Rational solitons of wave resonant-interaction models

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(Received 24 May 2013; published 20 November 2013)

Integrable models of resonant interaction of two or more waves in $1+1$ dimensions are known to be of applicative interest in several areas. Here we consider a system of three coupled wave equations which includes as special cases the vector nonlinear Schrödinger equations and the equations describing the resonant interaction of three waves. The Darboux-Dressing construction of soliton solutions is applied under the condition that the solutions have rational, or mixed rational-exponential, dependence on coordinates. Our algebraic construction relies on the use of nilpotent matrices and their Jordan form. We systematically search for *all bounded* rational (mixed rational-exponential) solutions and find a broad family of such solutions of the three wave resonant interaction equations.

DOI: 10.1103/PhysRevE.88.052914 PACS number(s): 05.45.Yv, 02.30.Ik, 42.65.Tg

I. INTRODUCTION

Integrable partial differential equations which model nonlinear wave propagation in $1+1$ dimension have been largely investigated because of their applicative relevance. In fact, even if approximate, some of them capture important nonlinear effects because they can be derived, as amplitude modulation equations, by multiscale perturbation methods from various kind of (not necessarily integrable) wave equations with the assumption of weak dispersion and nonlinearity (see, for instance, Ref. [1], and references therein). The universality of these integrable models has been well recognized [2,3]. The best known and simplest example of such models is the nonlinear Schrödinger (NLS) equation for the evolution of the amplitude of a quasimonochromatic wave with wave number k and frequency ω, as given by the linear dispersion function $\omega = \omega(k)$. Many physical applications require, however, that integrable models be extended to wave coupling. One important instance is in regard to resonance phenomena. If the dispersion relation allows for resonances, multiscale perturbation methods show that the amplitudes of two or more monochromatic waves couple to each other leading to (possibly integrable) systems of nonlinear partial differential equations. The simplest of such integrable systems is the vector nonlinear Schrödinger (VNLS) system of equations (see, e.g., Ref. [4]), given by the following two coupled equations (a subscript denotes partial differentiation):

$$\begin{aligned} u_t^{(1)} &= i\big[u_{xx}^{(1)} - 2(s_1\,|u^{(1)}|^2 + s_2\,|u^{(2)}|^2)u^{(1)}\big],\\ u_t^{(2)} &= i\big[u_{xx}^{(2)} - 2(s_1\,|u^{(1)}|^2 + s_2\,|u^{(2)}|^2)u^{(2)}\big], \end{aligned} \tag{1}$$

where, because of the integrability condition, $s_1^2 = s_2^2 = 1$. This system, also known as Manakov system, follows from the *weak* resonant condition that two quasimonochromatic waves, with wave numbers k_1 and k_2, have the same group velocity, i.e., $\omega'(k_1) = \omega'(k_2)$ $[\omega'(k) = d\omega/dk]$. In (1) $u^{(1)}(x,t), u^{(2)}(x,t)$ are the amplitudes of these two resonant waves. We note that all three integrable cases (i.e., $s_1 = s_2 = \pm 1$ and $s_1 = -s_2$) are of physical interest. For applications to propagation in elliptically birefringent optical fibers see Ref. [5], and for modeling crossing sea waves see Ref. [6]. A different type of phenomenon occurs when the medium nonlinearity includes quadratic terms and the dispersion relation $\omega(k)$ allows for the two wave numbers k_1 and k_2 to satisfy the *strong* resonant condition $\omega(k_1 + k_2) = \omega(k_1) + \omega(k_2)$. In this case a third wave is generated with amplitude $w(x,t)$, and the three amplitudes $u^{(1)}$, $u^{(2)}$, and w couple to each other according to the system of equations

$$\begin{aligned} u_t^{(1)} &= \big[-c_1 u_x^{(1)} - s_1\, w^* u^{(2)}\big],\\ u_t^{(2)} &= \big[-c_2 u_x^{(2)} + s_2\, w u^{(1)}\big],\\ 0 &= w_x + s_1\, s_2\, (c_1 - c_2)\, u^{(1)*}\, u^{(2)}. \end{aligned} \tag{2}$$

Also this system, with different choices of the signs s_1,s_2, applies to various physical settings and phenomena as in fluid dynamics, see, e.g., Ref. [7] and in optics, see, e.g., Ref. [8]. In this paper we construct particular solutions of both the systems (1) and (2). In the construction method, the physical meaning of the wave amplitudes and of the independent variables x,t does not play any essential role. On the other hand, the results given here are likely to be of applicative relevance in a rather broad range of different physical contexts (e.g., fluid dynamics, nonlinear optics, plasma physics, Bose-Einstein condensate) so it should be kept in mind that the actual meaning of all variables may vary according to context. In particular, for the system (2), if x is the evolution (e.g., time) variable, then this system is the the well-known three-wave resonant interaction (3WRI) equation [8] where the three characteristic velocities are $1/c_1$, $1/c_2$, and 0; otherwise, if the evolution variable is t, this system models the nonlocal interaction of two waves (NL2W) [9,10]. Here rescaling transformations have been used to give the equations (1) and (2) a neat form in terms of their coefficients.

As for the solutions presented below, we observe that elementary symmetries of equations (1) and (2) (such as gauge transformations and coordinate translations) and linear transformations of the (x,t) plane can be used also to eliminate some of the free parameters which may appear in analytic expressions. Indeed, these parameters will be considered in the

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following as unessential since they can be easily introduced through simple transformations.

The kind of solutions we construct here are usually referred to as *rational solitons* with the following specifications: (1) they are localized pulses nonlinearly superimposed to a plane wave and (2) they are solitons since they are spectrally characterized by the vanishing of the continuous spectrum component; however, the discrete spectrum eigenvalues are so special that their corresponding solutions have a *rational* dependence on the variables x,t, in contrast with the standard soliton whose expression is given in terms of exponentials. Rational solutions of multicomponent wave equations such as (1) and (2) generically have a dependence on coordinates which is richer than in the scalar case by possibly having a mixed *rational and exponential* expression. Despite this feature, in the following we term *rational solitons* all these kinds of solutions. Pole singularities in the x,t variables cannot be avoided, these being the zeros of the denominator of the rational expression. However, if these singularities occur only for complex (i.e., strictly nonreal) values of x and t, these solutions are bounded and gain physical relevance.

Rational solutions of integrable partial differential equations attracted immediate mathematical interest in the 1970s, first for the Korteweg-de Vries equation, the motion of the poles being associated with integrable many-body dynamics. Then quite a number of papers have been devoted to rational solutions of various integrable equations for one dependent variable, such as the Boussinesq equation [11], Hirota equation [12], Kadomtsev-Petviashvili equation [13], and NLS equation (see, e.g., Refs. [14–17]). Recently further investigations of rational solutions were extended to integrable systems of two coupled differential equations. In this direction a number of such solutions have been found for the VNLS (1) [18–20] and for two coupled Hirota equations [21]. Similar extension has been reported also for three coupled NLS equations [22].

The starting motivation of such a surge of research work goes back to the observation by Peregrine [23] that the simplest rational solution of the focusing NLS equation may well model an ocean rogue wave (for a recent survey, see Ref. [24]). This solution describes a localized lump over a background with a peak amplitude which is three times higher than the surrounding background itself and with a finite lifetime. On the physical side, these new *nonlinear objects* were soon recognized as ubiquitous rather than just ocean events and maritime disasters. Rogue waves have been observed not only in water tanks [25] but also in fiber optics [26] and in plasma [27]. They are predicted in the atmosphere [28], in superfluids [29], in Bose-Einstein condensates [30], and in capillary waves [31].

In this paper we systematically search for *all bounded* rational (mixed rational-exponential) solutions of both the VNLS equation (1) and of the 3WRI equation (2). We adopt a formalism such that these two equations are simultaneously treated by using an appropriate Lax pair. As happens to all integrable wave equations, in the present cases as well the boundedness condition necessarily requires that rational solitons exist only on continuous and *unstable* wave backgrounds. Our method of construction is based on the standard Darboux-Dressing transformation (DDT) as presented in Refs. [32,33] and briefly summarized in Sec. II. Section III describes the algebraic algorithm we use to obtain rational solutions. In Sec. IV we finally display examples of such solutions. The polynomials which appear in some of the expressions of rational solitons are given in the Appendix.

II. LAX PAIR AND DARBOUX-DRESSING TRANSFORMATION

Equations (1) and (2) are integrable models and as such admit a Lax representation (a Lax pair). For convenience, we introduce a Lax pair which combines both. Let

$$\psi_x = X\psi, \quad \psi_t = T\psi, \tag{3}$$

where ψ, X, and T are 3×3 square matrices, $\psi=\psi(x,t,k)$ being a common solution of the two linear ordinary differential matrix equations (3), while $X=X(x,t,k)$ and $T=T(x,t,k)$ depend on the variables x,t and the complex spectral parameter k according to the definitions

$$X(x,t,k) = ik\sigma + Q(x,t), \tag{4a}$$

$$T(x,t,k) = \alpha\, T_{nls}(x,t,k) + \beta\, T_{3w}(x,t,k), \tag{4b}$$

where $Q(x,t)$ contains the dynamical variables $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ and introduces two *signs* s_1, s_2, $s_1^2=s_2^2=1$,

$$Q = \begin{pmatrix} 0 & s_1 u^{(1)*} & s_2 u^{(2)*} \\ u^{(1)} & 0 & 0 \\ u^{(2)} & 0 & 0 \end{pmatrix}, \tag{5}$$

while $\sigma = \mathrm{diag}\{1,-1,-1\}$ is a constant diagonal matrix. The matrices T_{nls} and T_{3w} are defined by

$$T_{nls} = 2ik^2\sigma + 2kQ + i\sigma(Q^2 - Q_x), \tag{6}$$

$$T_{3w} = 2ikC - \sigma W + \sigma[C,Q], \tag{7}$$

where W contains the field $w(x,t)$

$$W = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -s_1 w^* \\ 0 & s_2 w & 0 \end{pmatrix}. \tag{8}$$

$C=\mathrm{diag}\{0,c_1,c_2\}$ is a real diagonal matrix, while α and β are real parameters such that, for $\alpha=1$, $\beta=0$, (3) is the Lax pair corresponding to the VNLS (Manakov) equation (1), and for $\alpha=0$, $\beta=1$ (3) is the Lax pair corresponding to the 3WRI equation (2). Indeed, the compatibility conditions yield the evolution equations

$$\begin{aligned} u_t^{(1)} &= i\alpha\big[u_{xx}^{(1)} - 2(s_1|u^{(1)}|^2 + s_2|u^{(2)}|^2)u^{(1)}\big] \\ &\quad + \beta\big[-c_1 u_x^{(1)} - s_1\, w^* u^{(2)}\big], \\ u_t^{(2)} &= i\alpha\big[u_{xx}^{(2)} - 2(s_1|u^{(1)}|^2 + s_2|u^{(2)}|^2)u^{(2)}\big] \\ &\quad + \beta\big[-c_2 u_x^{(2)} + s_2\, w u^{(1)}\big], \\ 0 &= \beta[w_x + s_1 s_2 (c_1 - c_2)\, u^{(1)*} u^{(2)}]. \end{aligned} \tag{9}$$

In the search for novel rational solutions of (9) we use the Darboux-Dressing construction, as developed in Ref. [32] (where the interested reader can find additional references).

For completeness, we briefly recall here the essential steps towards a new solution, starting from a known (*seed*) solution: given a solution $u_0^{(1)}$, $u_0^{(2)}$, w_0 of (9), let Ψ_0 be a corresponding fundamental matrix solution of (3). Then, if χ is strictly complex ($\chi \neq \chi^*$),

$$\Psi(x,t,k) = \left[\mathbf{1} + \left(\frac{\chi - \chi^*}{k-\chi}\right) P(x,t)\right] \Psi_0(x,t,k) \quad (10)$$

is a solution of (3) with

$$\begin{pmatrix} u^{(1)}(x,t) \\ u^{(2)}(x,t) \end{pmatrix} = \begin{pmatrix} u_0^{(1)}(x,t) \\ u_0^{(2)}(x,t) \end{pmatrix} + \frac{2i(\chi-\chi^*)\zeta^*}{|\zeta|^2 - s_1|z_1|^2 - s_2|z_2|^2}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad (11a)$$

$$w(x,t) = w_0(x,t) - \frac{2is_1s_2(c_1-c_2)(\chi-\chi^*)z_1^* z_2}{|\zeta|^2 - s_1|z_1|^2 - s_2|z_2|^2}, \quad (11b)$$

where the vector

$$Z(x,t) = \begin{pmatrix} \zeta(x,t) \\ z_1(x,t) \\ z_2(x,t) \end{pmatrix} = \Psi_0(x,t,\chi^*)Z_0 \quad (12)$$

is a solution of (3) with $k = \chi^*$ (Im$\chi \neq 0$) and Z_0 is an arbitrary, constant and complex vector. Moreover in (10) the projector matrix $P(x,t)$ is

$$P(x,t) = \frac{ZZ^\dagger}{|\zeta|^2 - s_1|z_1|^2 - s_2|z_2|^2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -s_1 & 0 \\ 0 & 0 & -s_2 \end{pmatrix}. \quad (13)$$

Here the condition that the parameter χ is *not real* is crucial. Indeed, the Darboux-Dressing transformation which adds one real pole to the solution $\Psi_0(x,t,k)$ in the k plane at $k = \chi = \chi^*$ is given by a different formula, as detailed in Ref. [32]. However this real-pole transformation is not considered here because it yields rational (or semirational) solutions which are singular (i.e., unbounded). The seed solution $u_0^{(1)}(x,t)$, $u_0^{(2)}(x,t)$, $w_0(x,t)$ of (9) is the plane wave

$$\begin{pmatrix} u_0^{(1)}(x,t) \\ u_0^{(2)}(x,t) \end{pmatrix} = \begin{pmatrix} a_1 e^{i(qx-\nu_1 t)} \\ a_2 e^{-i(qx+\nu_2 t)} \end{pmatrix}, \quad (14a)$$

$$w_0(x,t) = is_1s_2(c_2-c_1)\frac{a_1a_2}{2q}e^{-i[2qx+(\nu_2-\nu_1)t]}, \quad (14b)$$

with

$$\nu_1 = \alpha\left[q^2 + 2\left(s_1a_1^2 + s_2a_2^2\right)\right] + \beta\left[c_1q + s_2\frac{a_2^2}{2q}(c_1-c_2)\right],$$

$$\nu_2 = \alpha\left[q^2 + 2\left(s_1a_1^2 + s_2a_2^2\right)\right] + \beta\left[-c_2q + s_1\frac{a_1^2}{2q}(c_1-c_2)\right]. \quad (15)$$

Remark 1. With no loss of generality the amplitudes a_1 and a_2 can be taken to be real. Moreover, because of Galilei invariance, one may choose the wave numbers q and $-q$ of these two plane waves [see (14a)] to have opposite sign.

