Modulational instability and breathing motion in the two-dimensional nonlinear Schrödinger equation with a one-dimensional harmonic potential

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Modulational instability and breathing motion are studied in the two-dimensional nonlinear Schrödinger (NLS) equation trapped by the one-dimensional harmonic potential. The trapping potential is uniform in the y direction and the wave function is confined in the x direction. A breathing motion appears when the initial condition is close to a stationary solution which is uniform in the y direction. The amplitude of the breathing motion is larger in the two-dimensional system than that in the corresponding one-dimensional system. Coupled equations of the one-dimensional NLS equation and two variational parameters are derived by the variational approximation to understand the amplification of the breathing motion qualitatively. On the other hand, there is a breathing solution in the x direction which is uniform in the y direction to the two-dimensional NLS equation. It is shown that the modulational instability along the y direction is suppressed when the breathing motion is sufficiently strong, even if the norm is above the critical value of the collapse.

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I. INTRODUCTION

Solitons in the Bose-Einstein condensates (BECs) and the optical fibers [1] are described by the one-dimensional nonlinear Schrödinger (NLS) equation,

$$i\frac{\partial\phi}{\partial t}=-\frac{1}{2}\frac{\partial^2\phi}{\partial y^2}-g|\phi|^2\phi. \tag{1}$$

The one-dimensional NLS equation has various types of solutions other than solitons. The Peregrine solitons are solutions localized along both the t and x axes [2], and the Akhmediev breathers are solutions localized along the t axis [3]. These temporary localized solutions are considered to be related to the rogue waves: sudden exceptionally large waves.

There is a stationary solution $\phi=A_0\exp(-i\epsilon t)$ to Eq. (1), where $\epsilon=-gA_0^2$. To study the stability of the stationary solution, we assume the solution of the form $\phi=\{A_0+\delta B(t)\cos(ky)\}\exp(-i\epsilon t)$, where $\delta B=\delta B_r+i\delta B_i$ is the complex variable. The linearized equations for δB_r and δB_i around the stationary solution are expressed as

$$\begin{aligned}\frac{d\delta B_r}{dt}&=(k^2/2)\delta B_i,\\ \frac{d\delta B_i}{dt}&=\left(-k^2/2+2gA_0^2\right)\delta B_r.\end{aligned} \tag{2}$$

The two equations yield

$$\frac{d^2\delta B_r}{dt^2}=(k^2/4)\left(-k^2+4gA_0^2\right)\delta B_r, \tag{3}$$

when $k<k_c=\sqrt{4g}A_0$ and δB_r grows exponentially. This type of instability is called the modulational instability.

The Akhmediev breathers are closely related to the modulational instability and the generation of supercontinuum in optics [4]. The Akhmediev breather has a form

$$A(y,t)=A_0\frac{(1-4a)\cosh(bt)+ib\sinh(bt)+\sqrt{2a}\cos(ky)}{\sqrt{2a}\cos(ky)-\cosh(bt)}, \tag{4}$$

where k is the wave number, $2a=1-(k/k_c)^2$, and $b=\{8a(1-2a)\}^{1/2}$ with $k_c=\sqrt{4g}A_0$. The Akhmediev breather has only one peak near $t=0$ in the t axis and it is periodic in the y direction. A constant solution with amplitude A_0 is unstable against the modulational perturbation with wave number k satisfying $k<k_c$. The nonlinear time evolution of the modulational instability can be studied from an initial condition such as $\phi(y,t)=A_0(1+\delta\cos ky)$ at $t=0$. If δ is assumed to be $\delta=\mu\{1+i(2b/k)^2\}$, a breathing solution close to the Akhmediev breather appears from the initial condition, although the breathing motion repeats periodically [4].

Bright and dark solitons of BECs were experimentally found in the condensate trapped in strongly elongated traps [5,6]. Although the one-dimensional NLS equation is often used in theoretical studies, a three-dimensional NLS equation in a two-dimensional trapping potential is better to describe such systems. We study a two-dimensional NLS equation in a one-dimensional trapping potential as a simpler model in this paper because the model is suitable to investigate the effect of an additional dimension, and the direct numerical simulation of the model equation is easier than the three-dimensional model. We studied various soliton dynamics in the two-dimensional NLS equations trapped by a one-dimensional external potential [7,8]. In this paper, we will study the modulational instability and breathing motion in the two-dimensional NLS equation trapped in the one-dimensional harmonic potential. Such systems can be experimentally realized using a highly anisotropic trap such that $U=(1/2)\Omega_x^2x^2+(1/2)\Omega_y^2y^2+(1/2)\Omega_z^2z^2$ with $\Omega_y=0$ and $\Omega_z\gg\Omega_x$. That is, BECs are strongly confined in the z direction and moderately confined in the x direction, and can move freely in the y direction. If the degree of the motion in the z direction is neglected, we get a system of the two-dimensional NLS equation with the one-dimensional harmonic potential.

Various types of modulational instabilities and transverse instabilities in the NLS equation and its generalization in two

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or three dimensions were reviewed in [9]. We focus on the two-dimensional NLS equation trapped in a one-dimensional harmonic potential. In Sec. II, the model equation is shown and coupled equations of the one-dimensional NLS equation and two variational parameters are derived. In Sec. III, we will show numerical results and discuss the amplification of the breathing motion in the two-dimensional system compared to the corresponding one-dimensional NLS equation. On the other hand, there are solutions that are breathing in the x direction and uniform in the y direction. In Sec. IV, we will discuss the suppression of the modulational instability along the y direction for the solutions breathing in the x direction.

