETRIC OSCILLATOR

Here we provide some necessary details regarding the quantum control Hamiltonian for realizing STA. In our main text we consider a quantum parametric oscillator whose Hamiltonian is given by $\hat{H}_0(t) = \frac{\hat{p}^2}{2m} + \frac{m}{2}\omega^2(t)\hat{q}^2$ with a time-dependent angular frequency $\omega(t)$. The calculations for \hat{H}_C can be found from Ref. [15]. For a self-containing comparison with our classical theory, we perform a similar calculation here. Obviously

$$E_n - E_m = \hbar\omega(n - m) \quad (\text{A1})$$

and

$$\partial_t \hat{H}_0 = m\dot{\omega}\hat{q}^2. \quad (\text{A2})$$

Note also that

$$\hat{q} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger), \quad (\text{A3})$$

where \hat{a} and \hat{a}^\dagger are the annihilation and creation operators, respectively. The quantum transitionless driving Hamiltonian derived in Ref. [2] is

$$\begin{aligned} \hat{H}_C(t) &= i\hbar \frac{\partial U(t,0)}{\partial t} U^\dagger(t,0) - \hat{H}_0[\lambda(t)] \\ &= i\hbar \sum_n \sum_{m \neq n} \frac{|m\rangle \langle m| \partial_t \hat{H}_0[\lambda(t)] |n\rangle \langle n|}{E_n - E_m}, \end{aligned} \quad (\text{A4})$$

from which we have

$$\begin{aligned} \hat{H}_C(t) &= i\hbar \sum_i \sum_{j \neq i} \frac{|j\rangle \langle j| \partial_t \hat{H}_0(t) |i\rangle \langle i|}{E_i - E_j} \\ &= \frac{i\hbar\dot{\omega}}{2\omega} \sum_i \sum_{j \neq i} \frac{|j\rangle \langle j| (\hat{a}^2 + \hat{a}^{\dagger 2} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) |i\rangle \langle i|}{i - j} \\ &= \frac{i\hbar\dot{\omega}}{2\omega} \sum_i \frac{1}{2} (\sqrt{(i-1)i} |i-2\rangle \langle i| \\ &\quad - \sqrt{(i+1)(i+2)} |i+2\rangle \langle i|) \\ &= \frac{i\hbar\dot{\omega}}{2\omega} \sum_i \frac{1}{2} (\hat{a}^2 - \hat{a}^{\dagger 2}) |i\rangle \langle i| \\ &= -\frac{\dot{\omega}}{4\omega} (\hat{q}\hat{p} + \hat{p}\hat{q}). \end{aligned} \quad (\text{A5})$$

This is just the expression of $\hat{H}_C(t)$ given in Ref. [15].

APPENDIX B: H_C FOR A CLASSICAL PARAMETRIC OSCILLATOR

For a classical parametric oscillator $H_0 = \frac{p^2}{2m} + \frac{m}{2}\omega^2(t)q^2$ we have [20]

$$H_0(p, q, \omega) = \tilde{H}_0(I, \omega) = \omega I. \quad (\text{B1})$$

In addition, a type-II generating function relates (p, q) and (I, θ) through

$$p = \frac{\partial F_2(I, q, \omega)}{\partial q} \quad (\text{B2})$$

and

$$\theta = \frac{\partial F_2(I, q, \omega)}{\partial I}. \quad (\text{B3})$$

Equation (B2) leads to

$$\tilde{H}_0 = \omega I = \frac{1}{2m} \left(\frac{\partial F_2(I, q, \omega)}{\partial q} \right)^2 + \frac{1}{2} m \omega^2 q^2, \quad (\text{B4})$$

$$F_2(I, q, \omega) = \int \sqrt{2m\omega I - m^2\omega^2 q^2} dq.$$

Therefore,

$$\theta = \frac{\partial F_2(I, q, \omega)}{\partial I} = \arcsin \left(\sqrt{\frac{m\omega}{2I}} q \right) + \text{const}, \quad (\text{B5})$$

$$q = \left(\sqrt{\frac{2I}{m\omega}} \right) \sin(\theta - \text{const}),$$

where the constant is arbitrary and we set it to zero, which then gives

$$p = \sqrt{2m\omega I - m^2\omega^2 q^2} = \sqrt{2m\omega I} \cos \theta. \quad (\text{B6})$$