In order to construct the transformation (11) in the case where the seed solution $u_0^{(1)}$, $u_0^{(2)}$, w_0 of (9) is given by (14), one has to construct first the solution Ψ_0 of the Lax equations (3). To this aim we observe that, once (14) is fixed, the corresponding Q_0 and W_0 take the form

$$Q_0 = G\begin{pmatrix} 0 & s_1a_1 & s_2a_2 \\ a_1 & 0 & 0 \\ a_2 & 0 & 0 \end{pmatrix}G^{-1}, \quad (16)$$

$$W_0 = G\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & is_2\frac{a_1a_2}{2q}(c_2-c_1) \\ 0 & is_1\frac{a_1a_2}{2q}(c_2-c_1) & 0 \end{pmatrix}G^{-1}, \quad (17)$$

with

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i(qx-\nu_1 t)} & 0 \\ 0 & 0 & e^{-i(qx+\nu_2 t)} \end{pmatrix}. \quad (18)$$

It follows then that

$$\Psi_0(x,t,k) = G(x,t)\Phi(x,t,k), \quad (19)$$

and the Lax pair reads

$$\Phi_x = i\Lambda(k)\Phi, \quad \Phi_t = -i\Omega(k)\Phi, \quad (20)$$

where

$$\Lambda(k) = \begin{pmatrix} k & -is_1a_1 & -is_2a_2 \\ -ia_1 & -k-q & 0 \\ -ia_2 & 0 & -k+q \end{pmatrix} \quad (21)$$

and $\Omega(k) = \alpha\,\Omega_{nls}(k) + \beta\,\Omega_{3w}(k)$, with

$$\Omega_{nls} = \begin{pmatrix} -2k^2 - s_1a_1^2 - s_2a_2^2 & is_1a_1(2k-q) & is_2a_2(2k+q) \\ ia_1(2k-q) & 2k^2 - q^2 - s_1a_1^2 - 2s_2a_2^2 & s_2a_1a_2 \\ ia_2(2k+q) & s_1a_1a_2 & 2k^2 - q^2 - 2s_1a_1^2 - s_2a_2^2 \end{pmatrix}, \quad (22a)$$

$$\Omega_{3w} = \begin{pmatrix} 0 & -is_1c_1a_1 & -is_2c_2a_2 \\ -ic_1a_1 & -c_1(2k+q) - s_2\frac{a_2^2}{2q}(c_1-c_2) & s_2\frac{a_1a_2}{2q}(c_1-c_2) \\ -ic_2a_2 & s_1\frac{a_1a_2}{2q}(c_1-c_2) & -c_2(2k-q) - s_1\frac{a_1^2}{2q}(c_1-c_2) \end{pmatrix}. \quad (22b)$$

Since $[\Lambda(k),\Omega(k)] = 0$, the solution Ψ_0 can be written as

$$\Psi_0(x,t,k) = G(x,t)e^{i(\Lambda(k)x - \Omega(k)t)}. \quad (23)$$

Finally, the vector $Z(x,t)$ [see (12)] which appears in the Darboux-Dressing transformation (11) reads

$$Z(x,t) = G(x,t)e^{i(\Lambda(\chi^*)x - \Omega(\chi^*)t)}Z_0. \quad (24)$$

Remark 2. The Darboux-Dressing transformation (11) may lead to a singular solution of (9) due to zeros of the denominator $|\zeta|^2 - s_1|z_1|^2 - s_2|z_2|^2$. The condition that the signs s_1, s_2 are both negative ($s_1 = s_2 - 1$) is sufficient, but not necessary, for this solution to be bounded (i.e., nonsingular). Thus we will keep the signs s_1, s_2 arbitrary.

Remark 3. The parameter q, other than the signs s_1, s_2, turns out to be relevant to the stability of the plane wave solution (14). Despite the importance of this point we do not discuss it here.

III. RATIONAL SOLUTIONS

No rational dependence on x, t of the solution (11) exists if the two matrices $\Lambda(k)$ and $\Omega(k)$ (for $k = \chi^*$) are diagonalizable. Indeed, this statement stems from the expressions (11), together with (14) and (24), which imply that, in this generic case, the explicit expression (11) of the solution contains only exponential functions of x and t. Therefore we have to search for those particular, *critical*, values k_c of k, such that the two matrices $\Lambda(k_c)$ and $\Omega(k_c)$ are not diagonalizable but are instead similar to a Jordan form (two matrices A and B are similar to each other if $AT = TB$, the transformation matrix T being nonsingular, i.e., $\det T \neq 0$). Indeed, this form is generically the sum of a diagonal matrix and a nonvanishing nilpotent matrix, and our starting elementary observation is that, if N is a nilpotent matrix, e.g., $N^{m+1} = 0$ and $N^m \neq 0$ for an integer m, then $\exp(zN)$ is a matrix-valued polynomial of z of degree m. Moreover, in order to apply the Darboux-Dressing formula (11), the critical value k_c is required to be strictly complex, namely, to lie off the real axis of the complex k plane. Therefore through our investigation we disregard all those values of k which are real even if the corresponding matrices $\Lambda(k)$ and $\Omega(k)$ are similar to a Jordan form. Though the matrices $\Lambda(k)$ and $\Omega(k)$ play a similar role, it is convenient to focus first on $\Lambda(k)$ and its characteristic polynomial

$$P_\Lambda(\lambda) = \det[\lambda - \Lambda(k)] = \lambda^3 + A_2(k)\lambda^2 + A_1(k)\lambda + A_0(k) \tag{25}$$

whose coefficients take the expression [see (21)]

$$\begin{aligned} A_2(k) &= k, \quad A_1(k) = -k^2 - q^2 + s_1 a_1^2 + s_2 a_2^2, \\ A_0(k) &= -k^3 + k\left(q^2 + s_1 a_1^2 + s_2 a_2^2\right) + q\left(s_2 a_2^2 - s_1 a_1^2\right). \end{aligned} \tag{26}$$

The following proposition holds true:

Proposition 1. If $\lambda_1(k), \lambda_2(k), \lambda_3(k)$ are the three roots of the characteristic polynomial (25), then a *necessary* condition for $\Lambda(k_c)$ to be similar to a Jordan form Λ_J,

$$\Lambda(k_c) = T\,\Lambda_J\,T^{-1}, \tag{27}$$

is that either one of them, e.g., λ_3, is simple and $\lambda_1 = \lambda_2$ is double, or $\lambda_1 = \lambda_2 = \lambda_3$. T denotes the similarity transformation matrix. In the first case, $\Lambda(k_c)$ is similar to a Jordan form Λ_J if and only if $\lambda_1 = \lambda_2$ is geometrically simple,

$$\Lambda_J = \begin{pmatrix} \lambda_1 & \mu & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad \mu \neq 0; \tag{28}$$

while in the second case, $\Lambda(k_c)$ is similar to a Jordan form Λ_J if $\lambda_1 = \lambda_2 = \lambda_3$ is geometrically simple,

$$\Lambda_J = \begin{pmatrix} \lambda_1 & \mu_1 & 0 \\ 0 & \lambda_1 & \mu_1 \\ 0 & 0 & \lambda_1 \end{pmatrix}, \quad \mu_1 \neq 0. \tag{29}$$

Remark 4. The case in which $\lambda_1 = \lambda_2 = \lambda_3$ is geometrically double is the particular case of (28) for $\lambda_1 = \lambda_3$.

Remark 5. We point out for future reference that, in our notation (28) and (29), for dimensional reasons we prefer to leave the entry μ in (28) and μ_1 in (29) as free *nonvanishing* parameters rather than giving them the unit value, $\mu = \mu_1 = 1$, as commonly in use.

As for the second matrix $\Omega(k_c)$, since it commutes with $\Lambda(k_c)$, it is consequently taken by the same similarity transformation

$$\Omega(k_c) = T\,\widehat{\Omega}\,T^{-1} \tag{30}$$

into a matrix $\widehat{\Omega}$ which commutes with Λ_J, but it is not necessarily a Jordan form. Indeed, if ω_1, ω_2, ω_3 are the three eigenvalues of $\Omega(k_c)$, in the first case (i.e., $\lambda_1 = \lambda_2$), it necessarily follows that $\omega_1 = \omega_2$, so that

$$\widehat{\Omega} = \begin{pmatrix} \omega_1 & \rho & 0 \\ 0 & \omega_1 & 0 \\ 0 & 0 & \omega_3 \end{pmatrix}, \tag{31}$$

which is still a Jordan form if $\rho \neq 0$, while in the second case (i.e., $\lambda_1 = \lambda_2 = \lambda_3$),

$$\widehat{\Omega} = \begin{pmatrix} \omega_1 & \rho_1 & \rho_2 \\ 0 & \omega_1 & \rho_1 \\ 0 & 0 & \omega_1 \end{pmatrix}. \tag{32}$$

On the other hand, the values of ρ in (31) and of ρ_1 and ρ_2 in (32) have no *a priori* conditions.

Once a critical value k_c has been found, setting in (24) $\chi = k_c^*$ yields the expression

$$Z(x,t) = G(x,t)V(x,t), \tag{33}$$

$$V(x,t) = \begin{pmatrix} v(x,t) \\ v_1(x,t) \\ v_2(x,t) \end{pmatrix} = T e^{i(\Lambda_J x - \widehat{\Omega} t)} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix}, \tag{34}$$

where $\gamma_1, \gamma_2, \gamma_3$ are arbitrary complex constants. Due to the nilpotent part of Λ_J and $\widehat{\Omega}$, this last expression yields a dependence of $V(x,t)$ on x and t which is partially rational. Indeed, by inserting (28) and (31) into (33) yields the *semirational* dependence

$$V(x,t) = T \begin{pmatrix} (\gamma_1 + \gamma_2 \xi) e^{i(\lambda_1 x - \omega_1 t)} \\ \gamma_2 e^{i(\lambda_1 x - \omega_1 t)} \\ \gamma_3 e^{i(\lambda_3 x - \omega_3 t)} \end{pmatrix}, \quad \xi = i(\mu x - \rho t) \tag{35}$$

in the case λ_3 and ω_3 are (algebraically) simple. In the alternative case in which λ_1 and ω_1 are (algebraically) triple, the expression of V follows from (29) and (32), and it reads

$$V(x,t) = e^{i(\lambda_1 x - \omega_1 t)} T \begin{pmatrix} \gamma_1 + \gamma_2 \xi_1 + \gamma_3 \zeta \\ \gamma_2 + \gamma_3 \xi_1 \\ \gamma_3 \end{pmatrix}, \quad \xi_1 = i(\mu_1 x - \rho_1 t), \quad \zeta = \frac{1}{2}\xi_1^2 - i\rho_2 t. \tag{36}$$

As a consequence of (33), the expression (11) of the solution $u^{(1)}, u^{(2)}, w$ of (9) can be written in the more explicit form:

$$\begin{pmatrix} u^{(1)}(x,t) \\ u^{(2)}(x,t) \end{pmatrix} = \begin{pmatrix} e^{i(qx-\nu_1 t)} & 0 \\ 0 & e^{-i(qx+\nu_2 t)} \end{pmatrix} \left[\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \frac{2i(k_c^* - k_c)v^*}{|v|^2 - s_1|v_1|^2 - s_2|v_2|^2} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right], \tag{37a}$$

$$w(x,t) = i s_1 s_2 (c_2 - c_1) e^{-i[2qx + (\nu_2 - \nu_1)t]} \times \left[\frac{a_1 a_2}{2q} + \frac{2(k_c^* - k_c) v_1^* v_2}{|v|^2 - s_1|v_1|^2 - s_2|v_2|^2} \right]. \tag{37b}$$

Expressions (37) readily show that, if the three eigenvalues λ_j are all the same, $\lambda_1 = \lambda_2 = \lambda_3$, then the solution (37) is purely rational as its expression does not contain any exponentials [see (36)]. In the alternative case, $\lambda_1 = \lambda_2 \neq \lambda_3$, the expression (35) shows that the solution (37) is generically expressed in terms of both exponential and rational functions. Nongenerically, however, the dependence on coordinates is purely rational if $\gamma_3 = 0$ while it contains only exponentials if $\gamma_2 = 0$. We summarize the step-by-step construction of all such solutions of (9) as follows: once a critical value k_c off the real axis is found, one computes the corresponding eigenvalues λ_j, ω_j, the off-diagonal entries ρ or ρ_1, ρ_2, and the corresponding similarity matrix T. Finally, using the formula (37) yields the expression of the solution. The following two subsections describe the computation of the critical values k_c and of the corresponding similarity transformation matrix T.