II. MODEL EQUATION AND VARIATIONAL APPROXIMATION

The model equation of the two-dimensional NLS equation with a one-dimensional harmonic potential is written as

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\nabla^2\psi - |\psi|^2\psi + \frac{1}{2}\Omega^2x^2\psi, \tag{5}$$

where Ω corresponds to Ω_x in Sec. I. The matter wave denoted by ψ is trapped in the x direction with the harmonic potential $(1/2)\Omega^2x^2$. The system is uniform along the y direction. If $\Omega = 0$, there is a uniform solution $\psi = A_0\exp(-i\epsilon t)$. The uniform solution exhibits the modulational instability for the perturbations with wave number k satisfying $|k| < k_c = 2A_0$, which is similar to the one-dimensional NLS equation. In a system of nonzero Ω, there is a stationary solution which is uniform along the y direction: $\psi(x,y,t) = u(x)e^{-i\epsilon t}$. This type of solution is a state of the lowest energy. We focus on the modulational instability of this state for the sake of simplicity, although it is meaningful to study the modulational instability for more general stationary solutions such as $\psi(x,y,t) = u(x)e^{ik_yy}e^{-i\epsilon t}$ with nonzero k_y. If the nonlinear term is absent, $u(x) = A_0(2\alpha_0/\pi)^{1/4}e^{-\alpha_0x^2}$ with $\alpha_0 = \Omega/2$ is the stationary solution for arbitrary A_0. The Gaussian approximation $u(x) = A_0(2\alpha_0/\pi)^{1/4}e^{-\alpha_0x^2}$ is a good approximation for the stationary solution of Eq. (5) which is uniform along the y direction, if the norm is not so large. However, the stationary solution might be unstable against some perturbation in the y direction. We will discuss the breathing motion after the modulational instability of this type of stationary solution.

Before showing several numerical simulations, we derive coupled equations by a variational method to discuss the breathing motion in the two-dimensional NLS equation approximately using the breathing solution in the one-dimensional NLS equation. The one-dimensional NLS equation is an integrable system and the dynamical behaviors are well studied. On the other hand, the two-dimensional NLS equation is not completely studied compared to the one-dimensional NLS equation. It is meaningful to qualitatively understand the breathing motion in the two-dimensional NLS equation using the knowledge of the one-dimensional NLS equation.

The variational method using the Lagrangian is one of the useful methods to discuss the nonintegrable nonlinear waves [10], although it is a heuristic approach and there are some problems such as false instabilities [11]. In some cases, the variational approach has been in good agreement with direct numerical simulations in finding soliton-type solutions and the stability analyses [12–15].

The Lagrangian for Eq. (5) is written as

$$L = \int\int\left\{\frac{i}{2}\left(\frac{\partial\psi}{\partial t}\psi^* - \frac{\partial\psi^*}{\partial t}\psi\right) - \frac{1}{2}\left|\frac{\partial\psi}{\partial x}\right|^2 - \frac{1}{2}\left|\frac{\partial\psi}{\partial y}\right|^2 + \frac{1}{2}|\psi|^4 - \frac{\Omega^2x^2}{2}|\psi|^2\right\}dxdy. \tag{6}$$

If we assume the ansatz for the solution, $\psi(x,y) = u_0e^{-[\alpha(t)+i\beta(t)]x^2}\phi(y,t)$ with $u_0 = (2\alpha/\pi)^{1/4}$, L is evaluated as

$$L = N_y\frac{\dot{\beta}}{4\alpha} + \int\frac{i}{2}\left(\frac{\partial\phi}{\partial t}\phi^* - \frac{\partial\phi^*}{\partial t}\phi\right)dy - N_y\frac{\alpha^2+\beta^2}{2\alpha} - \frac{1}{2}\int\left|\frac{\partial\phi}{\partial y}\right|^2dy + \sqrt{\frac{\alpha}{4\pi}}\int|\phi^4|dy - N_y\frac{\Omega^2}{8\alpha}, \tag{7}$$

where $N_y = \int|\phi|^2dy$.

The variational equations expressed as

$$\frac{\partial}{\partial t}\frac{\delta L}{\delta\dot{\alpha}} = \frac{\delta L}{\delta\alpha},\quad \frac{\partial}{\partial t}\frac{\delta L}{\delta\dot{\beta}} = \frac{\delta L}{\delta\beta},\quad \frac{\partial}{\partial t}\frac{\delta L}{\delta\dot{\phi}^*} = \frac{\delta L}{\delta\phi^*}$$

yield

$$\frac{d\alpha}{dt} = 4\alpha\beta, \tag{8}$$

$$\frac{d\beta}{dt} = 2(\beta^2-\alpha^2) + \sqrt{\frac{\alpha^3}{\pi}}\frac{N_{2y}}{N_y} + \frac{\Omega^2}{2}, \tag{9}$$

$$i\frac{\partial\phi}{\partial t} = -\frac{1}{2}\frac{\partial^2\phi}{\partial y^2} - g|\phi|^2\phi, \tag{10}$$

where $g = \sqrt{\alpha/\pi}$ and $N_{2y} = \int|\phi|^4dy$ and the system size in the y direction is assumed to be L_y. Periodic boundary conditions are assumed in the y direction.