The control Hamiltonian K_C defined in the main text is then given by

$$\begin{aligned} K_C[I, \theta, t] &= - \left[\left(\frac{\partial F_2(I, q, \lambda)}{\partial t} \right) \Big|_{I, q} \right] \Big|_{q=q(I, \theta, \lambda)} \\ &= - \frac{\dot{\omega} I}{2\omega} \left\{ \sin \left[2 \arcsin \left(\sqrt{\frac{m\omega}{2I}} q \right) \right] \right\} \Big|_{q=q(I, \theta, \omega)} \\ &= - \frac{\dot{\omega} I}{2\omega} \sin(2\theta). \end{aligned} \quad (\text{B7})$$

As such, one directly has $H_C(p, q) = -\frac{\dot{\omega}}{2\omega} pq$ as given in the main text.

APPENDIX C: WORK FUNCTION IN BARE NONADIABATIC PROCESSES

To compare with the work function in classical STA, we also present analytical results for the work functions in classical (bare) nonadiabatic processes, using a parametric oscillator as a specific system. To that end we need to specify $\omega(t)$. In particular, we consider a positive-value angular frequency

$$\omega(t) = \omega_i \sqrt{\frac{a^2 + 1}{2} - \frac{a^2 - 1}{2} \cos\left(\pi \frac{t}{\tau}\right)} \quad (\text{C1})$$

so that $H_C \propto \dot{\omega} = 0$ at $t = 0, \tau$. For brevity we assume $a \equiv \omega_f/\omega_i > 1$, i.e., ω is increasing during the entire protocol. Then we have

$$W = \int_0^\tau \frac{dH}{dt} dt = \int_0^\tau \frac{\partial H}{\partial t} dt = \int_0^\tau 2\dot{\omega}(t)\omega(t)q^2 dt, \quad (\text{C2})$$

which clearly indicates that the classical work for a protocol using a monotonically increasing $\omega(t)$ is positive definite. Interestingly, as seen in the main text, this is not true in the quantum case.

The equation of motion is

$$\ddot{q}(t) + \omega^2(t)q(t) = 0. \quad (\text{C3})$$

There are two linearly independent special solutions C and S with

$$\begin{aligned} C(0) &= 1, & \dot{C}(0) &= 0; \\ S(0) &= 0, & \dot{S}(0) &= 1. \end{aligned} \quad (\text{C4})$$

Under our specific choice of ω in Eq. (C1) (see also Ref. [28]), C and S are basic solutions to Mathieu's equation [31]:

$$\begin{aligned} C(t) &= w_I\left(\frac{\pi}{2\tau}t; c, d\right), \\ S(t) &= \frac{2\tau}{\pi} w_{II}\left(\frac{\pi}{2\tau}t; c, d\right), \end{aligned} \quad (\text{C5})$$

where $c = 2\tau^2\omega_i^2(a^2 + 1)/\pi^2$ and $d = \tau^2\omega_i^2(a^2 - 1)/\pi^2$. Then a general solution $q(t)$ with initial condition (p_0, q_0) is given by

$$q(t) = q_0 C(t) + \frac{p_0}{m} S(t). \quad (\text{C6})$$

Once we have the solution $q(t)$, our following derivations will be quite general.

