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Stabilizing saddles

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A synergetic control technique for stabilizing *a priori* unknown saddle steady states of dynamical systems is described. The method involves an unstable filter technique combined with a derivative feedback. The cut-off frequency of the filter is not limited by the damping of the system, and therefore can be set relatively high. This essentially increases the rate of convergence to the steady state. The synergetic technique is robust to the influence of unknown external forces, which change the coordinates of the steady state in the phase space.

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A variety of feedback methods for controlling unstable steady states of dynamical systems have been described during the past two decades. The first example is the derivative feedback technique, applied to stabilize a laser [1] and an electrochemical reaction [2]. Other examples are various tracking filter methods [3–7]. They have been tested in many experimental systems, including electrical circuits [3,4] and lasers [5,6]. Some more complicated methods, originally designed to stabilize periodic orbits, namely the time-delayed feedback [8] and the notch-filter [9,10] methods with properly chosen parameters can stabilize the steady states as well [4,10].

However, the above mentioned techniques, when applied to control steady states, are able to stabilize nodes and spirals only. They cannot stabilize the saddle states, i.e. the states characterized with an odd number of real positive eigenvalues. To solve the problem Pyragas *et al.* proposed using an unstable filter (UF) [11]. The UF technique has been demonstrated to stabilize saddles in several mathematical models [11,12] also applied to experimental systems, e.g., an electrochemical oscillator [11] and the Duffing-Holmes electrical circuit [12]. The UF method, however, works in the dissipative systems only, similarly to the Ott-Grebogi-Yorke method of controlling chaos [13], in the sense that it is not applicable to the Hamiltonian systems [14]. Later the UF method has been extended to conservative systems [15]. Another limitation of the simple UF method is its slow performance, especially when applied to weakly damped systems. It has been derived analytically from the Hurwitz criteria for a pendulum [11,12] and for the Duffing-Holmes oscillator [12] that the cut-off frequency of the UF should be set less than the damping coefficient of the system. For weak damping, consequently low cut-off frequency, the transients become very long [11,12].

In the previous works [11,12,15] the performance of the UF technique was demonstrated for the "installation" stage, i.e., the evolution from either the originally oscillatory and rotatory states [11] or from an originally stable steady state [12] to the saddle steady state have been considered. Whenever the steady state is stabilized, the control methods should guarantee robust performance under the unknown and unpredictable external perturbations, which change the coordinates of the steady state.

In this work, we suggest an efficient synergetic method, which combines the UF and the derivative techniques (we abbreviate it as UFD), for stabilizing saddle steady states and inspect the response of the overall system to the external *a priori* unknown force.

We consider the Duffing-Holmes autonomous damped oscillator [16] as an example:

$$\ddot{x} + b\dot{x} - x + x^3 = 0. \tag{1}$$

Here b is the damping coefficient. The oscillator has three steady states (x_0,$\dot{x}_0$): two symmetrical stable spirals or nodes (depending on b) at (±1,0) and a saddle at (0,0).

To stabilize the saddle we apply two methods for comparison, namely, the simple UF method [11] and the synergetic UFD method. The first one is given by

$$\dot{x} = y, \tag{2a}$$
$$\dot{y} = x - x^3 - by + k(u - x) + p, \tag{2b}$$
$$\dot{u} = \omega(u - x). \tag{2c}$$

The second method is described by

$$\dot{x} = y, \tag{3a}$$
$$\dot{y} = x - x^3 - by + k(u + \dot{u} - x - \dot{x}) + p, \tag{3b}$$
$$\dot{u} = \omega(k - 1)(u - x). \tag{3c}$$

In Eqs. (2) and (3) the p is an unknown perturbation. When linearized around the saddle steady state Eqs. (2) and (3) read

$$\dot{x} = y, \tag{4a}$$
$$\dot{y} = x - by + k(u - x), \tag{4b}$$
$$\dot{u} = \omega(u - x), \tag{4c}$$
$$\dot{x} = y, \tag{5a}$$
$$\dot{y} = x - by + k(u + \dot{u} - x - \dot{x}), \tag{5b}$$
$$\dot{u} = \omega^*(u - x), \tag{5c}$$

respectively, with $\omega^* = (k - 1)\omega$ and $k > 1$. Here we assumed $p = 0$ for simplicity without the loss of generality.