A stationary solution of the coupled equations (8)–(10) is expressed as $\alpha = \alpha_0$, $\beta = 0$, and $\phi(y,t) = A_0e^{-i\epsilon t}$. The parameter ϵ is given by

$$\epsilon = -\sqrt{\alpha_0/\pi}\,A_0^2. \tag{11}$$

The norms N_y and N_{2y} are expressed as $N_y = A_0^2L_y$ and $N_{2y} = A_0^4L_y$. From Eq. (9), α_0 is given by a solution of

$$-2\alpha_0^2 + \sqrt{\frac{\alpha_0^3}{\pi}}A_0^2 + \frac{\Omega^2}{2} = 0. \tag{12}$$

III. AMPLIFICATION OF BREATHING MOTION BY TWO-DIMENSIONAL EFFECT

We have performed three kinds of numerical simulations to study the modulational instability of the stationary solution which is uniform in the y direction. The parameter Ω was set to be 2.5. The first one is the direct numerical simulation of the two-dimensional NLS equation (5). The system size is set to $L_x \times L_y = 16 \times 4$. The solution to Eq. (12) is $\alpha_0 = 1.43$ for $A_0 = 1$. The initial condition for Eq. (5) is assumed to be

$$\psi(x,y,0) = (2\alpha_0/\pi)^{1/4}e^{-\alpha_0x^2}A_0[1+\mu\{1+i(2b/k)^2\}\cos ky],$$

where $A_0 = 1$, $\mu = 0.002$, $k = 2\pi/L_y$, $k_c = \sqrt{4(\alpha_0/\pi)^{1/2}}$, $2a = 1-(k/k_c)^2$, and $b = \{8a(1-2a)\}^{1/2}$. The parameter

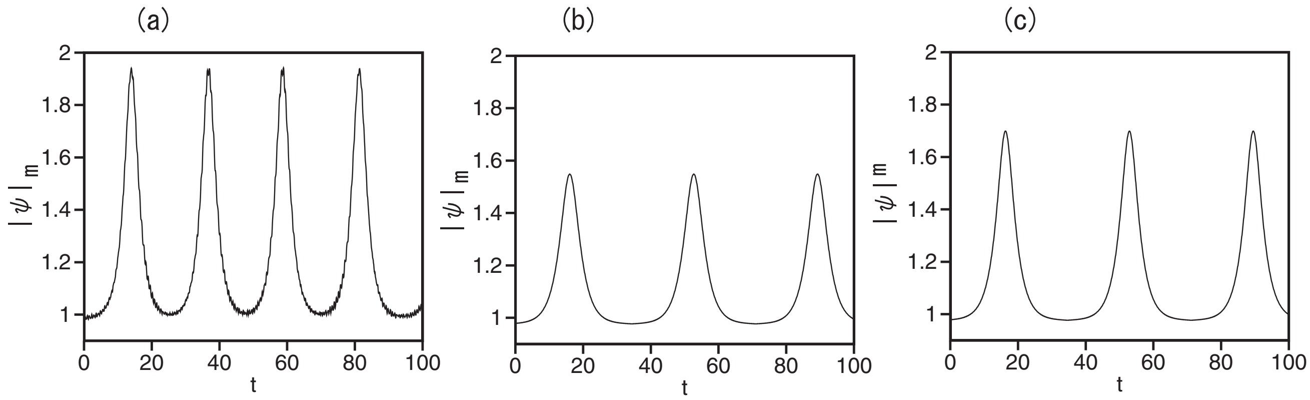

FIG. 1. (a) Time evolution of the peak amplitude $|\psi|_m$ in the two-dimensional NLS equation. (b) Time evolution of the peak amplitude $|\psi|_m = (2\alpha_0/\pi)^{1/4}|\phi|_m$ in the one-dimensional NLS equation (10) with a constant nonlinear parameter $g = \sqrt{\alpha_0/\pi}$. (c) Time evolution of the peak amplitude $|\psi|_m = (2\alpha(t)/\pi)^{1/4}|\phi|_m$ in the coupled equations (8)–(10).

μ denotes the magnitude of the perturbation. The modulational instability occurs because $k = 2\pi/4 = 1.57$ is smaller than $k_c = \sqrt{4(\alpha_0/\pi)^{1/2}} = 1.643$. Figure 1(a) shows the time evolution of the peak amplitude $|\psi|_m$ at $x = y = 0$. Owing to the modulational instability, a breathing motion appears. The peak-peak height of the breathing motion is evaluated at 0.94. The second one is the numerical simulation of the corresponding one-dimensional NLS equation (10), in which α is fixed to $\alpha_0 = 1.43$. The initial condition for the one-dimensional NLS equation is