The work done during a time duration τ is

$$\begin{aligned} W_\tau &= H_0[p(p_0, q_0, \tau), q(p_0, q_0, \tau), \tau] - H_0[p_0, q_0, 0] \\ &= \mathcal{K} \frac{\beta}{2m} p_0^2 + \mathcal{L} \frac{\beta m \omega_0^2}{2} q_0^2 + \mathcal{M} \beta \omega_0 p_0 q_0 \\ &= \mathcal{K} p'^2 + \mathcal{L} q'^2 + 2\mathcal{M} p' q', \end{aligned} \quad (\text{C7})$$

where $p' \equiv \sqrt{\frac{\beta}{2m}} p_0$, $q' \equiv \sqrt{\frac{\beta m \omega_0^2}{2}} q_0$, and

$$\begin{aligned} \mathcal{K} &\equiv \frac{1}{\beta} [\dot{S}^2(\tau) + \omega_f^2 S^2(\tau) - 1], \\ \mathcal{L} &\equiv \frac{1}{\beta} \left[\frac{\dot{C}^2(\tau)}{\omega_0^2} + \frac{\omega_f^2}{\omega_0^2} C^2(\tau) - 1 \right], \\ \mathcal{M} &\equiv \frac{1}{\beta \omega_0} [\dot{C}(\tau) \dot{S}(\tau) + \omega_f^2 C(\tau) S(\tau)]. \end{aligned} \quad (\text{C8})$$

Note that \mathcal{K} , \mathcal{L} , and \mathcal{M} defined above are independent of (p_0, q_0) and hence independent of p' and q' . Because W_τ in Eq. (C7) is expressed in a quadratic form, there always exist a two-dimensional orthogonal transformation such that

$$W_\tau = \mu_+ x^2 + \mu_- y^2, \quad (\text{C9})$$

with

$$\mu_\pm = \frac{1}{2} [(\mathcal{K} + \mathcal{L}) \pm \sqrt{(\mathcal{K} - \mathcal{L})^2 + 4\mathcal{M}^2}] \quad (\text{C10})$$

and (x, y) related to (p', q') by an orthogonal transformation (such that $x^2 + y^2 = p'^2 + q'^2$). Since the work is known to be positive definite [under our specific choice of $\omega(t)$], μ_\pm must be positive. The initial Gibbs ensemble ρ_0 under inverse temperature β then becomes

$$\rho_0(p_0, q_0) = \frac{1}{Z_0} e^{-\beta H_0[p_0, q_0, \omega_0]} = \frac{\beta \omega_0}{2\pi} e^{-(x^2 + y^2)}. \quad (\text{C11})$$

Finally, the classical work function is

$$\begin{aligned} P^c(W) &= \int_\Gamma \frac{\beta \omega_0}{2\pi} e^{-(x^2 + y^2)} \delta[W - \mu_+ x^2 - \mu_- y^2] dp_0 dq_0 \\ &= \int_\Gamma \frac{1}{\pi} e^{-(x^2 + y^2)} \delta[W - \mu_+ x^2 - \mu_- y^2] dx dy \\ &= \int \frac{1}{\pi} \exp\left[-r^2 \left(\frac{\cos^2 \phi}{\mu_+} + \frac{\sin^2 \phi}{\mu_-}\right)\right] \\ &\quad \times \delta[W - r^2] \frac{r}{\sqrt{\mu_+ \mu_-}} dr d\phi \\ &= \int_0^{2\pi} \frac{1}{2\pi \sqrt{\mu_+ \mu_-}} \exp\left[-W \left(\frac{\cos^2 \phi}{\mu_+} + \frac{\sin^2 \phi}{\mu_-}\right)\right] d\phi \\ &= \frac{1}{\sqrt{\mu_+ \mu_-}} \exp\left[-\frac{\mu_+ + \mu_-}{2\mu_+ \mu_-} W\right] I_0\left[\frac{\mu_+ - \mu_-}{2\mu_+ \mu_-} W\right], \end{aligned} \quad (\text{C12})$$

where a change of integration variable ($x = r \cos \phi / \sqrt{\mu_+}$ and $y = r \sin \phi / \sqrt{\mu_-}$) with $r \in (0, +\infty)$ and $\phi \in (0, 2\pi)$ is used.

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