A. The case $\lambda_1 = \lambda_2 = \lambda_3$

If the three roots of the characteristic polynomial (25) coincide, then $P_\Lambda(\lambda) = [\lambda - \lambda_1(k)]^3$, so that

$$\lambda_1(k) = \lambda_2(k) = \lambda_3(k) = \mathrm{tr}[\Lambda(k)]/3 = -k/3. \tag{38}$$

Moreover, by the Cayley theorem, $[\Lambda(k) + k/3]^3 = 0$ (we omit writing the identity matrix I where no confusion can arise). Therefore the requirement that the matrix $[\Lambda(k) + k/3]$ be nilpotent yields the critical values k_c. We disregard the case $[\Lambda(k) + k/3]^2 = 0$ because it leads to the strong reduction $a_1 a_2 = 0$ and to real critical values of k. Moreover the condition $[\Lambda(k) + k/3]^2 \neq 0$ excludes the limiting case in which (28) holds for $\lambda_1 = \lambda_3$ (see Remark 4). This way we compute all critical values k_c. By disregarding those values which are real, we are left with one case only, namely,

$$q \neq 0, \quad k_c = i\frac{\sqrt{27}}{2}\epsilon q, \quad s_1 = s_2 = -1, \quad a_1 = a_2 = 2q, \tag{39}$$

where ϵ is a sign, i.e., $\epsilon^2 = 1$. In this case the critical value k_c is imaginary and the free parameters are q (real) and ϵ; hence the Darboux-Dressing transformation (11) applies and the resulting solution will be considered below.

It now remains to provide the similarity transformation matrix T, as well as the two matrices Λ_J and $\hat{\Omega}$, namely ω_1 and ρ_1, ρ_2 [see (32)]. Λ_J is, however, already given by (29) with $\lambda_1(k_c) = -k_c/3$ (the nonvanishing parameter μ_1 can be fixed according to convenience). Needless to say, the expression of the similarity matrix T is not unique, and the one we give below may be changed, for instance, by a multiplication factor. In the present case in which $\lambda_1 = \lambda_2 = \lambda_3$ and $\Lambda - \lambda_1$ is nilpotent with $(\Lambda - \lambda_1)^2 \neq 0, (\Lambda - \lambda_1)^3 = 0$, the construction of the similarity transformation matrix T requires a tedious but straight computation, and we limit ourselves to give the final formula: $\lambda_1 = \lambda_2 = \lambda_3 = -i\frac{\sqrt{3}}{2}\epsilon q$ so that

$$\Lambda(k_c) = \lambda_1 + \mu_1 N, \quad N = \begin{pmatrix} \epsilon\sqrt{3} & 1 & 1 \\ -1 & \theta & 0 \\ -1 & 0 & \theta^* \end{pmatrix}, \quad \mu_1 = 2iq, \quad \theta = \frac{1}{2}(-\epsilon\sqrt{3} + i), \tag{40}$$

where the dimensionless matrix N is nilpotent and θ is a phase factor, namely, $|\theta| = 1$; in this case the similarity transformation (27), with (29), is provided by the matrix

$$T = \begin{pmatrix} \theta & 0 & -i \\ 1 & \theta^* & i\epsilon\sqrt{3} \\ i\theta^* & i & 0 \end{pmatrix} \tag{41}$$

whose *Jordanization* action is specified by the formula

$$N = T N_J T^{-1}, \quad N_J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{42}$$

As for the matrix Ω,

$$\omega_1 = \omega_2 = \omega_3 = \frac{\mathrm{tr}(\Omega)}{3} = \frac{11}{2}\alpha q^2 + \beta q[c_1 - c_2 - i\epsilon\sqrt{3}(c_1 + c_2)] \tag{43}$$

and

$$\Omega(k_c) = \omega_1 + 2\alpha q^2 \begin{pmatrix} 8 & 3\epsilon\sqrt{3} + i & 3\epsilon\sqrt{3} - i \\ -3\epsilon\sqrt{3} - i & -4 & -2 \\ -3\epsilon\sqrt{3} + i & -2 & -4 \end{pmatrix} + \beta q \times \begin{pmatrix} i\epsilon\sqrt{3}(c_1 + c_2) + c_2 - c_1 & 2ic_1 & 2ic_2 \\ -2ic_1 & i\epsilon\sqrt{3}(c_2 - 2c_1) - c_2 & -2(c_1 - c_2) \\ -2ic_2 & -2(c_1 - c_2) & i\epsilon\sqrt{3}(c_1 - 2c_2) + c_1 \end{pmatrix}, \tag{44}$$

while $\widehat{\Omega}$ has the expression (32), namely, $\widehat{\Omega} = \omega_1 + \rho_1 N_J + \rho_2 N_J^2$, which implies

$$\Omega(k_c) = \omega_1 + \rho_1 N + \rho_2 N^2, \tag{45}$$

where the matrix N has the expression (40). Comparing (45) with (40) yields

$$\begin{aligned} \rho_1 &= 4\alpha q^2 \epsilon\sqrt{3} + 2\beta q(\theta c_1 - \theta^* c_2), \\ \rho_2 &= 4\alpha q^2 + 2\beta q(c_1 - c_2). \end{aligned} \tag{46}$$

We now apply the Darboux-Dressing construction formula (37) with the naked solution appropriate to this case [namely, (14) with $a_1 = a_2 = 2q$], and the vector $V(x,t)$ as given by (36). Thus we arrive to the following expression of the solution:

$$\begin{pmatrix} u^{(1)}(x,t) \\ u^{(2)}(x,t) \end{pmatrix} = 2q \begin{pmatrix} e^{i(qx-\nu_1 t)} & 0 \\ 0 & e^{-i(qx+\nu_2 t)} \end{pmatrix} \times \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{3\epsilon\sqrt{3}A^*}{|A|^2 + |A_1|^2 + |A_2|^2} \begin{pmatrix} \theta^* A_1 \\ \theta A_2 \end{pmatrix} \right] \tag{47a}$$

$$w(x,t) = 2iq(c_2 - c_1)e^{-i[2qx+(\nu_2-\nu_1)t]} \times \left[1 + \frac{3\epsilon\sqrt{3}\theta^* A_1^* A_2}{|A|^2 + |A_1|^2 + |A_2|^2} \right] \tag{47b}$$

with the notation

$$\begin{aligned} \nu &= -15\alpha q^2 - \tfrac{3}{2}\beta q(c_1 - c_2), \\ \nu_1 &= \nu + \tfrac{1}{2}\beta q(c_1 + c_2), \quad \nu_2 = \nu - \tfrac{1}{2}\beta q(c_1 + c_2), \\ A &= \gamma_1 + \gamma_2 \xi_1 + \gamma_3(\zeta - i\theta^*), \\ A_1 &= \gamma_1 + \gamma_2(\xi_1 + \theta^*) + \gamma_3(\zeta + \theta^* \xi_1 + i\epsilon\sqrt{3}), \\ A_2 &= \gamma_1 + \gamma_2(\xi_1 + \theta) + \gamma_3(\zeta + \theta\xi_1), \end{aligned} \tag{48}$$

while ξ_1 and ζ are defined by (36) with $\mu_1 = 2iq$ [see (40)]. We observe that not all three complex parameters $\gamma_1, \gamma_2, \gamma_3$, as introduced via (33), are essential as one of them can be arbitrarily fixed and two more real parameters can be absorbed as translations of x and t. The analysis of this solution is detailed in Sec. IV.

B. The case $\lambda_1 = \lambda_2 \neq \lambda_3$

Here we consider the case in which, for a critical value $k = k_c$, one eigenvalue (e.g., λ_1) of $\Lambda(k)$ is algebraically double but geometrically simple, so that $\Lambda(k_c)$ is similar to a Jordan form; see (27) and (28). Since finding k_c generically requires computing the roots of a fourth order polynomial (see below), we postpone this computation and we construct first the similarity transformation matrix T with the assumption that $k = k_c$ is known. If $\lambda_1 = \lambda_1(k_c)$ and $\lambda_3 = \lambda_3(k_c)$ are the corresponding eigenvalues of Λ, we obtain the following general expression

$$T = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 \\ -\frac{i\phi_1 a_1}{(\lambda_1+k+q)} & -\frac{i\phi_2 a_1}{(\lambda_1+k+q)} + \frac{i\mu\phi_1 a_1}{(\lambda_1+k+q)^2} & -\frac{i\phi_3 a_1}{(\lambda_3+k+q)} \\ -\frac{i\phi_1 a_2}{(\lambda_1+k-q)} & -\frac{i\phi_2 a_2}{(\lambda_1+k-q)} + \frac{i\mu\phi_1 a_2}{(\lambda_1+k-q)^2} & -\frac{i\phi_3 a_2}{(\lambda_3+k-q)} \end{pmatrix}, \tag{49}$$

$k = k_c$, which turns out to depend on the three complex parameters ϕ_1, ϕ_2, ϕ_3, arbitrary except for the condition that the matrix T be nonsingular. Since the determinant

$$\det T = 2\phi_1^2 \phi_3 q \mu a_1 a_2 \frac{(\lambda_1 - \lambda_3)^2}{[(\lambda_3 + k)^2 - q^2][(\lambda_1 + k)^2 - q^2]^2} \tag{50}$$

does not depend on ϕ_2, we may take $\phi_2 = 0$ and conveniently set $\phi_1 = (\lambda_1 + k)^2 - q^2$ and $\phi_3 = (\lambda_3 + k)^2 - q^2$. With this choice of the parameters the matrix T takes the expression ($k = k_c$)

$$T = \begin{pmatrix} (\lambda_1 + k)^2 - q^2 & 0 & (\lambda_3 + k)^2 - q^2 \\ -ia_1(\lambda_1 + k - q) & i\mu a_1 \frac{\lambda_1+k-q}{\lambda_1+k+q} & -ia_1(\lambda_3 + k - q) \\ -ia_2(\lambda_1 + k + q) & i\mu a_2 \frac{\lambda_1+k+q}{\lambda_1+k-q} & -ia_2(\lambda_3 + k + q) \end{pmatrix}, \tag{51}$$

where the condition of being invertible reads $q\mu a_1 a_2(\lambda_1 - \lambda_3) \neq 0$. We note that the derivation of this expression requires not only that $P_\Lambda(\lambda_1) = P_\Lambda(\lambda_3) = 0$ but also that $P'_\Lambda(\lambda_1) = 0$ where $P'_\Lambda(\lambda) = dP_\Lambda(\lambda)/d\lambda$. Since this matrix T becomes singular (i.e., noninvertible) if $q = 0$ [see (50)], before proceeding we prefer to first consider this separate case here below. In this respect we observe that we assume $a_1 a_2 \neq 0$ if $q \neq 0$ since if, for instance a_2 were vanishing, then the background parameter q would be irrelevant as it can be transformed away by a Galilei-type transformation, which yields therefore to the $q = 0$ case.

The assumption $q = 0$ leads to consider two separate cases, namely, either $s_1 a_1^2 + s_2 a_2^2 \neq 0$ or $s_1 a_1^2 + s_2 a_2^2 = 0$. We disregard this second case as our analysis shows that its corresponding solution becomes singular because of the vanishing of the denominator in the expression (37). Thus we treat here only the case in which $q = 0$ and $s_1 a_1^2 + s_2 a_2^2$ is strictly nonvanishing. With these assumptions the explicit expression of the roots of $P_\Lambda(\lambda)$ are

$$\begin{aligned} \lambda_1 &= \sqrt{k^2 - s_1 a_1^2 - s_2 a_2^2}, \\ \lambda_2 &= -\sqrt{k^2 - s_1 a_1^2 - s_2 a_2^2}, \quad \lambda_3 = -k, \quad q = 0. \end{aligned} \tag{52}$$

The conditions that $\lambda_1 = \lambda_2$ and that the value of k_c be not real leads to the condition $s_1 a_1^2 + s_2 a_2^2 < 0$. This inequality therefore excludes the choice $s_1 = s_2 = 1$ and leads to the two values $k = k_c = ip$, $p = \pm\sqrt{-s_1 a_1^2 - s_2 a_2^2}$, $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = -k_c = -ip$. We find, however, that the condition $s_1 s_2 = 1$ is necessary and sufficient for the solution (11a) to be nonsingular (in general singularities come from the zeros of the denominator which appears in this expression). We conclude therefore that only the (focusing) case $s_1 = s_2 = -1$ is worth considering. Thus in this particular (and interesting, see below) case the eigenvalues are

$$\lambda_1 = \lambda_2 = 0, \quad \lambda_3 = -ip, \quad p = \epsilon\sqrt{a_1^2 + a_2^2}, \quad \epsilon^2 = 1. \tag{53}$$