$$\phi(y,0) = A_0[1 + \mu\{1 + i(2b/k)^2\}\cos ky],$$

where $A_0 = 1$, $\mu = 0.002$, $k = 2\pi/L_y$, $k_c = \sqrt{4(\alpha_0/\pi)^{1/2}}$, $2a = 1 - (k/k_c)^2$, and $b = \{8a(1-2a)\}^{1/2}$. Figure 1(b) shows the time evolution of $|\psi|_m = (2\alpha_0/\pi)^{1/4}|\phi(y,t)|$ at $y = 0$, where $\phi(y,t)$ obeys the one-dimensional NLS equation. A breathing motion appears in this one-dimensional NLS equation owing to the modulational instability. The peak structure can be well approximated by the Akhmediev breather (4). The peak-peak height is 0.57. The peak-peak height is rather smaller than that in the two-dimensional NLS equation shown in Fig. 1(a). The third one is the numerical simulation of the coupled equations (8)–(10). The initial condition for the coupled equations is

$$\alpha = \alpha_0, \quad \beta = 0,$$
$$\phi(y,0) = A_0[1 + \mu\{1 + i(2b/k)^2\}\cos ky], \qquad (13)$$

where $A_0 = 1$, $\mu = 0.002$, $k = 2\pi/L_y$, $k_c = \sqrt{4(\alpha_0/\pi)^{1/2}}$, $2a = 1 - (k/k_c)^2$, and $b = \{8a(1-2a)\}^{1/2}$. Figure 1(c) shows the time evolution of the peak amplitude: $|\psi|_m = [2\alpha(t)/\pi]^{1/4}|\phi(y,t)|$ at $y = 0$. A similar type of breathing motion appears. The peak-peak height is evaluated at 0.73. The peak-peak height is larger than that in Fig. 1(b), but smaller than that in Fig. 1(a).

Figure 2(a) shows the time evolution of

$$\tilde{\alpha} = \frac{\int |\psi|^2 dx dy}{4\int x^2|\psi|^2 dx dy} \qquad (14)$$

obtained by the numerical simulation of the two-dimensional NLS equation (5). The width in the x direction is evaluated by $1/\sqrt{4\tilde{\alpha}}$. Figure 2(b) shows the time evolution of α obtained by the numerical simulation of the coupled equations (8)–(10). The periodic time evolution of α is caused by the modulational instability in the y direction. Owing to the increase of α, the effective nonlinear parameter $g = \sqrt{\alpha/\pi}$ in Eq. (10) increases, which amplifies the modulational instability. This is a mechanism of the amplification of the breathing motion in the two-dimensional NLS equation. However, Figs. 2(a)

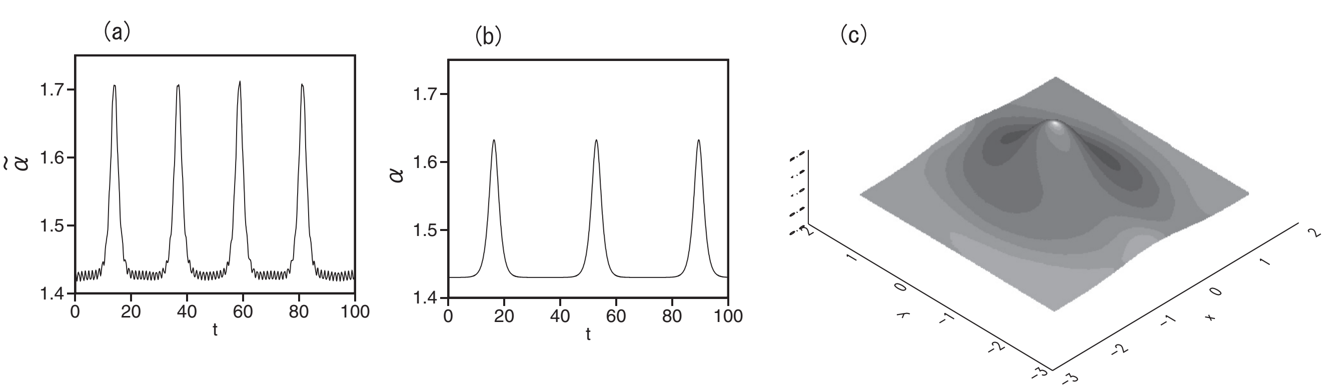

FIG. 2. (a) Time evolution of $\tilde{\alpha}$ in the two-dimensional NLS equation (5). (b) Time evolution of α in the coupled equations (8)–(10). (c) 3D plot of $|\psi|$ at $t = 13.7$ for the two-dimensional NLS equation (5).

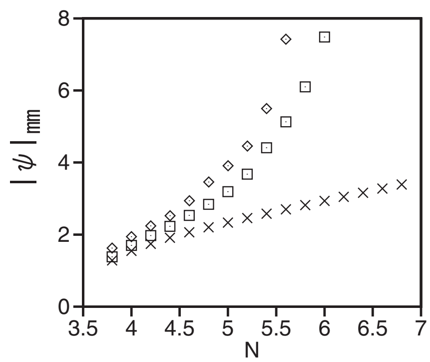

FIG. 3. Maximum peak amplitude $|\psi|_{mm}$ as a function of N for the two-dimensional NLS equation (rhombi), the coupled equations (8)–(10) (squares), and the one-dimensional NLS equation (×).

and 2(b) show that the width along the x direction is more strongly compressed in the two-dimensional NLS equation (5) than in the coupled equations (8)–(10). Figure 2(c) shows the three-dimensional (3D) plot of $|\psi|$ at $t = 13.7$ in the two-dimensional NLS equation when the peak amplitude takes the maximum value in Fig. 1(a). It is observed that the profile $|\psi|$ is strongly localized around $x = y = 0$ both in the x and y directions. The two-dimensional compression or the focusing to a point $(x,y) = (0,0)$ cannot be described well in the coupled equations (8)–(10), which is an origin of the differences between Figs. 1(a) and 1(c) or Figs. 2(a) and 2(b). In any case, it has been shown that the breathing motion in the two-dimensional NLS equation trapped in the one-dimensional harmonic potential is amplified compared to the Akhmediev breather in the one-dimensional NLS equation owing to the effect of the two-dimensional compression.