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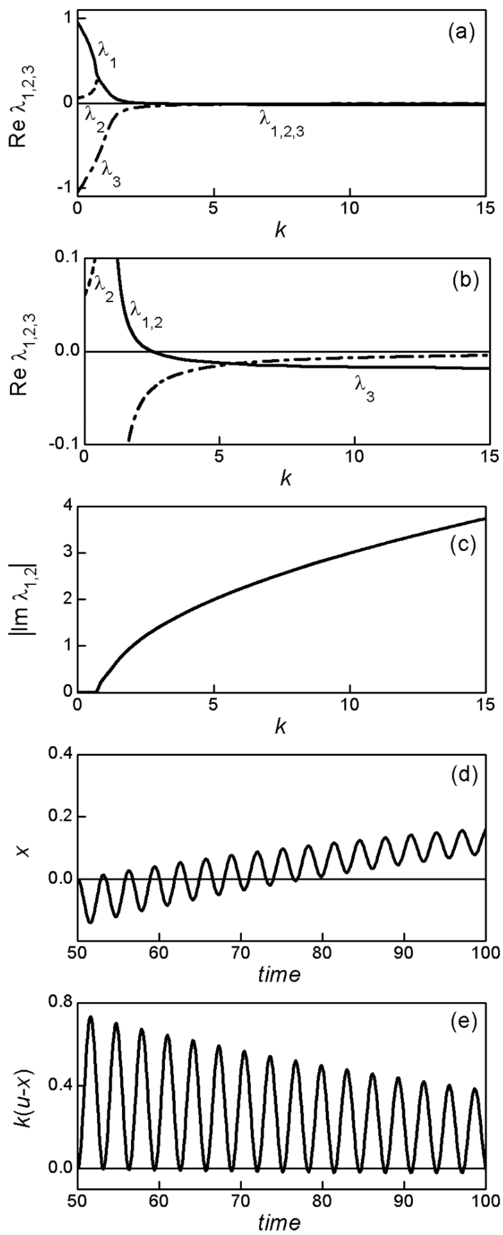

FIG. 1. Stabilizing the saddle by means of simple UF method, $b = 0.1$, $\omega = 0.06$ ($\omega < b$). (a) Real parts of the eigenvalues Re$\lambda_{1,2,3}$ versus the control gain k from Eq. (6). (b) Same as (a), but the vertical scale is zoomed by a factor of 10. (c) Imaginary parts of the eigenvalues Im$\lambda_{1,2}$ versus the control gain k from Eq. (6). (d) Variable x from Eq. (2). (e) Control term $k(u-x)$ from Eq. (2). In (d) and (e) $k = 5$, perturbation $p = -0.3$ is applied at $t = 50$.

The corresponding characteristic equations are

$$\lambda^3 + (b-\omega)\lambda^2 + (k-1-\omega b)\lambda + \omega = 0, \tag{6}$$

$$\lambda^3 + (b+k-\omega^*)\lambda^2 + (k-1-\omega^* b)\lambda + \omega^* = 0. \tag{7}$$

The overall system is stable, if the real parts of all three eigenvalues are negative. The necessary and sufficient conditions