Hence the matrix Λ reads

$$\Lambda(k_c) = \begin{pmatrix} ip & ia_1 & ia_2 \\ -ia_1 & -ip & 0 \\ -ia_2 & 0 & -ip \end{pmatrix} \tag{54}$$

and is taken into the Jordan form [here we set $\mu = -ip$; see (28)]

$$\Lambda_J = -ip \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{55}$$

by the similarity transformation (27) with

$$T = \begin{pmatrix} -p & p & 0 \\ a_1 & 0 & a_2 \\ a_2 & 0 & -a_1 \end{pmatrix}. \tag{56}$$

Moreover, since this case does not apply to the 3WRI equations [see (14b)], we set $\alpha = 1$ and $\beta = 0$ so that the matrix $\Omega(k_c)$ has the expression

$$\Omega(k_c) = \begin{pmatrix} 3p^2 & 2pa_1 & 2pa_2 \\ -2pa_1 & -p^2 + a_2^2 & -a_1a_2 \\ -2pa_2 & -a_1a_2 & -p^2 + a_1^2 \end{pmatrix}, \tag{57}$$

which is similar to the Jordan form $\widehat{\Omega}$ [see (30) and (31)] with

$$\omega_1 = \omega_2 = \omega = p^2, \quad \omega_3 = 0, \quad \rho = -2p^2. \tag{58}$$

These findings, together with the explicit expression (35) and the Darboux-Dressing formula (11), yield the semirational solution of the VNLS equations

$$\begin{pmatrix} u^{(1)}(x,t) \\ u^{(2)}(x,t) \end{pmatrix} = e^{2i\omega t}\left[\frac{L}{B}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \frac{M}{B}\begin{pmatrix} a_2 \\ -a_1 \end{pmatrix}\right], \tag{59}$$

where $L = \frac{3}{2} - 8\omega^2t^2 - 2p^2x^2 + 8i\omega t + |f|^2e^{2px}$, $M = 4f(px - 2i\omega t - \frac{1}{2})e^{(px+i\omega t)}$, $B = \frac{1}{2} + 8\omega^2t^2 + 2p^2x^2 + |f|^2e^{2px}$, and where f is a complex arbitrary constant. It should be remarked that the dressing construction has introduced γ_1, γ_2, γ_3 as arbitrary parameters [see (35)]. However, only the complex parameter γ_3 is left essential since the other parameters can be absorbed by translations of the coordinates x, t. In fact, the expression (59) is derived by setting $\gamma_1 = 1/2$, $\gamma_2 = 1$, and $\gamma_3 = -f$. We note also that the dependence of L, M, and B [see (59)] on x,t is both polynomial and exponential only through the dimensionless variables ax and ωt. Moreover the vector solution (59) turns out to be a combination of the two constant orthogonal vectors $(a_1, a_2)^T$ and $(a_2, -a_1)^T$.

Let us proceed further to the case in which $q \neq 0$, and let us maintain the assumption that k_c is known. We first aim to computing the Jordan matrices Λ_J (28) and $\widehat{\Omega}$ (31), which amounts to computing $\lambda_1, \lambda_3, \omega_1, \omega_3$, and ρ. We start from the observation that the eigenvalue λ_1 is a zero of both the polynomial $P_\Lambda(\lambda)$ and of its derivative [see (25)] $P'_\Lambda(\lambda) = 3\lambda^2 + 2A_2(k_c)\lambda + A_1(k_c) = 3(\lambda - \lambda_+)(\lambda - \lambda_-)$ where

$$\lambda_\pm = -\frac{1}{3}A_2 \pm \sqrt{\left(\frac{A_2}{3}\right)^2 - \frac{A_1}{3}}\ . \tag{60}$$

Therefore this readily implies the following proposition:

Proposition 2. Assume $k = k_c$, then if $P_\Lambda(\lambda_+) = 0$, the three roots of $P_\Lambda(\lambda)$ are

$$\lambda_1 = \lambda_2 = \lambda_+, \quad \lambda_3 = \tfrac{1}{2}(3\lambda_- - \lambda_+), \tag{61}$$

while if $P_\Lambda(\lambda_-) = 0$, the three roots of $P_\Lambda(\lambda)$ are

$$\lambda_1 = \lambda_2 = \lambda_-, \quad \lambda_3 = \tfrac{1}{2}(3\lambda_+ - \lambda_-). \tag{62}$$

The proof of these formulas is elementary and consistent with the fact that the discriminant of a generic third degree polynomial [see (25)] is proportional to the product $[P_\Lambda(\lambda_+)][P_\Lambda(\lambda_-)]$. The explicit expression of λ_1 and λ_3 finally obtains by inserting in (60) the coefficients A_2, A_1 in terms of k via (26).

As for the eigenvalues ω_1, ω_3 and the parameter ρ, see (31), we use the similarity property (30), the matrix transformation T being given by (51), and we obtain the expressions (with $k = k_c$)

$$\begin{aligned} \omega_i &= -\alpha\left\{2k\lambda_1 + s_1a_1^2 + s_2a_2^2 + q\left[\frac{s_1a_1^2}{(\lambda_1+k+q)} - \frac{s_2a_2^2}{(\lambda_1+k-q)}\right]\right\} \\ &\quad - \frac{\beta}{2}\left\{(c_1+c_2)(k-\lambda_1) + (c_1-c_2)\left[\frac{s_1a_1^2}{(\lambda_1+k+q)} - \frac{s_2a_2^2}{(\lambda_1+k-q)}\right]\right\}, \quad i = 1,2 \\ \omega_3 &= -\alpha\left\{2k\lambda_3 + s_1a_1^2 + s_2a_2^2 + q\left[\frac{s_1a_1^2}{(\lambda_3+k+q)} - \frac{s_2a_2^2}{(\lambda_3+k-q)}\right]\right\} \\ &\quad - \frac{\beta}{2}\left\{(c_1+c_2)(k-\lambda_3) + (c_1-c_2)\left[\frac{s_1a_1^2}{(\lambda_3+k+q)} - \frac{s_2a_2^2}{(\lambda_3+k-q)}\right]\right\}, \\ \rho &= -\alpha\mu\left\{2k - q\left[\frac{s_1a_1^2}{(\lambda_1+k+q)^2} - \frac{s_2a_2^2}{(\lambda_1+k-q)^2}\right]\right\} + \frac{\beta}{2}\left\{c_1 + c_2 + (c_1-c_2)\left[\frac{s_1a_1^2}{(\lambda_1+k+q)^2} - \frac{s_2a_2^2}{(\lambda_1+k-q)^2}\right]\right\}. \end{aligned} \tag{63}$$

The main task now is finding the critical values k_c which are in the complex k plane strictly off the real axis ($\mathrm{Im}k_c \neq 0$). These values are zeros of the discriminant of the polynomial $P_\Lambda(\lambda)$ (25). By taking into account the expression of the coefficients

(26), this discriminant turns out to be proportional to the fourth order monodic polynomial

$$\Delta(k) = k^4 + D_3k^3 + D_2k^2 + D_1k + D_0, \tag{64}$$

where the coefficients are

$$\begin{aligned} D_3 &= \left(s_2a_2^2 - s_1a_1^2\right)/(2q),\\ D_2 &= -\big[8q^4 - \left(s_1a_1^2 + s_2a_2^2\right)^2\\ &\quad + 20q^2\left(s_1a_1^2 + s_2a_2^2\right)\big]/(2^4q^2),\\ D_1 &= -9\left(s_2a_2^2 - s_1a_1^2\right)\left(2q^2 + s_1a_1^2 + s_2a_2^2\right)/(2^4q),\\ D_0 &= \left(q^2 - s_1a_1^2 - s_2a_2^2\right)^3/(2^4q^2) - \left(\tfrac{3}{4}\right)^3\left(s_2a_2^2 - s_1a_1^2\right)^2. \end{aligned} \tag{65}$$

Though the generic fourth degree algebraic equation is solvable, the explicit expression of its solutions is so complicate that its use does not make their computation any easier than just computing them numerically. One exception to this wisdom is the case in which this algebraic equation reduces to a second degree equation. This is the case if we assume the condition $s_1a_1^2 = s_2a_2^2$, which implies that $D_1 = D_3 = 0$ with the consequence that the vanishing of the polynomial (64) reads as the second degree equation

$$\Delta(k) = R(h) = h^2 + D_2h + D_0 = 0 \tag{66}$$

for the new variable $h = k^2$. Here the coefficients are

$$\begin{aligned} D_2 &= -\left(2q^4 - a_1^4 + 10sq^2a_1^2\right)/(2^2q^2),\\ D_0 &= \left(q^2 - 2sa_1^2\right)^3/(2^4q^2). \end{aligned} \tag{67}$$

In this special case the reality of a_1, a_2 implies the condition $s_1 = s_2 = s$ and $a_1^2 = a_2^2$, which has been used to pass from (65) to (67). The search for the critical values k_c in the parameter space, the parameters being q, a_1 and the sign s, is now simple since the four zeros of the discriminant (64) have the explicit expression

$$\begin{aligned} &k = k(\eta_1,\eta_2) = \eta_2\left(-\tfrac{1}{2}D_2 + \eta_1\sqrt{\tfrac{1}{4}D_2^2 - D_0}\right)^{1/2},\\ &\eta_1^2 = \eta_2^2 = 1, \end{aligned} \tag{68}$$

which is the starting point of our short discussion of the corresponding family of solutions we present in Sec. IV B2. We note here that these expressions of k_c are explicit because of the assumption $s_1a_1^2 = s_2a_2^2$. In the generic case in which $q \neq 0$ and $s_1a_1^2 - s_2a_2^2 \neq 0$, we prefer to compute k_c numerically as roots of the discriminant (64).

IV. ANALYSIS OF THE SOLUTIONS AND CONCLUSIONS

In the previous section we have shown the way of deriving a rich family of solutions of the system (9). In fact, we have constructed *all* the bounded (rational or semirational) solutions which can be obtained via the DDT method. The aim of this section is to select and detail some of such solutions.

We separately treat those which are solutions of the VNLS system (1) (by setting $\alpha = 1$, $\beta = 0$) and those which are solutions of the 3WRI equations (2) (by setting $\alpha = 0$, $\beta = 1$). As for the parameters which appear in the expressions of our solutions, some of them are structural coefficients which enter the partial differential equations (9), such as the signs s_1, s_2 and the characteristic velocities c_1, c_2; other parameters, i.e., q, a_1, a_2, originate from the background [see (14))] while others, γ_1, γ_2, γ_3, come from the DDT transformation. In this transformation there appears also the critical value k_c of the spectral variable k, which depends only on s_1, s_2, q, a_1, a_2. Although some of the parameters are not essential as they could be eliminated by using simple symmetries, in some cases we prefer to keep them because of their physical significance. We point out also that the background parameter q plays a distinctive role in our solutions as it has no counterpart in the scalar NLS equation. While our solutions of the 3WRI equations (2) are novel, some of the solutions of the VNLS equations (1) reported here have been constructed by Darboux method in Refs. [19,20]. However, they belong to the subset of our parameter space corresponding to the focusing case $s_1 = s_2 = 1$ only.

A. $\lambda_1 = \lambda_2 = \lambda_3$

In this case the solutions are rather peculiar as they are all purely rational. Only two critical values of k are possible, namely, $k_c = \pm iq\sqrt{27}/2$ as specified by (39). These solutions exist only if $s_1 = s_2 = -1$, which is the focusing case of the VNLS equations, together with the condition $a_1 = a_2 = 2q$ for the background amplitudes. The general expression of the corresponding solutions is (47). As for the three complex parameters γ_1, γ_2, γ_3, we omit considering $\gamma_2 = \gamma_3 = 0$ since in this case the expression (47) is trivially that of a plane wave. Thus we find it convenient to illustrate the dependence of the solution on these parameters by considering separately the two cases: (1) $\gamma_3 = 0$ and (2) $\gamma_2 = 0$. With no loss of generality because of translation invariance, one can set $\gamma_2 = 1$, $\gamma_1 = 0$ in the first case and $\gamma_3 = 1$, $\gamma_2 = 0$, while γ_1 remains arbitrary and complex, in the second case. Moreover the expression of the solution is the ratio of two polynomials of second degree in the first case (1) and of two polynomials of fourth degree in the second case (2). Figures 1–4 illustrate these two cases separately for the VNLS and for the 3WRI equations.

1. *Solutions of the VNLS*

Let $X = qx$ and $T = q^2t$ be rescaled variables; let $u^{(j)}(x,t) = qU^{(j)}(X,T)$, $j = 1,2$.

Case $\gamma_3 = 0$, $\gamma_2 = 1$, $\gamma_1 = 0$:

$$U^{(1)} = 2i\theta e^{i(X+15T)}\left[\frac{12X^2 + 144T^2 + (4\epsilon\sqrt{3} + 6i)X - 36iT - 1 + i\epsilon\sqrt{3}}{12X^2 + 144T^2 + 4\epsilon\sqrt{3}X + 2}\right]. \tag{69}$$

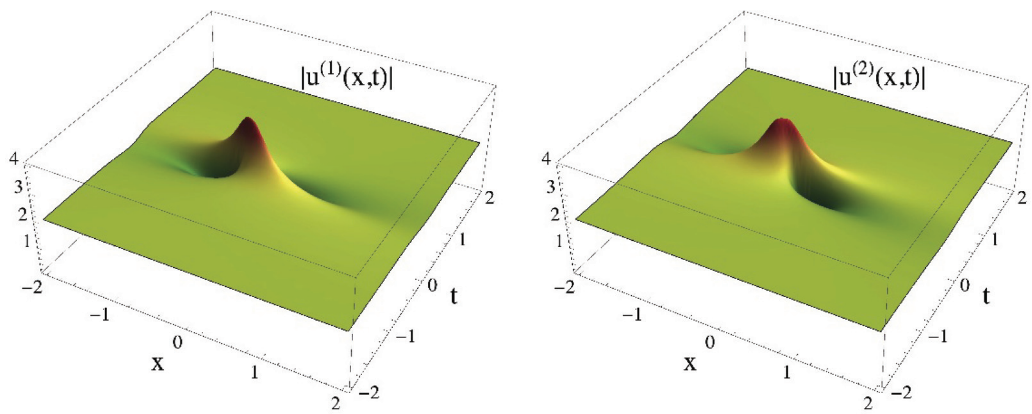

FIG. 1. (Color online) VNLS: $k_c = i\frac{\sqrt{27}}{2}$, $s_1 = s_2 = -1$, $a_1 = a_2 = 2$, $q = 1$, $\epsilon = 1$; $\gamma_2 = 1$, $\gamma_1 = \gamma_3 = 0$; see (69).