We have further performed the three kinds of numerical simulations for various values of A_0. Figure 3 compares the maximum peak values obtained by the three kinds of numerical simulations when the norm $N = A_0^2 L_y$ is changed. The rhombi in Fig. 3 denote the maximum peak amplitude $|\psi|_{mm}$ which is the maximum value in the periodic time evolution $|\psi|_m$ in the two-dimensional NLS equation, as shown in Fig. 1(a). The squares in Fig. 3 denote the maximum value $|\psi|_{mm}$ of the peak amplitude $|\psi|_m = [2\alpha(t)/\pi]^{1/4}|\phi(y,t)|$ at $y = 0$ as a function of N obtained by the numerical simulation of the coupled equations (8)–(10). The initial condition is the same as Eq. (13) and A_0 is changed as a control parameter. The crosses in Fig. 3 show the maximum value $|\psi|_{mm}$ of the peak amplitude $|\psi|_m = (2\alpha_0/\pi)^{1/4}|\phi(y,t)|$ at $y = 0$ obtained by the one-dimensional NLS equation (10) with $\alpha = \alpha_0$. Figure 3 shows that the approximation by the coupled equations (8)–(10) is fairly good to describe the breathing motion in the two-dimensional NLS equation.

It is known that the two-dimensional compression leads to the collapse in the two-dimension NLS equation (5) without the confinement potential, when the total norm $N = \int|\psi|^2 dxdy$ is beyond a critical value around $N = 5.85$. The collapse phenomena have been studied by many authors. [16,17]. The collapse occurs even in the two-dimensional NLS equation (5) with the confinement potential at the same threshold. The maximum peak amplitude denoted by rhombi increases rapidly near $N = 5.6$, which suggests the collapse in the two-dimensional NLS equation. We have checked that the maximum peak amplitude exhibits the divergence for $N > 5.9$. The peak amplitude $|\phi|_{mm}$ by the numerical simulation of Eqs. (8)–(10) is slightly smaller than that of Eq. (5); however, it grows rapidly near $N = 6$. We have checked that the coupled equations (8)–(10) also exhibit the divergence for $N > 6.3$. On the other hand, the collapse phenomenon does not occur in the one-dimensional NLS equation. Indeed, the maximum peak amplitude denoted by crosses for the one-dimensional NLS equation does not exhibit rapid growth, as shown in Fig. 3. The coupled equations (8)–(10) are fairly good for qualitatively describing the modulational instability in the two-dimensional NLS equation also from the viewpoint of the collapse.

IV. SUPPRESSION OF MODULATIONAL INSTABILITY BY BREATHING MOTION IN THE *x* DIRECTION

In previous sections, we have studied the modulational instability for the stationary solution which is uniform in the y direction. However, there are other types of solutions which are uniform in the y direction but breathing in the x direction. Such breathing solutions are obtained using different types of initial conditions, which are uniform in the y direction. In this section, we study the modulational instability along the y direction for such solutions breathing in the x direction. That is, we consider an initial condition of the following form for the two-dimensional NLS equation:

$$\psi(x,y,0) = (2\alpha_1/\pi)^{1/4}e^{-\alpha_1 x^2}A_0(1+\delta\cos ky),$$

where α_1 is deviated from the stationary solution α_0. This type of initial condition might be approximately realized as a ground state in an experiment by changing the strength of the trapping potential from Ω to Ω'. If Ω' is suddenly restored to the original value Ω, the state will start breathing motion in the trapping potential $(1/2)\Omega^2x^2$.

If $\delta = 0$ and the solution is uniform in the y direction, a breathing motion appears in the x direction when α_1 is deviated from α_0. First, we consider the modulational instability by the approximation of the coupled equations (8)–(10) to understand the modulational instability of the solution breathing in the x direction. In the description of the coupled equations (8)–(10), α and β obey

$$\frac{d\alpha}{dt} = 4\alpha\beta, \qquad \frac{d\beta}{dt} = 2(\beta^2-\alpha^2) + \sqrt{\frac{\alpha^3}{\pi}}A_0^2 + \frac{\Omega^2}{2}, \tag{15}$$

and $\phi(y,t)$ obeys

$$i\frac{\partial\phi}{\partial t} = -\frac{1}{2}\frac{\partial^2\phi}{\partial y^2} - g(t)|\phi|^2\phi, \tag{16}$$

where $g(t) = \sqrt{\alpha(t)/\pi}$. The stationary solution, which is uniform in the y direction for Eq. (16), has a form

$$\phi(y,t) = A_0\exp[-i\theta(t)],$$

where $\theta(t)$ satisfies $d\theta/dt = -g(t)A_0^2$.