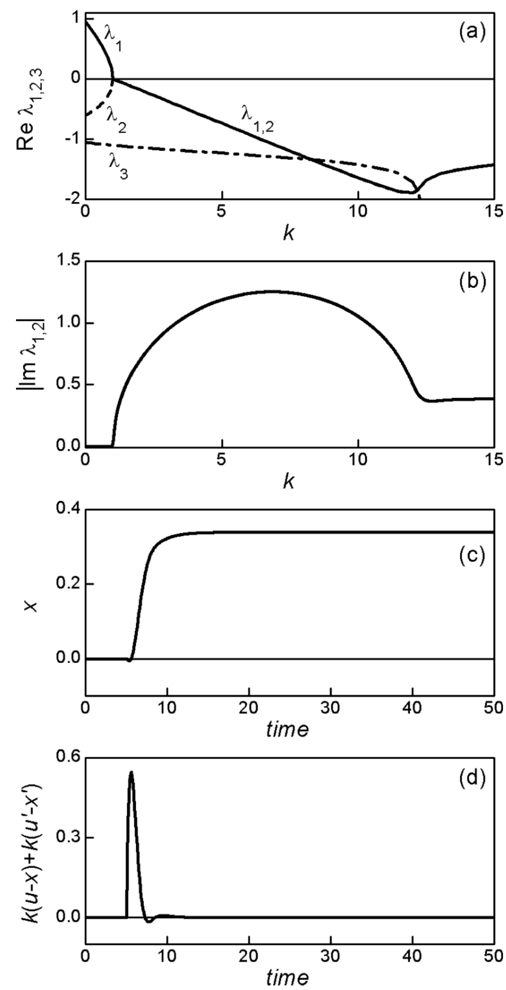

FIG. 2. Stabilizing the saddle by means of the synergetic UFD method, $b = 0.1$, $\omega = 0.6$. (a) Real parts of the eigenvalues Re$\lambda_{1,2,3}$ versus the control gain k from Eq. (7). (b) Imaginary parts of the eigenvalues Im$\lambda_{1,2}$ versus the control gain k from Eq. (7). (c) Variable x from Eq. (3). (d) Control term from Eq. (3). In (c) and (d) $k = 10$, perturbation $p = -0.3$ is applied at $t = 5$.

can be found from the Hurwitz matrices

$$H = \begin{pmatrix} b-\omega & \omega & 0 \\ 1 & k-1-\omega b & 0 \\ 0 & b-\omega & \omega \end{pmatrix}, \tag{8}$$

$$H^* = \begin{pmatrix} b+k-\omega^* & \omega^* & 0 \\ 1 & k-1-\omega^* b & 0 \\ 0 & b+k-\omega^* & \omega^* \end{pmatrix}. \tag{9}$$

The eigenvalues Re $\lambda_{1,2,3}$ are all negative if the diagonal minors of the H and H^* matrices are all positive. For the matrix H the diagonal minors are

$$\Delta_1 = b - \omega > 0, \tag{10a}$$

$$\Delta_2 = (b-\omega)(k-1-\omega b) - \omega > 0, \tag{10b}$$

$$\Delta_3 = \omega\Delta_2 > 0. \tag{10c}$$

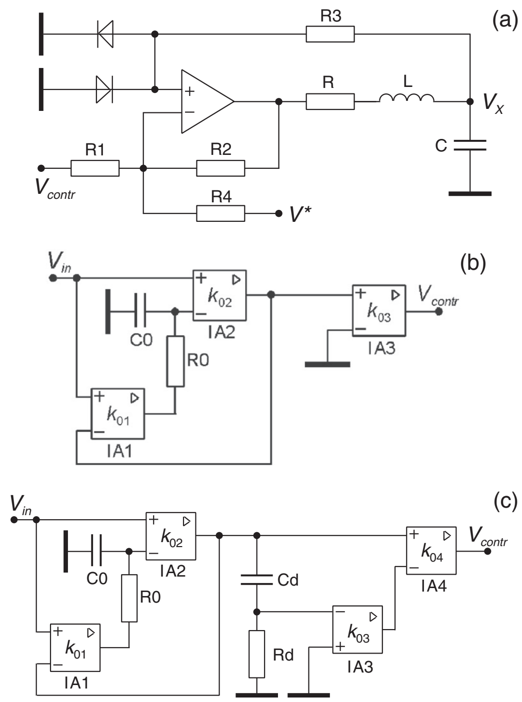

FIG. 3. Experimental circuit diagrams. (a) Duffing–Holmes oscillator. $L = 19$ mH, $C = 470$ nF, $\rho = \sqrt{L/C} = 200\ \Omega$, the characteristic frequency $f_0 = (2\pi\sqrt{LC})^{-1} \approx 1.7$ kHz, $R = 20\ \Omega$ ($b = R/\rho = 0.1$), $R_1 = R_2 = R_3 = 10\,\text{k}\Omega$, $R_4 = 1\,\text{M}\Omega$, operational amplifier is the NE5534 integrated circuit, diodes are the 1N4148 type devices (forward voltage drop $V_b \approx 0.5$ V at 0.1 mA), V_{contr} is control voltage, V^* is external perturbation voltage. (b) Simple UF controller, $R_0 = 10\,\text{k}\Omega$, C_0 is specified in Fig. 4, instrumentation amplifiers IA1...IA3 are the AD620 integrated circuits. (c) Synergetic UFD controller, $R_0 = R_d = 10\,\text{k}\Omega$, C_0 and C_d are specified in Fig. 4, IA1...IA4 are the same as in (b). $V_{\text{in}} = V_x$.