Since this solution satisfies the relation $u^{(2)}(x,t) = u^{(1)*}(x,-t)$ we report only the component $u^{(1)}(x,t) = qU^{(1)}(qx,q^2t)$; Fig. 1 displays the amplitudes $|u^{(1)}(x,t)|$ and $|u^{(2)}(x,t)|$ for a choice of parameters (see caption).

Case $\gamma_3 = 1$, $\gamma_2 = 0$, $\gamma_1 \neq 0$:

$$\begin{aligned} U^{(1)} &= 2i\theta e^{i(X+15T)}\frac{P_4^{(1)}}{P_4}, \\ U^{(2)} &= -2i\theta^* e^{-i(X-15T)}\frac{P_4^{(2)}}{P_4}, \end{aligned} \tag{70}$$

where the fourth degree polynomials $P_4^{(1)}$, $P_4^{(2)}$, P_4 are given in the Appendix. Figure 2 displays the amplitudes $|u^{(1)}(x,t)|$ and $|u^{(2)}(x,t)|$ (see caption).

2. *Solutions of the 3WRI*

Let $X = qx$ and $T = qt$ be rescaled variables; let $u^{(j)}(x,t) = qU^{(j)}(X,T)$, $j = 1,2$, $w(x,t) = qW(X,T)$.

Case $\gamma_3 = 0$, $\gamma_2 = 1$, $\gamma_1 = 0$:

$$\begin{aligned} U^{(1)} &= 2i\theta e^{i[X+T(c_1-2c_2)]}\frac{Q_2^{(1)}}{M_2}, \\ W &= 2\theta(c_1-c_2)e^{-i[2X-T(c_1+c_2)]}\frac{Q_2}{M_2}, \end{aligned} \tag{71}$$

where the second degree polynomials $Q_2^{(1)}$, Q_2, M_2 are given in the Appendix. Since this solution satisfies the relation $u^{(2)}(x,t,c_1,c_2) = u^{(1)*}(x,t,c_2,c_1)$ we report the expression of the components $u^{(1)}(x,t) = qU^{(1)}(qx,qt)$, $w(x,t) = qW(qx,qt)$ only. Figure 3 displays the amplitudes $|u^{(1)}(x,t)|$, $|u^{(2)}(x,t)|$, and $|w(x,t)|$ (see caption).

Case $\gamma_3 = 1$, $\gamma_2 = 0$, $\gamma_1 \neq 0$:

$$\begin{aligned} U^{(1)} &= 2i\theta e^{i[X+T(c_1-2c_2)]}\frac{Q_4^{(1)}}{M_4}, \\ U^{(2)} &= -2i\theta^* e^{-i[X+T(c_2-2c_1)]}\frac{Q_4^{(2)}}{M_4}, \\ W &= 2\theta(c_1-c_2)e^{-i[2X-T(c_1+c_2)]}\frac{Q_4}{M_4}, \end{aligned} \tag{72}$$

where the fourth degree polynomials $Q_4^{(1)}$, $Q_4^{(2)}$, Q_4, M_4 are given in the Appendix. Figure 4 displays the amplitudes $|u^{(1)}(x,t)|$, $|u^{(2)}(x,t)|$ and $|w(x,t)|$ (see caption).

B. The case $\lambda_1 = \lambda_2 \neq \lambda_3$

The expression (37), together with (35), shows that generically these solutions feature a dependence on coordinates which is both rational and exponential. In particular, however, if $\gamma_3 = 0$ the dependence is purely rational, while if $\gamma_2 = 0$ the solution has only exponential functions. In the following

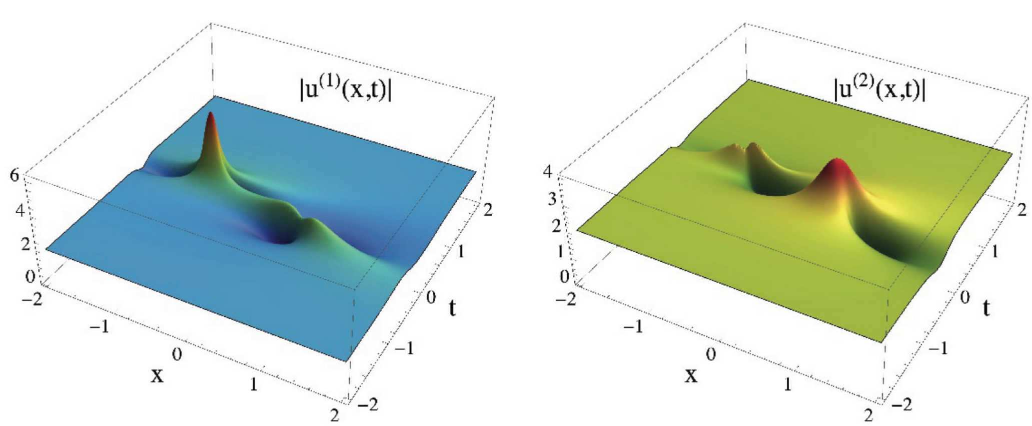

FIG. 2. (Color online) VNLS: $k_c = i\frac{\sqrt{27}}{2}$, $s_1 = s_2 = -1$, $a_1 = a_2 = 2$, $q = 1$, $\epsilon = 1$; $\gamma_1 = i$, $\gamma_2 = 0$, $\gamma_3 = 1$; see (70).

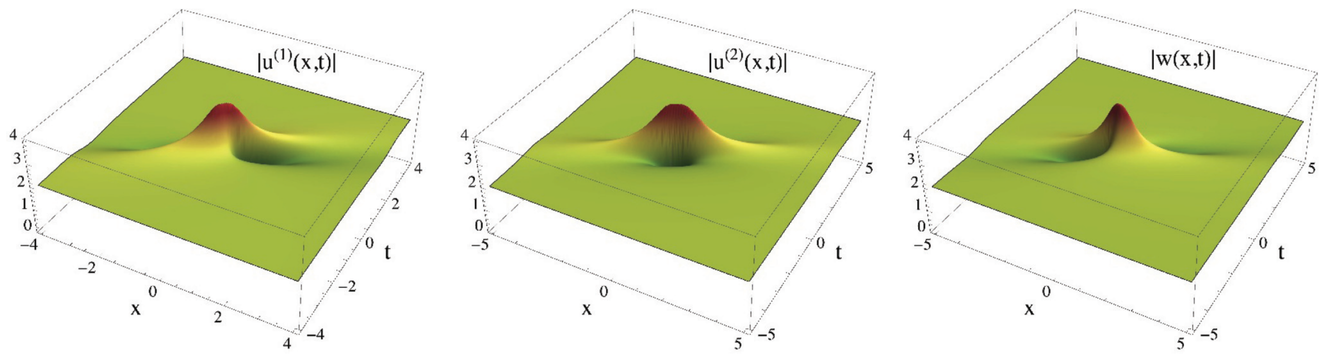

FIG. 3. (Color online) 3WRI: $k_c = i\frac{\sqrt{27}}{2}$, $s_1 = s_2 = -1$, $a_1 = a_2 = 2$, $q = 1$, $\epsilon = 1$, $c_1 = 1$, $c_2 = 2$; $\gamma_2 = 1$, $\gamma_1 = \gamma_3 = 0$; see (71).

we disregard this last case and consider only solutions with $\gamma_2 \neq 0$. Here we separately consider solutions corresponding to $q = 0$ and different background amplitudes, $a_1 \neq a_2$, with $q \neq 0$ but $a_1 = a_2$ and, finally, with $q \neq 0$ and $a_1 \neq a_2$. These distinctions are merely due to computational reasons. However, and interestingly enough, we numerically show below that in the last two cases (i.e., with $q \neq 0$) bounded rational solutions exist not only in the focusing case $s_1 = s_2 = -1$, as for the Peregrine soliton of the scalar NLS equation, but also in the defocusing case $s_1 = s_2 = 1$ and in the mixed case $s_1 s_2 = -1$.

1. $q = 0$ and vector Peregrine solutions

In this case the solution, which is well described by its expression (59), applies only to the VNLS equation. In this respect we first notice that this expression (59), with $f = 0$ and $a_2 = 0$, coincides with the Peregrine soliton of the scalar NLS equation. We further note that, since the two components $u^{(1)}(x,t,a_1,a_2)$, $u^{(2)}(x,t,a_1,a_2)$ satisfy the relation $u^{(2)}(x,t,a_1,a_2) = u^{(1)}(x,t,a_2,-a_1)$, we limit our attention only to $u^{(1)}(x,t)$. In the rescaled variables $X = x\sqrt{a_1^2 + a_2^2}$, $T = t(a_1^2 + a_2^2)$, this solution $u^{(1)}(x,t) = U^{(1)}(X,T)$ [see (59)] may be written as

$$U^{(1)} = e^{2iT}\left\{a_1\left[\frac{(2+8iT) + (4X^2+16T^2-8iT-1)\tanh(X-Z)}{4X^2+16T^2+1}\right] + a_2\frac{\sqrt{2}f}{4|f|}\left(\frac{8X-16iT-1}{\sqrt{4X^2+16T^2+1}}\right)\frac{1}{\cosh(X-Z)}\right\} \tag{73}$$

where the curve $X = Z(T)$ is the *trajectory* of the soliton as implicitly defined by the formula

$$2|f|^2 e^{2Z} = 4Z^2 + 16T^2 + 1. \tag{74}$$

As a consequence of these expressions, the large T asymptotic behavior along the curve $X = Z(T)$ is found to be

$$\begin{aligned} U^{(1)}(X,T) &\to e^{2iT}\left[a_1\tanh(X-Z) - ia_2\frac{\sqrt{2}f}{|f|}\frac{\operatorname{sign}T}{\cosh(X-Z)}\right], \\ Z(T) &\to \log|T| + \frac{1}{2}\log\left(\frac{8}{|f|^2}\right) + \mathrm{O}\left(\frac{\log|T|}{|T|}\right), \\ T &\to \pm\infty. \end{aligned} \tag{75}$$

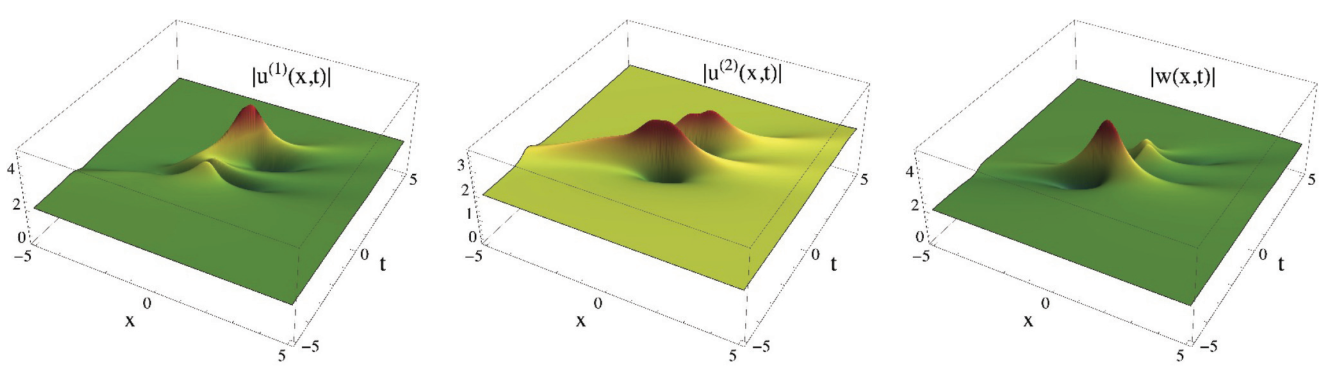

FIG. 4. (Color online) 3WRI: $k_c = i\frac{\sqrt{27}}{2}$, $s_1 = s_2 = -1$, $a_1 = a_2 = 2$, $q = 1$, $\epsilon = 1$, $c_1 = 1$, $c_2 = 2$; $\gamma_1 = i$, $\gamma_2 = 0$, $\gamma_3 = 1$; see (72).