If the initial value of α is deviated from α_0 in Eq. (15), the parameters α and β exhibit a periodic motion in Eq. (15). There is a conserved quantity H in the time evolution of Eq. (15):

$$H = \alpha + \frac{\beta^2}{\alpha} + \frac{\Omega^2}{4\alpha} - \sqrt{\frac{\alpha}{\pi}}A_0^2. \tag{17}$$

The equation of motion of only $\alpha(t)$ can be obtained by eliminating β using Eq. (17) [18]. This periodic motion of $\alpha(t)$ describes the breathing motion in the x direction. The amplitude $[2\alpha(t)/\pi]^{1/4}$ and the width $1/\sqrt{4\alpha(t)}$ of the localized structure change with time owing to the periodic motion of $\alpha(t)$.

To study the modulational instability along the y direction for the breathing solution in the x direction, we assume the solution of the form $\phi = \{A_0 + \delta B(t)\cos(ky)\}\exp[-i\theta(t)]$, where $\delta B = \delta B_r + i\delta B_i$ is the complex variable. Perturbations δB_r and δB_i around the breathing solution obey the linearized equations

$$\frac{d\delta B_r}{dt} = (k^2/2)\delta B_i, \quad \frac{d\delta B_i}{dt} = \{-k^2/2 + 2g(t)A_0^2\}\delta B_r, \tag{18}$$

where $g(t)$ is given by $\sqrt{\alpha(t)/\pi}$. The two equations yield

$$\frac{d^2\delta B_r}{dt^2} = (k^2/4)\{-k^2 + 4g(t)A_0^2\}\delta B_r. \tag{19}$$

This equation is a kind of Hill's equation with time-periodic coefficients. If $g(t)$ is constant in time, the modulational instability occurs for $k^2 < 4gA_0^2$. However, the time-periodic coefficients can facilitate or suppress the modulational instability, depending on the period and the amplitude of $g(t)$. The perturbation $|\delta B_r|$ increases as $\exp(\lambda t)$ with $\lambda > 0$ when the system is unstable. The nonlinear parameter $g(t) = \sqrt{\alpha(t)/\pi}$ changes with time, which is determined through Eq. (15) with the initial conditions $\alpha(0) = \alpha_1$ and $\beta(0) = 0$. We have performed numerical simulation of Eqs. (15) and (19) and determined the growth rate λ numerically. Figure 4(a) shows the relation of λ and α_1 for $A_0^2 = 1.6$. The norm is $N = A_0^2 L_y = 6.4$ for $L_y = 4$, which is beyond the threshold for the collapse. For $\alpha_1 < 0.065$, λ becomes 0. This implies that the modulational instability along the y direction is suppressed by the breathing motion in the x direction. The collapse is initiated by the modulational instability, therefore, the stabilization of the modulational instability can suppress the collapse. Figure 4(b) shows the time evolution of the peak amplitude $|\psi|_m = [2\alpha(t)/\pi]^{1/4}|\phi(y,t)|$ at $y = 0$ for $\alpha_1 = \alpha(0) = 0.05$ in the coupled equations (8)–(10). The initial conditions for the coupled equations is $\alpha(0) = 0.05, \beta(0) = 0$ and $\phi(y,0) = A_0 + 0.001i\cos(2\pi y/L_y)$ with $A_0 = \sqrt{1.6}$. The peak amplitude exhibits the oscillation owing to the breathing motion along the x direction, but the modulational instability in the y direction does not occur. Figure 4(c) shows the time evolution of $[2\alpha(t)/\pi]^{1/4}|\phi(y,t)|$ for $\alpha(0) = 0.08$. The modulational instability along the y direction occurs because $\alpha(0)$ is larger than the critical value 0.065. Indeed, the peak amplitude $|\psi|_m|$ exhibits oscillation, and the amplitude of the oscillation grows in time.

If the initial value α_1 is deviated from α_0 but the difference is small, the breathing motion in the x direction is well described by the coupled equations (15). However, if α_1 is far from α_0, the breathing motion cannot be approximated by the simple equations (15) using only two parameters α and β, and the breathing motion becomes more complicated. Then, we need to return to the two-dimensional NLS equation (5). If the solution is uniform in the y direction but the profile in the x direction is more general, $\psi(x,y,t)$ can be assumed to be $\psi = A_0 u(x,t)$. u satisfies another one-dimensional NLS equation:

$$i\frac{\partial u}{\partial t} = -\frac{1}{2}\frac{\partial^2 u}{\partial x^2} - A_0^2|u|^2 u + \frac{1}{2}\Omega^2 x^2 u, \tag{20}$$

where A_0 is a constant amplitude and $N_x = \int |u|^2 dx = 1$ is assumed. The initial condition of $u(x,t)$ for Eq. (20) is assumed to be $u(x,0) = (2\alpha_1/\pi)^{1/4} e^{-\alpha_1 x^2}$. The profile of $u(x,t)$ exhibits complicated breathing motion for small α_1. Figure 5(a) shows time evolution of $|u(x,t)|$ for $\alpha_1 = 0.03$. The modulus $|u|$ exhibits a breathing motion, but the profile cannot be approximated by the Gaussian function. Figure 5(b) shows the time evolution of the peak amplitude $|\psi|_m = |u|_m A_0$ with $A_0 = \sqrt{1.6}$ for $\alpha_1 = 0.03$ for the one-dimensional NLS equation (20) in the x direction. Here, $|u|_m$ is the maximum value of the modulus $|u|$. The peak amplitude does not exhibit regular periodic motion but seems to decay slowly for $t < 20$, in contrast to the periodic motion shown in Fig. 4(b). The peak

