Analytical investigation of the faster-is-slower effect with a simplified phenomenological model

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We investigate the mechanism of the phenomenon called the "faster-is-slower"effect in pedestrian flow studies analytically with a simplified phenomenological model. It is well known that the flow rate is maximized at a certain strength of the driving force in simulations using the social force model when we consider the discharge of self-driven particles through a bottleneck. In this study, we propose a phenomenological and analytical model based on a mechanics-based modeling to reveal the mechanism of the phenomenon. We show that our reduced system, with only a few degrees of freedom, still has similar properties to the original many-particle system and that the effect comes from the competition between the driving force and the nonlinear friction from the model. Moreover, we predict the parameter dependences on the effect from our model qualitatively, and they are confirmed numerically by using the social force model.

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I. INTRODUCTION

Collective motion of self-driven particles has captured much interest for the past few decades from the perspective of nonequilibrium dynamics. The concept of self-driven particles has been widely used in many fields (e.g., traffic flow [1–4], active matter [5–7], and granular media [8,9]) and has enhanced interdisciplinary collaboration. Pedestrian dynamics has drawn the attention of physicists as a collective motion of self-driven particles since it shows a wide variety of self-organized phenomena [1], such as lane formation, segregation, and flow oscillations at a bottleneck. Studies of these phenomena are also expected to contribute to the safe evacuation of people in an emergency in the real world. In particular, the flow of particles through a bottleneck has been studied intensively with many approaches, such as the social force (SF) model [10], the lattice gas model [11], cellular automata (CA) [12,13], and real experiments [14,15].

The "faster-is-slower"effect [16] is a well-known phenomenon observed in certain systems of self-driven particles. Let us consider the discharge of self-driven particles from a square room through a narrow exit. In such a system, the discharge flow rate is a monotonically increasing function of the self-driven force when the force is relatively weak. However, counterintuitively, the flow rate begins to decrease when the driving force exceeds a critical value. This faster-is-slower effect was shown in a numerical study using the SF model [16]. The SF model is based on ordinary differential equations, and detailed numerical studies have been performed using this model and its variations [17–21]. It is also known that the CA model shows a similar effect [12]. In this model, each particle moves in a discrete manner with given probabilities that include the effect of friction between two particles. The CA model gives some insights into the role of the friction and the effect of an obstacle on the evacuation problem. Both numerical approaches indicate that the friction is crucial in this phenomenon, but a theoretical understanding of the phenomenon has not been attained.

We also should note that we have a gap between the SF model and a real system. Although the SF model shows the faster-is-slower effect, we have no definite evidence that shows the existence of the phenomenon in a real pedestrian system. Some systematic experiments have been performed so far [15,22]; they do not show any supportive evidence with the exception of a biological experiment [23]. A better strategy to understand the reason of the gap is to know which factors included in the SF model generate faster-is-slower effect through the model-aided analysis. This approach allows us to discuss the effect of each parameter and the relation among them more clearly. In the present paper, we propose a simplified phenomenological model to clarify the reason why the original SF model proposed by Helbing *et al.* shows the faster-is-slower effect, or, more generally, what are the physical causes of the phenomenon. We perform an analytical investigation of our model, and give comparisons to the result of the SF model. From this study, we expect to gain a better understanding of the microscopic mechanism of the phenomenon. Describing a possible scenario of the phenomenon would be helpful to understand the gap between the SF simulations and the reality.

II. REVIEW OF THE SF MODEL

First, a brief review of the SF model is given [10,16]. It is widely used in the study of pedestrian flow due to its notable ability to reproduce many interesting phenomena [1,10,24]. In the SF model, each particle has a self-driven force, a social force, linear elasticity, and friction. The equations of motion for N particles are given by

$$m\frac{d\boldsymbol{v}_i}{dt} = \frac{m}{\tau}(v_0\boldsymbol{e}_i - \boldsymbol{v}_i) + \sum_j^N \left(Ae^{\frac{r_{ij}-d_{ij}}{B}} + kg(r_{ij}-d_{ij})\right)\boldsymbol{n}_{ij} + \sum_j^N \kappa g(r_{ij}-d_{ij})[(\boldsymbol{v}_j-\boldsymbol{v}_i)\cdot\boldsymbol{t}_{ij}]\boldsymbol{t}_{ij} + \boldsymbol{F}_{iw}, \quad (1)$$

where $\boldsymbol{v}_i$ is the velocity of the particle i, d_{ij} is the distance between particles i and j, $r_{ij} = r_i + r_j$ is the sum of the radii of the particles, $\boldsymbol{n}_{ij}$ is the normal unit vector from j to i, $\boldsymbol{t}_{ij}$