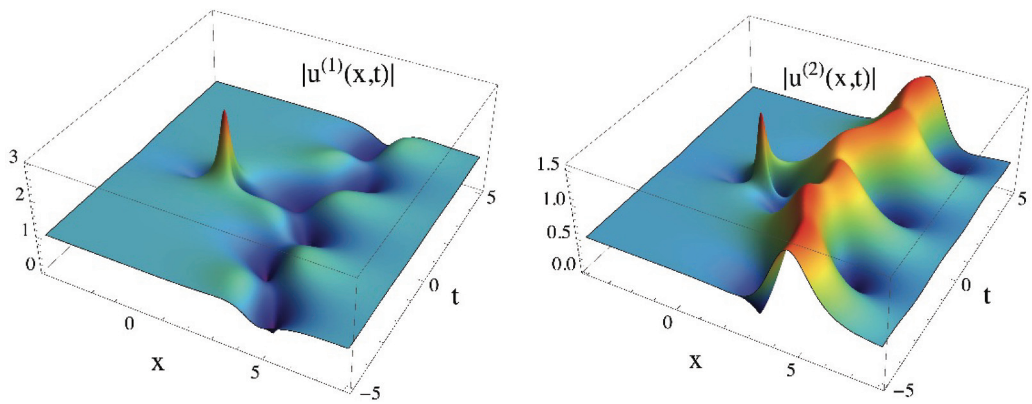

FIG. 5. (Color online) VNLS: $k_c = i\frac{\sqrt{5}}{2}$, $s_1 = s_2 = -1$, $q = 0$, $a_1 = 1$, $a_2 = 0.5$, $f = 0.1i$; see (73).

We observe that, as suggested by (73) and explicitly indicated by the asymptotic expression of $U^{(1)}(X,T)$ in (75), the amplitude a_1 multiplies a kink-type profile while the amplitude a_2 multiplies a bright-type pulse. Moreover the asymptotic motion [see $Z(T)$ in (75)] is that of a particle which comes from $x = +\infty$ and goes back to $x = +\infty$ where it "stops" since its velocity asymptotically vanishes, namely, $dZ(T)/dT \to 1/T + \mathrm{O}(\log|T|/T^2)$. Figure 5 shows an instance (see caption) of the amplitudes $|u^{(1)}(x,t)|$ and $|u^{(2)}(x,t)|$. Further instances of this solution (73) are reported in Refs. [18,19].

2. $q \neq 0$ *and* $a_1 = a_2$

This family of solutions possesses two novel features with respect to those discussed in the previous subsections. First, the choice $s_1 = s_2 = 1$ is compatible with the boundedness of solutions (see below). Second, the conditions on the parameter set for the existence of a critical value k_c lead to threshold phenomena for the dimensionless positive parameter $m = a_1^2/q^2$. As implied by the explicit expression (68) of the zeros of the discriminant (64), alias (66), we state the following proposition:

Proposition 3. Assume $s_1 = s_2 = 1$:

(1) If $q^2 \geqslant 2a_1^2$ then the four zeros $k(\eta_1,\eta_2)$, see (68), are real and no (complex) critical value k_c exists.

(2) If $q^2 < 2a_1^2$ then the two zeros $k(1,\eta_2)$ are real and the other two $k(-1,\eta_2)$ are imaginary. Therefore in this subset of the parameter plane (a_1, q) there are two critical values with opposite sign, i.e., $k_c = k(-1,\eta_2)$ or, explicitly,

$$k_c = k(-1,\eta_2) = i\eta_2\left(\tfrac{1}{2}D_2 + \sqrt{\tfrac{1}{4}D_2^2 - D_0}\right)^{1/2},\quad \eta_2^2 = 1, \tag{76}$$

where D_0 and D_2 are given by (67) with $s = 1$.

Proposition 4. Assume $s_1 = s_2 = -1$:

(1) If $q^2 > \frac{1}{4}a_1^2$ then the four zeros $k(\eta_1,\eta_2)$ [see (68)] are strictly complex (namely $\mathrm{Im}[k] \neq 0$), and therefore there are four critical values $k_c = k(\eta_1,\eta_2)$.

(2) If $q^2 \leqslant \frac{1}{4}a_1^2$ then the four zeros are imaginary, and the critical values are again $k_c = k(\eta_1,\eta_2)$.

Once k_c is computed, its corresponding solution of the equations (9) is obtained through the following chain of steps: (1) use Proposition 2 to compute the eigenvalues λ_1 and λ_3, (2) compute ω_1, ω_3, ρ according to (63), (3) insert the expression (51) of the similarity matrix T in (35) to compute the vector V, and (4) finally apply the Darboux-Dressing formula (37). Instances of solutions in this special class are shown in Figs. 6–10. Precisely the solution of the VNLS equations (1) in Fig. 6 is of particular interest since it refers to the defocusing case $s_1 = s_2 = 1$ which has no counterpart in the scalar NLS equation. We believe that its existence is made possible by the

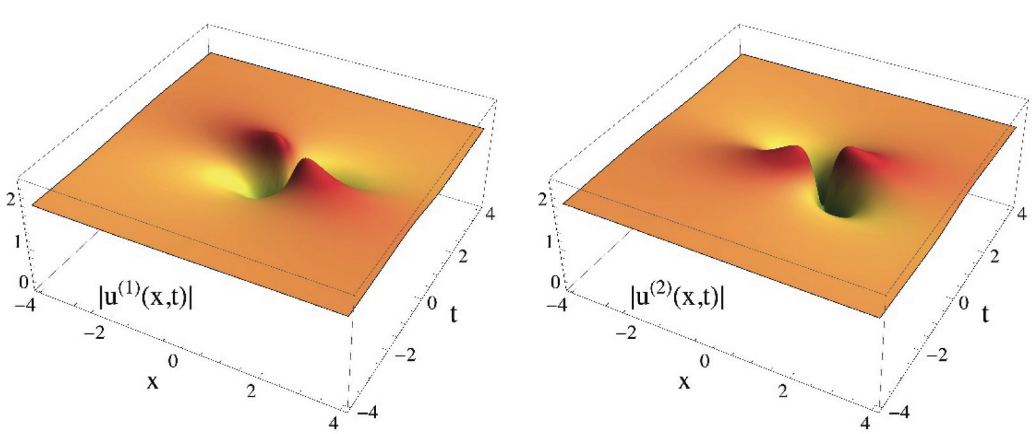

FIG. 6. (Color online) VNLS: $k_c = \frac{i}{2}\sqrt{-13 + 16\sqrt{2}}$, $s_1 = s_2 = 1$, $q = 1$, $a_1 = a_2 = 2$; $\gamma_2 = 1$, $\gamma_1 = \gamma_3 = 0$; see Sec. IV B2.

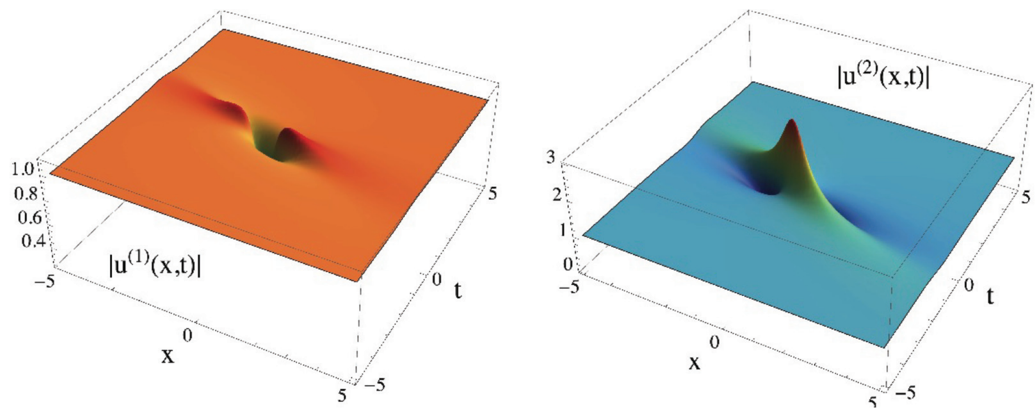

FIG. 7. (Color online) VNLS: $k_c = \sqrt{\frac{3}{8}}\sqrt{-3+i\sqrt{3}}$, $s_1 = s_2 = -1$, $q = 1$, $a_1 = a_2 = 1$; $\gamma_2 = 1$, $\gamma_1 = \gamma_3 = 0$; see Sec. IV B2.

nonvanishing of the background parameter q. Figure 7 shows instead a solution of the VNLS in the focusing case $s_1 = s_2 = -1$, and so also Fig. 8 but for different values of the parameters $\gamma_1, \gamma_2, \gamma_3$ (see captions). Similarly Figs. 9 and 10 show two solutions of the 3WRI equations (2) again for $s_1 = s_2 = 1$ and for $s_1 = s_2 = -1$, respectively; the first one features rational amplitudes as it follows from the condition $\gamma_3 = 0$, while the second one, with $\gamma_3 \neq 0$, proves to have a mixture of exponential and rational dependence on coordinates.

3. $q \neq 0$ and $a_1 \neq a_2$

We explore this case by first computing k_c numerically. Then the step-by-step method of construction of solutions, as detailed in the preceding section, leads to the three plots of solutions of the VNLS equations (1) as in Figs. 11–13, and to the plot (see Fig. 14) of one solution of the 3WRI equations (2). For all these solutions the parameter γ_3 is chosen to vanish ($\gamma_3 = 0$, see captions) with the implication that the plotted amplitudes clearly show a rational dependence on coordinates. Moreover we should point out that Figs. 11 and 12, respectively, show a solution in the defocusing case $s_1 = s_2 = 1$ and focusing case $s_1 = s_2 = -1$. The other Figure 13 shows instead a solution of the VNLS equations in the mixed case $s_2 = -s_1 = 1$. Finally a solution of the 3WRI equations (2) is shown in Fig. 14 with $s_1 = s_2 = 1$.

C. Conclusions

In this paper we have devised a method of construction of solutions of two integrable systems of partial differential equations of interest in a variety of applications. These systems, the VNLS equations and the 3WRI equations, model the coupling of two waves and, respectively, of three waves. Our construction is specially tailored to yield solutions which feature a rational, or mixed rational-exponential, dependence on the independent variables. While rational solutions of integrable partial differential equations attracted mathematical interest since the 1970s and consequently this type of solutions were derived for a number of integrable wave equations, it was only recently that further investigations of rational solutions were extended to integrable systems of two or three coupled differential equations. The main motivation of such a renewed interest goes back to the observation by Peregrine that the simplest rational solution of the focusing NLS equation may model an ocean rogue wave. In a variety of physical contexts it was however soon recognized that, several waves, rather than a single one, should be considered in order to account for important resonant interaction processes. For integrable partial differential equations, according to personal taste, various, yet equivalent, approaches have been adopted: spectral transform and dressing techniques, Wronskian and Hirota methods, and Darboux transformations as considered

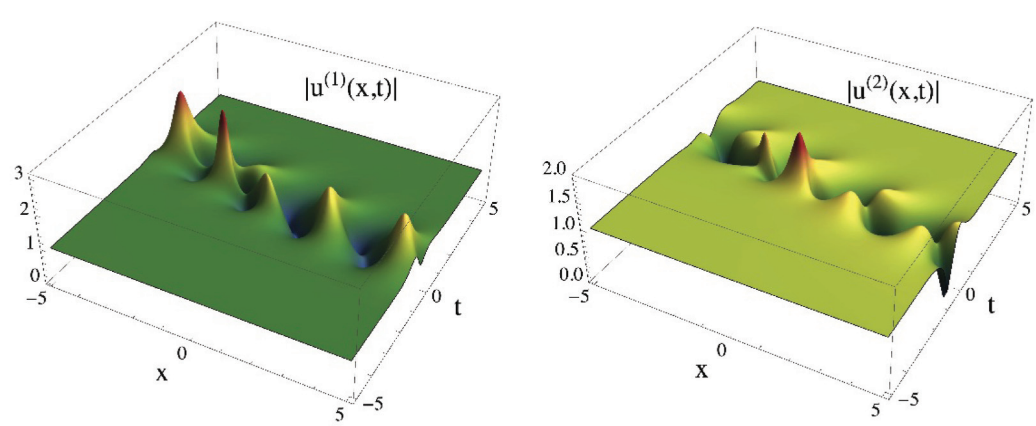

FIG. 8. (Color online) VNLS: $k_c = \sqrt{\frac{3}{8}}\sqrt{-3+i\sqrt{3}}$, $s_1 = s_2 = -1$, $q = 1$, $a_1 = a_2 = 1$; $\gamma_1 = \gamma_2 = \gamma_3 = 1$; see Sec. IV B2.

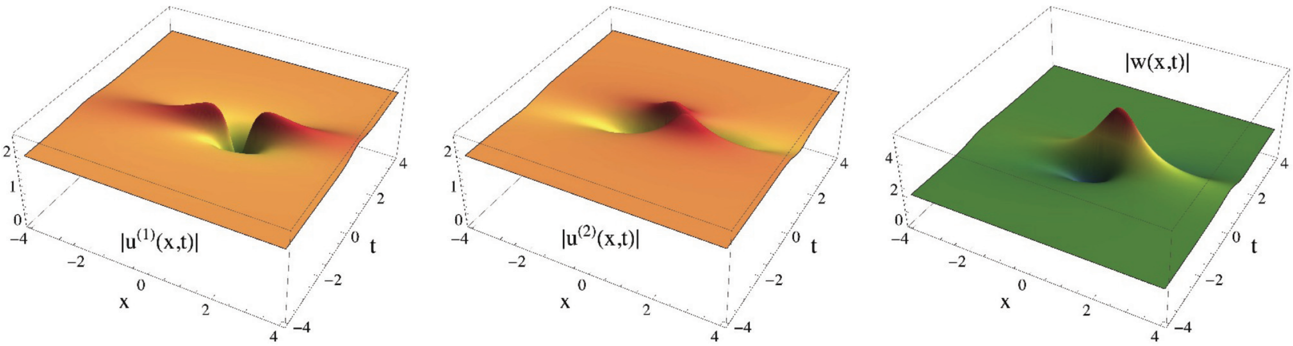

FIG. 9. (Color online) 3WRI: $k_c = \frac{i}{2}\sqrt{-13+16\sqrt{2}}$, $s_1 = s_2 = 1$, $q = 1$, $a_1 = a_2 = 2$, $c_1 = 1$, $c_2 = 2$; $\gamma_2 = 1$, $\gamma_1 = \gamma_3 = 0$; see Sec. IV B2.