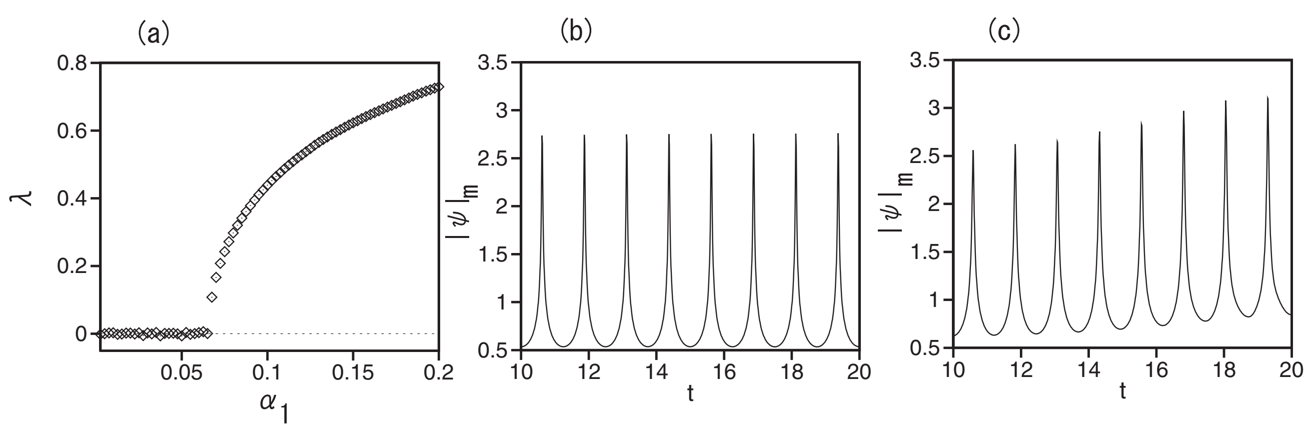

FIG. 4. (a) Stability exponent λ as a function of α_1 for the coupled equations. (b) Time evolution of the peak amplitude $|\psi|_m = [\alpha(t)/\pi]^{1/4}|\phi|_m$ for $\alpha(0) = 0.05$ in the coupled equations (8)–(10). (c) Time evolution of the peak amplitude $|\psi|_m = [\alpha(t)/\pi]^{1/4}|\phi|_m$ for $\alpha(0) = 0.08$.

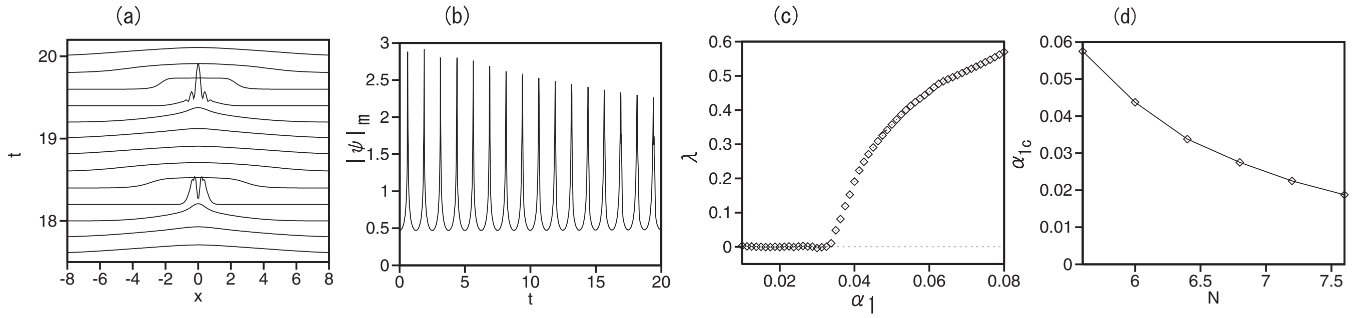

FIG. 5. (a) Time evolution of $|u(x,t)|$ for Eq. (20). (b) Time evolution of the peak amplitude $A_0|u|_m$ obtained by using Eq. (20) for $\alpha_1 = 0.03$. (c) Stability exponent λ as a function of α_1 for Eq. (21). (d) Critical values of α_{1c} for the modulational instability as a function of the norm N calculated by Eq. (21).

amplitude does not decay to zero, but exhibits a complicated quasiperiodic motion for $t > 50$.