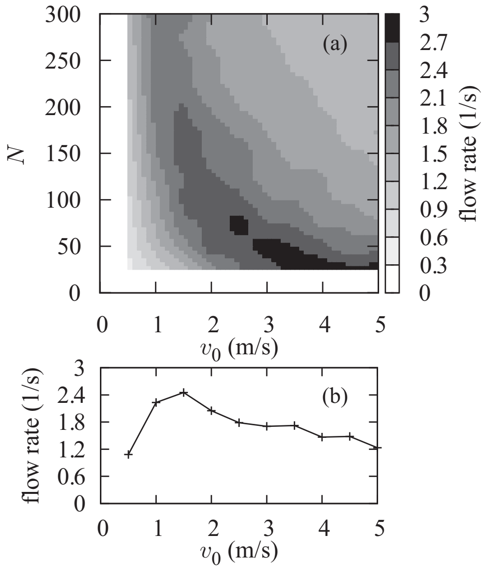

FIG. 1. Numerical results for the stationary flow rate obtained from the SF model simulation. A higher flow rate corresponds to a faster evacuation time. (a) v_0 and N dependence of the flow rate. The flow rate was not calculated for small values of v_0 and N. (b) The flow rate with a fixed N of 200.

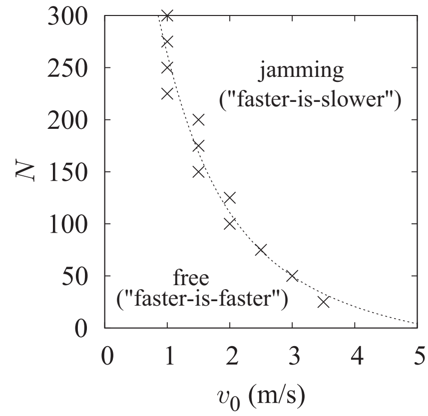

FIG. 2. Schematic phase diagram. The area below the rough additional line is the "free state,"where the flow rate increases as the driving force increases. Each point ($\times$) shows the maximum point for each N. The area above the line is the "jamming state,"where the "faster-is-slower"effect occurs. The critical v_0 value depends on N, and this suggests that the effect begins to slow down evacuation when N is relatively large.

is the tangential unit vector, and $v_0\boldsymbol{e}_i$ is the desired velocity, which is always directed toward the exit. The particle i has mass m, time constant τ, social force parameters A and B, elasticity k, and friction coefficient κ. The function g is defined as $g(x) = 0$ if $x < 0$ and as $g(x) = x$ if $x \geqslant 0$. The interaction with the nearest wall $\boldsymbol{F}_{iw}$ has the same form as the two-body interaction.

Let us solve Eq. (1) to obtain the flow rate as a function of v_0 and N. As a boundary condition, we consider a large square room that has a narrow exit in the center of one of its walls. In this calculation, we consider the stationary state. Since the flow rate depends on N [25], we need to keep N constant. For this purpose, particles are appropriately supplied from the side of the room opposite the exit [18,19]. Here, we use the following parameters: the width of the exit is 1.2 m, $r_i = 0.3 \pm 0.05$ m, $m = 80$ kg, $\tau = 0.5$ s, $A = 2000$ N, $B = 0.08$ m, $k = 120\,000$ kg/s^2, and $\kappa = 240\,000$ kg/(m s). Basically, the parameters are based on Helbing *et al.* [16]. Although it is known that these parameters lead to unrealistic behavior [26], we use these values in this paper since our intention is to identify the cause of the "faster-is-slower" effect rather than to seek more suitable model parameters. The resulting stationary flow rate is given by Fig. 1, where the flow rate is calculated from the time needed for 500 evacuation events. When v_0 increases under fixed N, there exists a critical point above which the flow rate decreases. The schematic phase diagram is given in Fig. 2, which shows that (v_0, N) space is divided into qualitatively different regions.