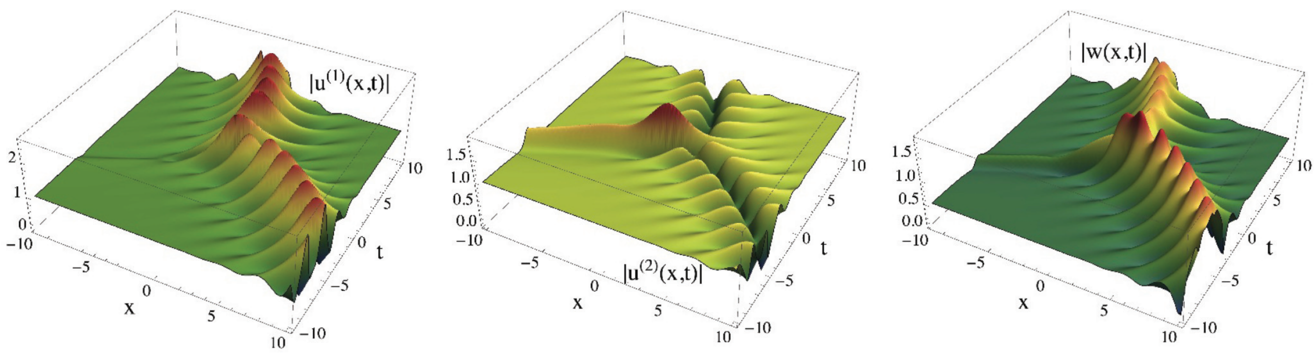

FIG. 10. (Color online) 3WRI: $k_c = \sqrt{\frac{3}{8}}\sqrt{-3+i\sqrt{3}}$, $s_1 = s_2 = -1$, $q = 1$, $a_1 = a_2 = 1$, $c_1 = 1$, $c_2 = 2$; $\gamma_1 = \gamma_2 = \gamma_3 = 1$; see Sec. IV B2.

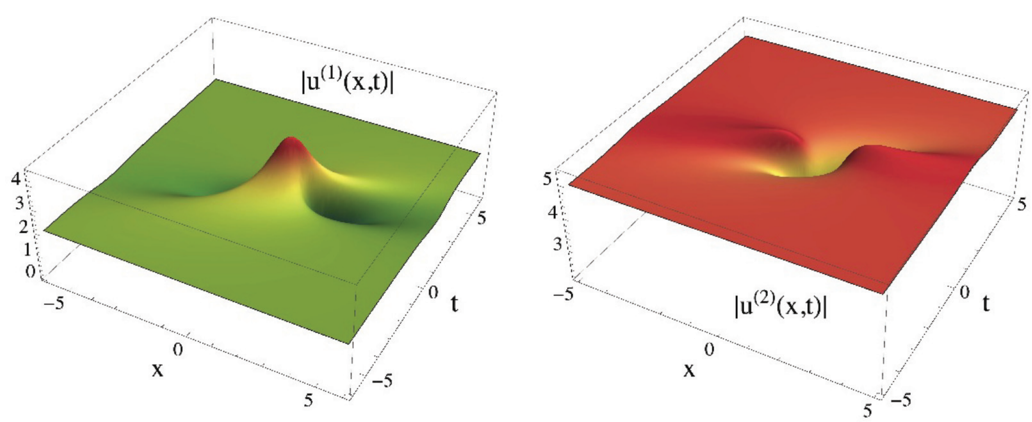

FIG. 11. (Color online) VNLS: $k_c = -5.600 + 4.655i$, $s_1 = s_2 = 1$, $q = 1$, $a_1 = 2$, $a_2 = 5$; $\gamma_2 = 1$, $\gamma_1 = \gamma_3 = 0$; see Sec. IV B3.

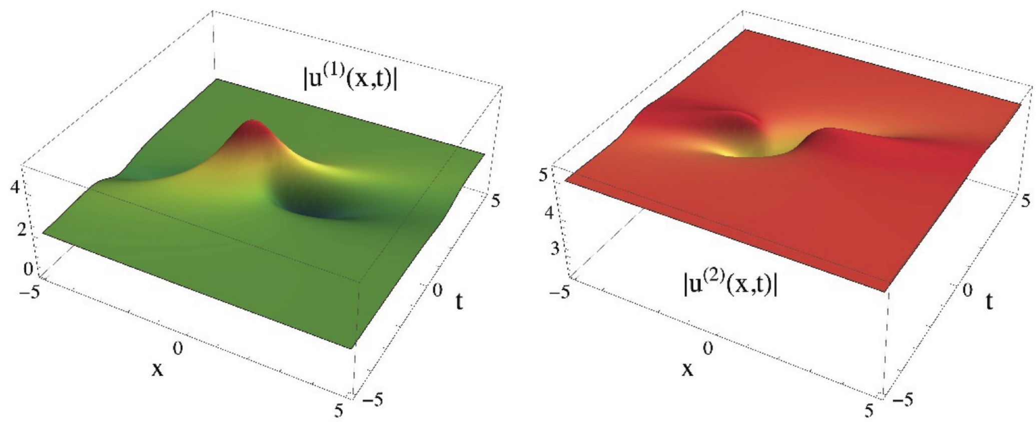

FIG. 12. (Color online) VNLS: $k_c = 4.876 + 5.343i$, $s_1 = s_2 = -1$, $q = 1$, $a_1 = 2$, $a_2 = 5$; $\gamma_2 = 1$, $\gamma_1 = \gamma_3 = 0$; see Sec. IV B3.

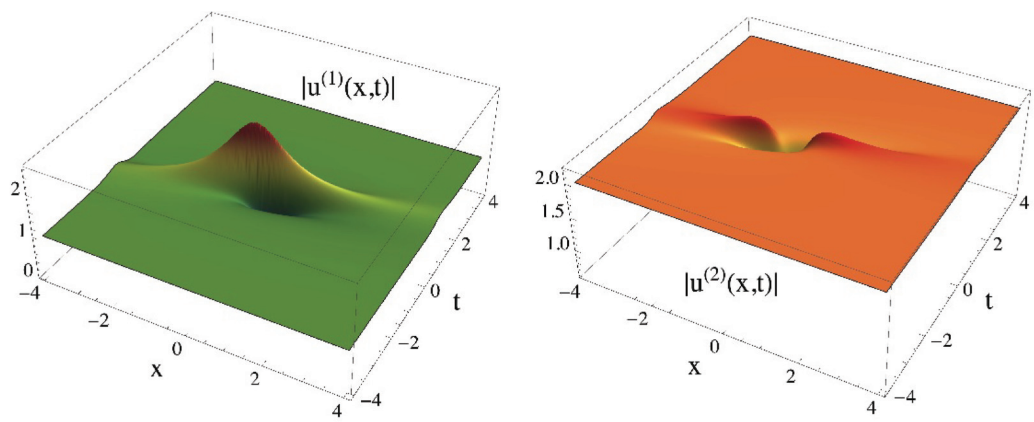

FIG. 13. (Color online) VNLS: $k_c = -1.242 + 0.636i$, $s_1 = -1$, $s_2 = 1$, $q = 1$, $a_1 = 1$, $a_2 = 2$; $\gamma_2 = 1$, $\gamma_1 = \gamma_3 = 0$; see Sec. IV B3.

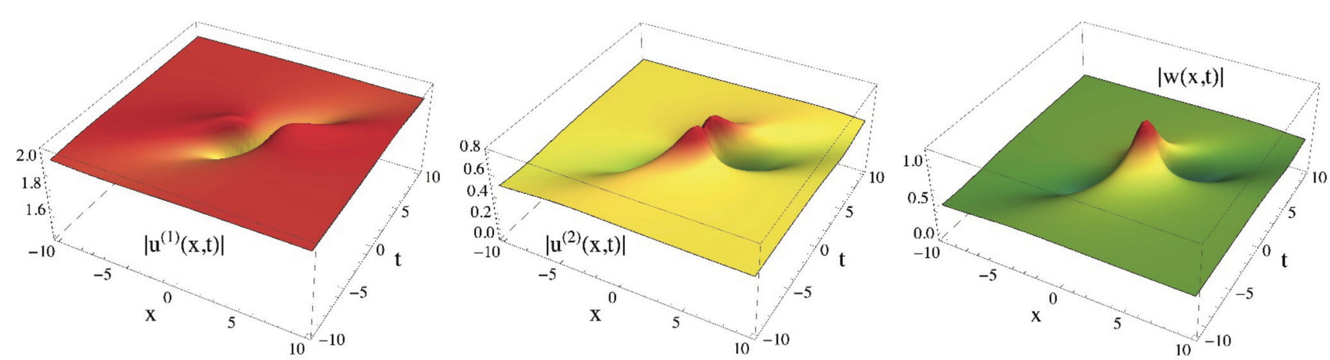

FIG. 14. (Color online) 3WRI: $k_c = 1.319 + 0.256i$, $s_1 = s_2 = 1$, $q = 1$, $a_1 = 2$, $a_2 = 0.5$, $c_1 = 1$, $c_2 = 2$; $\gamma_2 = 1$, $\gamma_1 = \gamma_3 = 0$; see Sec. IV B3.

here. These solutions are all soliton solutions since their corresponding spectral data on the continuos spectrum vanish. Moreover the strategy of computation may depend on whether the soliton is superimposed to the vacuum (i.e., the vanishing solution) or to a plane wave background. Here we deal with this second type of solitons. In most of the constructions discussed in the literature, the way to obtain polynomials out of (a linear combination of) exponentials goes through an appropriate limit process by making a number of eigenvalues of the Lax equations coalesce to get all the same value. Our approach is instead based on the exponentiation of nondiagonalizable matrices. This construction naturally leads to consider those critical values k_c of the spectral variable k such that the matrices which appear as exponent are similar to a Jordan form. There is therefore no need to take the limit in which different eigenvalues coalesce. We believe that our investigation is able to capture all possible solutions in this class. We are confident that the broad family of solutions presented here add a contribution to the understanding of rogue wave phenomena in novel physical situations where wave resonant interactions are relevant.

ACKNOWLEDGMENT

A.D. gratefully acknowledges partial financial support by INFN, Italy, project RM41.

APPENDIX: POLYNOMIALS

$$\begin{aligned}P_4^{(1)} = {} & 12X^4 + 1728T^4 + 288X^2T^2 + 4(3i + 2\sqrt{3}\epsilon)X^3 - 864iT^3 - 72iX^2T + 48\,XT^2(3i - 2\sqrt{3}\epsilon) \\ & + 3\,X^2(4\mathrm{Re}(\gamma_1) + 3i\sqrt{3}\epsilon - 1) - 12\,T^2(12\mathrm{Re}(\gamma_1) + 5i\sqrt{3}\epsilon + 9) + 12\,TX(4\sqrt{3}\epsilon\mathrm{Im}(\gamma_1) - i\sqrt{3}\epsilon + 9) \\ & + 6\,T(2i\sqrt{3}\epsilon\mathrm{Im}(\gamma_1) + 6i\mathrm{Re}(\gamma_1) - \sqrt{3}\epsilon + 3i) + 2X(-3i\sqrt{3}\epsilon\mathrm{Im}(\gamma_1) + 2\sqrt{3}\epsilon\mathrm{Re}(\gamma_1) + 3i\mathrm{Re}(\gamma_1) - 2\sqrt{3}\epsilon - 3i) \\ & + 3|\gamma_1|^2 + \tfrac{1}{2}(1 + 5i\sqrt{3}\epsilon)\mathrm{Re}(\gamma_1) + \tfrac{9}{2}(\sqrt{3}\epsilon - i)\mathrm{Im}(\gamma_1) + \tfrac{5}{2}(1 - i\sqrt{3}\epsilon),\end{aligned} \tag{A1a}$$

$$\begin{aligned}P_4^{(2)} = {} & 12X^4 + 1728T^4 + 288X^2T^2 + 4(-3i + 2\sqrt{3}\epsilon)X^3 - 864iT^3 - 72iX^2T - 48\,XT^2(3i + 2\sqrt{3}\epsilon) \\ & + 3\,X^2(4\mathrm{Re}(\gamma_1) - 3i\sqrt{3}\epsilon - 1) - 12\,T^2(12\mathrm{Re}(\gamma_1) - 5i\sqrt{3}\epsilon + 9) + 12\,TX(4\sqrt{3}\epsilon\mathrm{Im}(\gamma_1) - i\sqrt{3}\epsilon + 3) \\ & + 6\,T(-2i\sqrt{3}\epsilon\mathrm{Im}(\gamma_1) + 6i\mathrm{Re}(\gamma_1) + \sqrt{3}\epsilon - 3i) + 2X(-3i\sqrt{3}\epsilon\mathrm{Im}(\gamma_1) + 2\sqrt{3}\epsilon\mathrm{Re}(\gamma_1) - 3i\mathrm{Re}(\gamma_1) - 2\sqrt{3}\epsilon - 6i) \\ & + 3|\gamma_1|^2 + \tfrac{1}{2}(1 - 5i\sqrt{3}\epsilon)\mathrm{Re}(\gamma_1) + \tfrac{3}{2}(\sqrt{3}\epsilon - 3i)\mathrm{Im}(\gamma_1) - 2(1 + i\sqrt{3}\epsilon),\end{aligned} \tag{A1b}$$