Next, we study the modulational instability of this solution breathing in the x direction. If the solution to Eq. (5) is assumed as $\psi = u(x,t)A_0 + \delta C(x,t)\cos(ky)$, then δC satisfies the linear equation

$$i\frac{\partial \delta C}{\partial t} = -\frac{1}{2}\frac{\partial \partial^2 C}{\partial x^2} - \frac{k^2}{2}\delta C - 2A_0^2|u|^2\delta C - A_0^2 u^2 \delta C^* + \frac{1}{2}\Omega^2 x^2 \delta C. \tag{21}$$

We can evaluate numerically the linear growth rate λ for δC using Eq. (21) in which u is determined by Eq. (20). We have calculated λ at various values of α_1 in the initial condition. Figure 5(c) shows the relationship between λ and α_1. The relationship is analogous to the one of the coupled equations (15) and (19) shown in Fig. 4(a), although the critical value is smaller for Eq. (21). For $\alpha_1 < \alpha_{1c} = 0.034$, λ becomes 0. It implies that the modulational instability is suppressed by the breathing motion in the x direction, even though the Gaussian approximation is not assumed. Figure 5(d) shows the critical value α_{1c} as a function of the norm $N = A_0^2 L_y$. The critical value α_{1c} decreases with N.

We have checked the stability of the solution breathing in the x direction with the direct numerical simulation of the two-dimensional NLS equation (5). Figure 6(a) shows the time evolution of the peak amplitude $|\psi|_m$ for the initial condition $\psi(x,y,0) = \{\sqrt{1.6} + 0.001i\cos(2\pi y/L_y)\}(2\alpha_1/\pi)^{1/4}e^{-\alpha_1 x^2}$ at $\alpha_1 = 0.03$. The peak amplitude exhibits damping oscillation for $0 < t < 20$ which is almost the same as Fig. 5(b). This result suggests that the perturbation in the y direction does not grow, that is, the modulational instability in the y direction does not occur. Figure 6(b) shows the time evolution of the amplitude of the perturbation $\Delta = |\int \psi(x,y,t)\cos(2\pi y/L_y)dxdy|$ at $\alpha_1 = 0.03$. This result confirms that the modulational instability does not occur in this two-dimensional NLS equation owing to the breathing motion in the x direction. The suppression of the collapse by the externally driven time-periodic nonlinear parameter was studied previously [19,20]. In our system, the time-periodic nonlinear parameter is internally induced by the breathing motion in the x direction. It implies that the collapse can be suppressed in the two-dimensional NLS equation if the initial condition is suitably chosen.

V. SUMMARY AND DISCUSSION

We have studied the modulational instability and the breathing motion due to perturbations in the y direction of two types of solutions to the two-dimensional NLS equation trapped in the one-dimensional harmonic potential. The modulational instability is amplified by the two-dimensional effect, if the initial condition is close to the stationary solution. If the initial condition is far from the stationary solution, the breathing motion occurs in the x direction, and the strong breathing motion can suppress the modulational instability along the y direction. We think that the suppression of the modulational instability and the collapse by controlling initial conditions is an important result, which might be applied in other systems. We have shown that these effects are qualitatively described by the coupled equations derived from the variational approximation. The breathing motion in

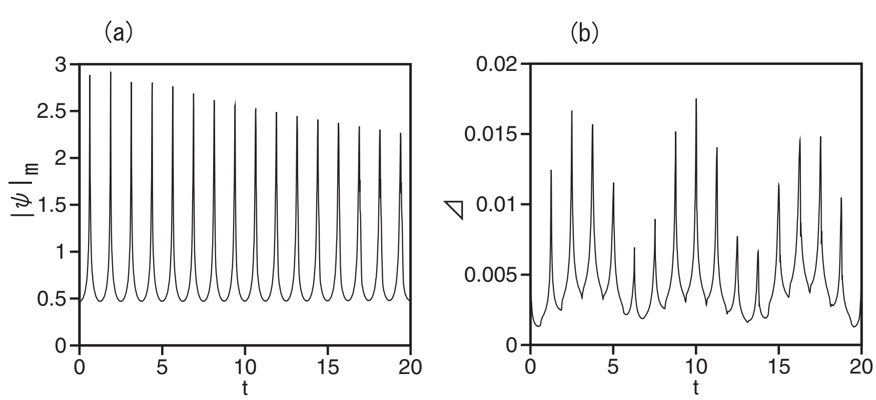

FIG. 6. (a) Time evolution of the peak amplitude $|\psi|_m$ for $\alpha_1 = 0.03$ in the two-dimensional NLS equation (5). (b) Time evolution of the amplitude Δ of the perturbation in the two-dimensional NLS equation (5).

a two-dimensional system with a one-dimensional harmonic potential could be realized in experiments of BECs by suitably designing the trap potential. An experiment for the stabilization of the solution breathing in the x direction might be possible by constructing such breathing state as an initial condition through the sudden change of the trap potential in the x direction.

We have studied the modulational instability for the solution which is uniform in the y direction in a two-dimensional NLS equation trapped in the harmonic potential $(1/2)\Omega x^2$. The modulational instability for more general solutions such as $\phi = u(x,t)e^{ik_y y}e^{-i\theta(t)}$ is left to future study. It is also left in the future to study a three-dimensional model in a trapping potential $U = (1/2)\Omega_x^2 x^2 + (1/2)\Omega_y^2 y^2 + (1/2)\Omega_z^2 z^2$ with $\Omega_y = 0$ or $\Omega_y \ll \Omega_x$. The case of $\Omega_x = \Omega_z$ and $\Omega_y \ll \Omega_x$ is important because the system is relevant to experiments studied in Refs. [5] and [6].

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