III. MODEL

To investigate the origin of the phenomenon analytically, we introduce a qualitative phenomenological model inspired by the SF model to describe the discharge velocity of the many-particle system. The derivation of the model is as follows (see Fig. 3). To simplify the original many-particle system, we made the following assumptions. The discharge property is determined by the motion of particles in the immediate vicinity of the exit. The flow is governed by the radial motion toward the exit [27] and has no significant angular dependence on long-time average. The total number of particles inside the room is fixed. We introduce virtual particles for each direction that obey the above assumptions and have the same type of interactions in the SF model (the social force, elasticity, and friction). The particles around the exit are also affected by the force from behind,

$$h(v_0, N) = b v_0 \sqrt{N}, \tag{2}$$

where b is the dimensionless constant. An intuitive interpretation of the above formula is as follows. In the original many-particle system, the particles rush into the exit and form a semicircular shape around the exit due to the clogging. The force driving the particles is proportional to v_0. We assume that the pressure around the exit is proportional to the radius

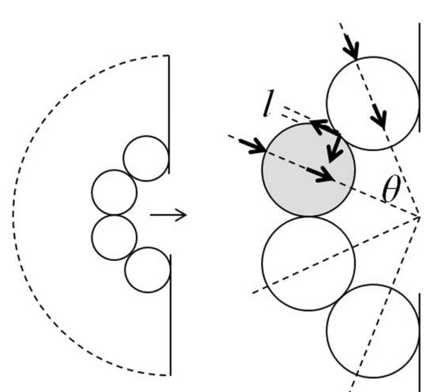

FIG. 3. Conceptual images of the model. (Left) The typical formation of the arch near the exit. (Right) The forces acting on the virtual particles.

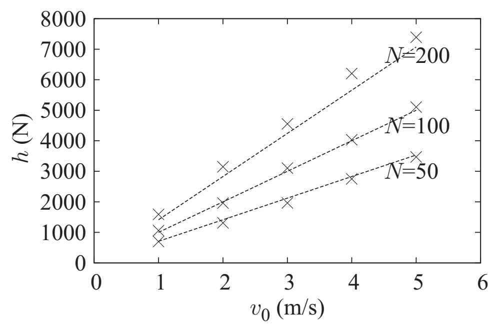

FIG. 4. Force applied to the particle at the center of the semicircle for each N. The results are obtained from the SF simulation with zero-width exit. Dashed lines represent Eq. (2) with the dimensionless parameter $b = 0.625$.

of the semicircle, which is dependent on $\sqrt{N}$. As a result, we obtain Eq. (2). The validity of this assumption, at least as a first approximation, can be confirmed by performing a numerical calculation using the SF model in which the width of the exit is nearly zero (see Fig. 4). Next, we assume that the repetitive formation and breakup of the four-particle arch in front of the exit is the most frequently observed dynamic structure and the major cause of the clogging. According to an experimental study of granular media [28] in which the discharge of particles from a rectangular-shape orifice is investigated, the size of the typical arch depends on the width of the exit, $\eta = 1.41 + 1.15R$, where η is the number of particles in the arch and R is the width of the exit measured by the diameter of particles. Using this relation here would be valid since the original SF model is very similar to the model of granular media. By applying this relation to our system ($R = 2$), we obtain $\eta \sim 4$ and assume that the four-particle arch is the dominant formation in the process of evacuation considered here. Finally, we simplify the process of the breakup of the arch. Namely, one of the particles in the arch moves toward the exit and other particles in the arch stand still during the motion. It is not important which particle in the arch moves as far as we consider long-time averaged flow. The equation of motion of the virtual particles that obey the above assumption is given by