$$\begin{aligned}P_4 = {} & 12X^4 + 1728T^4 + 288X^2T^2 + 8\sqrt{3}\epsilon X^3 - 96\sqrt{3}\epsilon XT^2 + 6X^2(1 + 2\mathrm{Re}(\gamma_1)) + 72T^2(1 - 2\mathrm{Re}(\gamma_1)) \\ & + 12XT(6 + 4\sqrt{3}\epsilon\mathrm{Im}(\gamma_1)) + 2X\sqrt{3}\epsilon(1 + 2\mathrm{Re}(\gamma_1)) + 3|\gamma_1|^2 - \mathrm{Re}(\gamma_1) + 3\sqrt{3}\epsilon\mathrm{Im}(\gamma_1) + 4,\end{aligned} \tag{A1c}$$

$$\begin{aligned}Q_2^{(1)} = {} & 12X^2 + 12T^2\big(c_1^2 - c_1c_2 + c_2^2\big) - 12XT(c_1 + c_2) + 2X(2\sqrt{3}\epsilon + 3i) \\ & - 2\,T[c_1(\sqrt{3}\epsilon - 3i) + c_2(\sqrt{3}\epsilon + 6i)] + i\sqrt{3}\epsilon - 1,\end{aligned} \tag{A2a}$$

$$Q_2 = 12X^2 + 12\,T^2\big(c_1^2 - c_1c_2 + c_2^2\big) - 12\,XT(c_1 + c_2) + 4\,X(\sqrt{3}\epsilon - 3i) - 2\,T(c_1 + c_2)(\sqrt{3}\epsilon - 3i) - 2i\sqrt{3}\epsilon - 1, \tag{A2b}$$

$$M_2 = 12X^2 + 12T^2\big(c_1^2 - c_1c_2 + c_2^2\big) - 12XT(c_1 + c_2) + 4\sqrt{3}\epsilon X - 2\sqrt{3}\epsilon T(c_1 + c_2) + 2, \tag{A2c}$$

$$\begin{aligned}Q_4^{(1)} = {} & 12X^4 + 12T^4\big(c_1^2 - c_1c_2 + c_2^2\big)^2 + 36X^2T^2\big(c_1^2 + c_2^2\big) - 24X^3T(c_1 + c_2) - 24XT^3\big(c_1^3 + c_2^3\big) + 4X^3(2\sqrt{3}\epsilon + 3i) \\ & - 4T^3\big[c_1^3(4\sqrt{3}\epsilon - 3i) - 3c_1^2c_2(\sqrt{3}\epsilon - 3i) - 3c_1c_2^2(\sqrt{3}\epsilon + 3i) + 2c_2^3(2\sqrt{3}\epsilon + 3i)\big] \\ & - 12X^2T[\sqrt{3}\epsilon c_1 + c_2(\sqrt{3}\epsilon + 3i)] + 12XT^2\big[3\sqrt{3}\epsilon c_1^2 - 4\sqrt{3}\epsilon c_1c_2 + 3c_2^2(\sqrt{3}\epsilon + i)\big] \\ & + 3X^2(4\mathrm{Re}(\gamma_1) + 3i\sqrt{3}\epsilon - 1) - 3T^2\big\{c_1^2[2\mathrm{Re}(\gamma_1) - \sqrt{3}\epsilon(2\mathrm{Im}(\gamma_1) - i) - 11] + 2c_1c_2(-4\mathrm{Re}(\gamma_1) + i\sqrt{3}\epsilon + 7) \\ & + 2c_2^2[\mathrm{Re}(\gamma_1) + \sqrt{3}\epsilon(\mathrm{Im}(\gamma_1) - 3i) - 1]\big\} - 6XT\{2c_1[\sqrt{3}\epsilon(\mathrm{Im}(\gamma_1) - i) + \mathrm{Re}(\gamma_1) + 2] \\ & + c_2[\sqrt{3}\epsilon(-2\mathrm{Im}(\gamma_1) + 5i) + 2\mathrm{Re}(\gamma_1) - 5]\} + X[\mathrm{Re}(\gamma_1)(4\sqrt{3}\epsilon + 6i) - 2\sqrt{3}\epsilon(2 + 3i\mathrm{Im}(\gamma_1)) - 6i] \\ & - 2T\{c_1[\mathrm{Re}(\gamma_1)(\sqrt{3}\epsilon + 6i) + 9\mathrm{Im}(\gamma_1) + 5\sqrt{3}\epsilon - 6i] + c_2[\mathrm{Re}(\gamma_1)(\sqrt{3}\epsilon - 3i) - 3\mathrm{Im}(\gamma_1)(3 + i\sqrt{3}\epsilon) - 7\sqrt{3}\epsilon + 3i]\} \\ & + 3|\gamma_1|^2 + \tfrac{9}{2}\sqrt{3}\epsilon\mathrm{Im}(\gamma_1) - \tfrac{9}{2}i\mathrm{Im}(\gamma_1) + \tfrac{5}{2}i\sqrt{3}\epsilon\mathrm{Re}(\gamma_1) + \tfrac{1}{2}\mathrm{Re}(\gamma_1) - \tfrac{5}{2}i\sqrt{3}\epsilon + \tfrac{5}{2},\end{aligned} \tag{A3a}$$

$$\begin{aligned}Q_4^{(2)} = {} & 12X^4 + 12T^4\big(c_1^2 - c_1c_2 + c_2^2\big)^2 + 36T^2X^2\big(c_1^2 + c_2^2\big) - 24TX^3(c_1 + c_2) - 24T^3X\big(c_1^3 + c_2^3\big) + 4X^3(2\sqrt{3}\epsilon - 3i) \\ & - 4T^3\big[2c_1^3(2\sqrt{3}\epsilon - 3i) - 3c_1^2c_2(\sqrt{3}\epsilon - 3i) - 3c_1c_2^2(\sqrt{3}\epsilon + 3i) + c_2^3(4\sqrt{3}\epsilon + 3i)\big] - 12X^2T[c_1(\sqrt{3}\epsilon - 3i) + \sqrt{3}\epsilon c_2] \\ & + 12XT^2\big[3c_1^2(\sqrt{3}\epsilon - i) - 4\sqrt{3}\epsilon c_1c_2 + 3\sqrt{3}\epsilon c_2^2\big] + 3X^2(4\mathrm{Re}(\gamma_1) - 3i\sqrt{3}\epsilon - 1) \\ & - 3T^2\big\{2c_1^2[\mathrm{Re}(\gamma_1) - \sqrt{3}\epsilon(\mathrm{Im}(\gamma_1) - 3i) - 4] - 2c_1c_2(4\mathrm{Re}(\gamma_1) + i\sqrt{3}\epsilon - 7) \\ & + c_2^2[2\mathrm{Re}(\gamma_1) + \sqrt{3}\epsilon(2\mathrm{Im}(\gamma_1) - i) - 5]\big\} - 6XT\{c_1[2\mathrm{Re}(\gamma_1) + \sqrt{3}\epsilon(2\mathrm{Im}(\gamma_1) - 5i) + 1]\end{aligned}$$

$$+2c_2[\mathrm{Re}(\gamma_1)-\sqrt{3}\epsilon(\mathrm{Im}(\gamma_1)-i)-1]\}+2X[\mathrm{Re}(\gamma_1)(2\sqrt{3}\epsilon-3i)-\sqrt{3}\epsilon(3i\mathrm{Im}(\gamma_1)+2)-6i]$$
$$+2T\{-c_1[\mathrm{Re}(\gamma_1)(\sqrt{3}\epsilon+3i)+3\mathrm{Im}(\gamma_1)(3-i\sqrt{3}\epsilon)+2\sqrt{3}\epsilon-12i]+c_2[\mathrm{Re}(\gamma_1)(-\sqrt{3}\epsilon+6i)+9\mathrm{Im}(\gamma_1)$$
$$+4\sqrt{3}\epsilon-6i]\}+3|\gamma_1|^2+\tfrac{1}{2}\mathrm{Re}(\gamma_1)-\tfrac{5}{2}i\sqrt{3}\epsilon\mathrm{Re}(\gamma_1)+\tfrac{3}{2}\sqrt{3}\epsilon\mathrm{Im}(\gamma_1)-\tfrac{9}{2}i\mathrm{Im}(\gamma_1)-2i\sqrt{3}\epsilon-2, \tag{A3b}$$

$$Q_4=12X^4+12T^4\left(c_1^2-c_1c_2+c_2^2\right)^2+36X^2T^2\left(c_1^2+c_2^2\right)-24X^3T(c_1+c_2)-24XT^3\left(c_1^3+c_2^3\right)+8X^3(\sqrt{3}\epsilon-3i)$$
$$-4(c_1+c_2)T^3\left[\left(c_1^2+c_2^2\right)(4\sqrt{3}\epsilon-3i)+c_1c_2(-7\sqrt{3}\epsilon+3i)\right]-12(c_1+c_2)X^2T(\sqrt{3}\epsilon-3i)$$
$$+12XT^2\left[3\left(c_1^2+c_2^2\right)(\sqrt{3}\epsilon-i)-4\sqrt{3}\epsilon c_1c_2\right]+6X^2(2\mathrm{Re}(\gamma_1)-3i\sqrt{3}\epsilon-2)$$
$$-3T^2\left\{c_1^2[2\mathrm{Re}(\gamma_1)-\sqrt{3}\epsilon(2\mathrm{Im}(\gamma_1)-5i)-11]-4c_1c_2(2\mathrm{Re}(\gamma_1)+i\sqrt{3}\epsilon-5)\right.$$
$$\left.+c_2^2[2\mathrm{Re}(\gamma_1)+\sqrt{3}\epsilon(2\mathrm{Im}(\gamma_1)+5i)-5]\right\}-6XT\{c_1[2\mathrm{Re}(\gamma_1)+\sqrt{3}\epsilon(2\mathrm{Im}(\gamma_1)-3i)+1]$$
$$+c_2[2\mathrm{Re}(\gamma_1)-\sqrt{3}\epsilon(2\mathrm{Im}(\gamma_1)+3i)-5]\}+2X[2\mathrm{Re}(\gamma_1)(\sqrt{3}\epsilon-3i)-5\sqrt{3}\epsilon-3i]$$
$$+2T\{c_1[\mathrm{Re}(\gamma_1)(-\sqrt{3}\epsilon+3i)-3\mathrm{Im}(\gamma_1)(3-i\sqrt{3}\epsilon)-2\sqrt{3}\epsilon+6i]+c_2[\mathrm{Re}(\gamma_1)(-\sqrt{3}\epsilon+3i)$$
$$+3\mathrm{Im}(\gamma_1)(3-i\sqrt{3}\epsilon)+7\sqrt{3}\epsilon-3i]\}+3|\gamma_1|^2-4\mathrm{Re}(\gamma_1)-5i\sqrt{3}\epsilon\mathrm{Re}(\gamma_1)+3\sqrt{3}\mathrm{Im}(\gamma_1)+2i\sqrt{3}\epsilon-2, \tag{A3c}$$

$$M_4=12X^4+12T^4\left(c_1^2-c_1c_2+c_2^2\right)^2+36T^2X^2\left(c_1^2+c_2^2\right)-24X^3T(c_1+c_2)-24XT^3\left(c_1^3+c_2^3\right)+8\sqrt{3}\epsilon X^3$$
$$-4\sqrt{3}\epsilon T^3\left(4c_1^3-3c_1^2c_2-3c_1c_2^2+4c_2^3\right)-12\sqrt{3}\epsilon(c_1+c_2)X^2T+12\sqrt{3}\epsilon XT^2\left(3c_1^2-4c_1c_2+3c_2^2\right)$$
$$+6X^2(2\mathrm{Re}(\gamma_1)+1)-6T^2\left[c_1^2(\mathrm{Re}(\gamma_1)-\sqrt{3}\epsilon\mathrm{Im}(\gamma_1)-7)+2c_1c_2(5-2\mathrm{Re}(\gamma_1))+c_2^2(\mathrm{Re}(\gamma_1)+\sqrt{3}\epsilon\mathrm{Im}(\gamma_1)-4)\right]$$
$$-12XT[c_1(\mathrm{Re}(\gamma_1)+\sqrt{3}\epsilon\mathrm{Im}(\gamma_1)+2)+c_2(\mathrm{Re}(\gamma_1)-\sqrt{3}\epsilon\mathrm{Im}(\gamma_1)-1)]+2\sqrt{3}\epsilon X(2\mathrm{Re}(\gamma_1)+1)-2T\{c_1[9\mathrm{Im}(\gamma_1)$$
$$+\sqrt{3}\epsilon(\mathrm{Re}(\gamma_1)+5)]+c_2[\sqrt{3}\epsilon(\mathrm{Re}(\gamma_1)-4)-9\mathrm{Im}(\gamma_1)]\}+3|\gamma_1|^2+3\sqrt{3}\epsilon\mathrm{Im}(\gamma_1)-\mathrm{Re}(\gamma_1)+4. \tag{A3d}$$

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