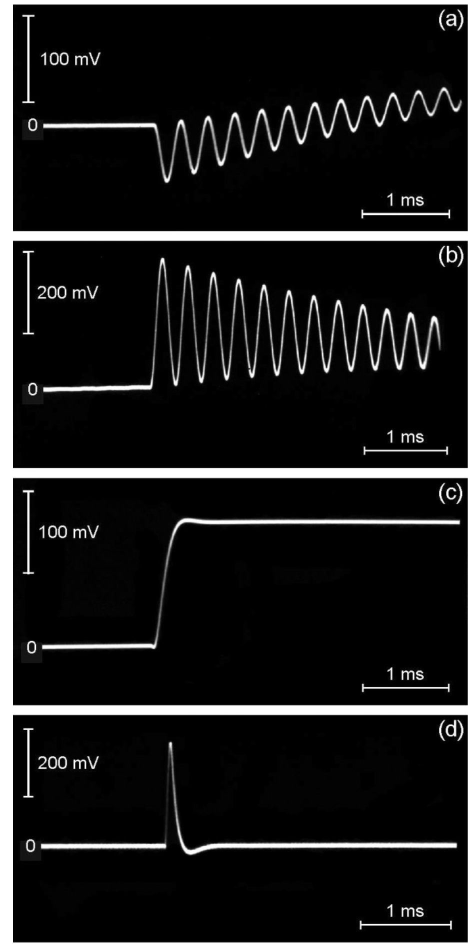

FIG. 4. Stabilizing the saddle in the Duffing–Holmes oscillator. Perturbation $V^* = 15$ V; $p = -(R_2/R_4)(V^*/V_b) = -0.3$. (a,b) Simple UF method, $C_0 = 175$ nF ($\omega = \sqrt{LC}/(R_0C_0) = 0.06$), $k_{01} = 1$, $k_{02} = 2$, $k_{03} = 2.5$, $k = k_{02}k_{03} = 5$. (c,d) Synergetic UFD method, $C_0 = 16$ nF ($\omega = 0.6$), $C_d = 330$ pF ($R_dC_d \approx 3.10^{-6}s \ll \sqrt{LC} \approx 10^{-4}s$), $k_{01} = 1$, $k_{02} = 10$, $k_{03} = 30$, $k_{04} = 1$, $k = k_{02}k_{04} = 10$, $(R_dC_d/\sqrt{LC})k_{03} \approx 1$. (a,c) Output signals V_x, (b,d) Control signals V_{contr}.

These inequalities are satisfied if

$$\omega < b, \quad k > k_{\text{th}} = \frac{b}{b-\omega} + \omega b. \tag{11}$$

For example, at $b = 0.1$ and $\omega = 0.06$ the $k_{\text{th}} \approx 2.5$. On one hand, the ω could be only slightly less than b. On the other hand, it should not be too close to b, because small value of the denominator $b - \omega$ would heavily increase the stabilization threshold k_{th}.

Numerical solution of the characteristic equation is plotted in Fig. 1. The largest eigenvalues $\text{Re}\,\lambda_1 = \text{Re}\,\lambda_2$ cross zero axis at $k \approx 2.5$ in good agreement with the analytical result. We note very small absolute values of the largest Re $\lambda_{\max}$ at $k > k_{\text{th}}$. In the full scale [Fig. 1(a)] the curve lays almost on the abscissa. Only the zoomed in plot [Fig. 1(b)] reveals the negative values. However, even at $k = k_{\text{opt}} = 5$ the $|\text{Re}\,\lambda_{\max}| = 0.01$. Such a low value, related to small parameters b and ω, results in slow convergence to the steady state. This is a serious shortcoming of the UF method, especially if applied to weakly damped ($b \leqslant 0.1$) and Hamiltonian dynamical systems. Numerical results of the control dynamics under the influence of an *a priori* unknown external constant force p, which changes the position of the saddle steady state, are shown in Figs. 1(d) and 1(e).

The diagonal minors of the matrix H^* are

$$\Delta_1^* = b + k - \omega^* > 0, \tag{12a}$$

$$\Delta_2^* = (b + k - \omega^*)(k - 1 - \omega^* b) - \omega^* > 0, \tag{12b}$$

$$\Delta_3^* = \omega^* \Delta_2^* > 0. \tag{12c}$$

Inequalities (12) provide the following stability criteria:

$$\omega < \frac{b+k}{k-1}, \quad k > k_{\text{th}} \approx 1. \tag{13}$$

For large k the required cut-off frequency of the filter $\omega < 1$ and in contrast to the UF method does not depend on b. In (13) k_{th} is derived for weak damping ($|b| \ll 1$); the value $k_{\text{th}} \approx 1$ is in good agreement with the numerical results [Fig. 2(a)].