$$\frac{dv_r}{dt} = (v_0 - av_r) - 2\kappa g[l(v_0,N)]v_r + h(v_0,N), \tag{3}$$

and the balance of the forces along the arch is given by

$$[v_0 + h(v_0,N)]\sin\theta = Ae^{l(v_0,N)} + kg[l(v_0,N)], \tag{4}$$

where v_r is the radial flow velocity, a is the deceleration parameter which represents the effect of collisions, $\theta = \pi/\eta$ is the angle between two particles, and $l(v_0,N)$ is the characteristic overlap length between particles, which is assumed to be a monotonically increasing function of v_0 and N. The function $h(v_0,N)$ represents an external force driving the particles from behind. The other symbols have the same meanings as in the SF model. Note that these equations are scaled and dimensionless: for example, $A^* = \frac{\tau^2}{Bm}A$ and we write A^* as A for simplicity. The other parameters and variables are also scaled in the same manner. The parameters need to satisfy the condition $a(k + A) < 2\kappa A$, the meaning of which is that the friction coefficient is sufficiently large.

The equations describe the motion of virtual particles that represent the original many-particle system. In the stationary state, the average discharge velocity can be interpreted as being its velocity. We assume that the discharge flow in the original system is characterized by the stationary solution of Eq. (3).

Note that our simplified model is based on the flow velocity, not the flow rate. Nevertheless, our model was able to describe the faster-is-slower effect. It is known that the maximum bulk kinetic energy corresponds to the maximum discharge flow rate when numerical experiments are performed with various values for the desired speed v_0 [19]. Therefore, we assume that the maximum discharge velocity in the model corresponds to the maximum discharge flow rate.

IV. ANALYSIS

Let us consider the stationary state of Eqs. (3) and (4). We set $\dot{v}_r = 0$. The solution is given by

$$v_r(v_0,N) = \frac{v_0 + h(v_0,N)}{a + 2\kappa g[l(v_0,N)]}, \tag{5}$$

$$l(v_0,N) = \begin{cases} \ln\frac{[v_0+h(v_0,N)]\sin\theta}{A} & (v_0 < v_c), \\ \frac{[v_0+h(v_0,N)]\sin\theta - A}{k+A} & (v_0 \geqslant v_c), \end{cases} \tag{6}$$

where $v_c = A/(1 + b\sqrt{N})\sin\theta$. Here, the Taylor expansion is used to derive Eq. (6). The approximation error of the expansion is less than 4% (see Fig. 6) in the case of $v_0 \leqslant 5$ and $N \leqslant 300$, so that the analytical expression above is valid in the discussion presented here. The solution is shown in Fig. 5. In this paper, we choose $\theta = 45^\circ$, $a = 10$, and $b = 0.625$. The parameter a is chosen to realize appropriately small outflow velocity, and b is determined from the numerical data fitting shown in Fig. 4. The other parameters used here are the same as those in Eq. (1) but scaled. As can be seen, the model presented

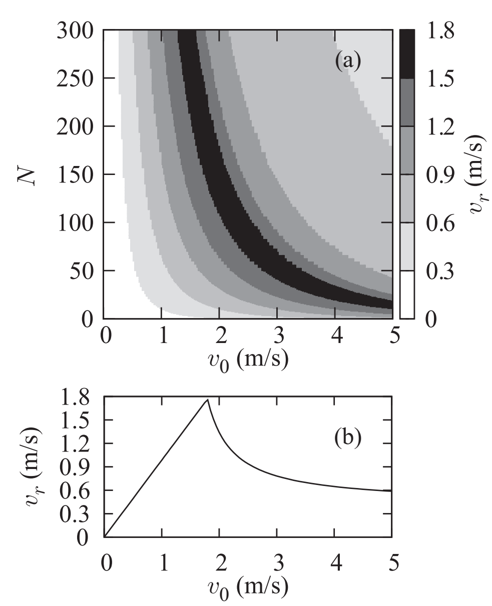

FIG. 5. Stationary flow velocity obtained from the model. (a) The v_0 and N dependence of the flow velocity. (b) The flow velocity with a fixed N of 200.