There are three main advantages of the UFD method over the UF technique. First, in the case of the UF method at the optimal control gain $k = k_{\text{opt}} = 5$ the $\text{Re}\,\lambda_{\max} = -0.01$ [Fig. 1(b)], while in the case of the UFD method at $k = k_{\text{opt}} = 12.5$ the $\text{Re}\,\lambda_{\max} \approx -2$, which is about 200 times larger than for the UF technique, resulting in extremely fast convergence to the steady state. Secondly, the $|\mathrm{Im}\,\lambda_{1,2}|$ for the UFD method is several times smaller than for the UF method. Finally, in the case of the UFD method there is no negative drop of the $x(t)$ at the time moment when the perturbation is applied.

Circuit diagrams of the Duffing-Holmes electronic oscillator and the controllers are sketched in Fig. 3. The electronic oscillator actually is a modified version of the low frequency Young-Silva oscillator [17]. It operates in the kilohertz range. The oscillator has been used previously to illustrate switching from a stable spiral to the saddle point [12] also to demonstrate chaos control in a nonautonomous (periodically driven) Duffing-Holmes system by means of the time-delayed feedback [18]. The experimental results, presented in Fig. 4, coincide rather well with the corresponding numerical simulations.

To better understand the reasons for the enhanced performance of the UFD method we replace in Eq. (5b) $\dot{x}$ with y from Eq. (5a) and $\dot{u}$ with $\omega^*(u-x)$ from Eq. (5c). Then the following explicit form is obtained:

$$\dot{x} = y, \tag{14a}$$

$$\dot{y} = x - (b+k)y + (1+\omega^*)k(u-x), \tag{14b}$$

$$\dot{u} = \omega^*(u-x). \tag{14c}$$

One can see that Eq. (14) for the UFD method has exactly the same form as Eq. (4) for the UF method, but with the effective parameters $b^* = b + k$, $k^* = (1+\omega^*)k$, and $\omega^* = (k-1)\omega$. The most important issue is, that the effective damping coefficient b^* is increased considerably due to the summand $+k$. The effective cut-off frequency ω^* is increased by a factor of $(k-1)$. The inequality $\omega^* < b^*$ can be still satisfied, like $\omega < b$ in (11) for the UF method, but at essentially higher values of both b^* and ω^*. Eventually, the $\omega^* < b^*$ yields the first inequality in the stability criteria (13), where for large k the actual cut-off frequency $\omega \approx 1$ is much higher than in the common UF method ($\omega < b \ll 1$). This is the main reason for the faster performance of the UFD method.

In conclusion, we have proposed a synergetic UFD control method for stabilizing *a priori* unknown saddle steady states of dynamical systems. The controller is model independent and reference-free. It requires neither the mathematical model nor the coordinates of the steady state, but automatically tracks and stabilizes the state. The numerical and the experimental results have been presented for the Duffing-Holmes oscillator only. However, the general form of a saddle given by Eq. (5) indicates that the UFD technique can be applied to many other dynamical systems as well. The suggested UFD controller is essentially faster than the simple UF version [11,12]. Moreover, it is suitable to stabilize saddle steady states also in dynamical system with zero ($b = 0$) and negative ($b < 0$) damping. In contrast to the simple UF technique the cut-off frequency of the UFD controller ω can be set relatively high and is independent of the damping of the dynamical system; the effective frequency is further increased by a factor of $(k-1)$ in Eq. (3c). The UFD controller exhibits robust performance in the presence of external unknown forces, which change the coordinates of the steady state in the phase space.

Recently an attempt was made to improve the UF method of stabilizing saddle steady states by combining two filters in the feedback, namely, an unstable filter and a stable one [19]. The enhancement was noticeable, but not so good as expected. Though the transients in weakly damped systems became shorter ($\approx$1 ms in the experiment), the main variables and the control signals still exhibited ringing effects. In contrast, the UFD method ensures very short transient ($\approx$0.2 ms), which is close to the intrinsic response time of the oscillator $\sqrt{LC} \approx 0.1$ ms, and practically no ringing is observed. From a mathematical point of view, an additional filter in the enhanced UF method [19] increases dimension of the overall system from three to four, thus making analysis of the 4×4 Hurwitz matrix and its four diagonal minors extremely complicated. Whereas in the UFD method the stability criterium (13) is easy to derive and has a very simple compact form: $k > 1 > \omega$.

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