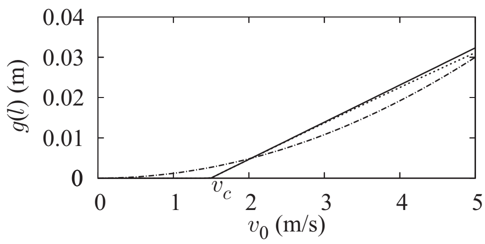

FIG. 6. Explicit representation of the function $g[l(v_0, N = 300)]$. (Solid) The analytical expression based on Eq. (6), (dotted) the numerical result calculated directly from Eq. (4), and (chain) the approximation curve $g[l(v_0,N)] \sim v_0^n$.

here has a solution that shows the faster-is-slower effect, and it mimics the v_0 and N dependence of the flow rate qualitatively.

The existence of the faster-is-slower solution can be understood in terms of the competition between the driving force and the nonlinear friction. For simplicity, we fix the number of particles to, e.g., $N = 200$. The flow velocity is given by Eq. (5), which means that the flow velocity is determined by the driving force and the drag force. From the model, the total driving force $v_0 + h(v_0,N)$ is linear with respect to v_0. On the other hand, the friction term is a nonlinear function of v_0. The concrete representation of g in Eq. (5) is given by the following nonlinear function:

$$g[l(v_0,N)] = \begin{cases} 0 & (v_0 < v_c), \\ \frac{[v_0+h(v_0,N)]\sin\theta - A}{k+A} & (v_0 \geqslant v_c), \end{cases} \tag{7}$$

which is shown in Fig. 6. Equation (7) behaves like a nonlinear function of v_0 around v_c, and it could be approximated by $g[l(v_0,N)] \sim v_0^n$, for $n > 1$. The competition between the linear driving force and friction that has such a nonlinearity causes the solution (5) to have a maximum value.

The nonlinearity of the friction term originates from the coupling of the social force with the linear friction. The critical point in Eq. (7) is determined by the coupling constant of the social force A. Below the critical driving force, there is no friction because no overlap between particles exists due to a weak driving force. Note that the friction becomes linear if $A = 0$. From these points, we can say that the social force plays the role of a barrier. A packing force that is strong enough to break the barrier is needed for the emergence of contact friction. The piecewise function in Eq. (7) shows such a barrier effect. The convexity of Eq. (7) comes from the combination of the linear behavior of the friction and the existence of the social force.

V. SIMULATION

If the above analysis is correct, we can immediately make the following statement. The outflow shows no faster-is-slower property (i) if $\kappa = 0$, (ii) if r_i is small, or (iii) if $A = 0$. If the size of the particle is small, the overlap between particles does not exist [that is, $l(v_0,N) < 0$], then $g[l(v_0,N)] = 0$ in Eq. (5) so that (ii) holds. We can also state that (iv) if $A = 0$ but the friction term maintains its nonlinearity for some reason, the faster-is-slower effect still exists. One of the ways of realizing such a condition is the introduction of a shift parameter c only in the friction term in Eq. (1), such that it becomes $g(r_i + r_j - d_{ij} - c)$. The argument of the g in the elastic term in Eq. (1) remains unchanged. The procedure is equivalent to giving a frictionless soft skin to the particles and causes an effect similar to the social force barrier. Under this implementation, such a result is expected even if there is no social force.

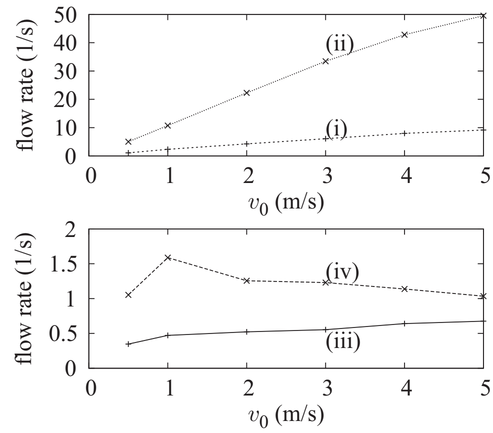

FIG. 7. Numerical results of the SF model simulation corresponding to each conjecture. Only (iv) shows a maximum.

Now we perform the numerical simulation using the SF model [Eq. (1)] to test the above predictions. We use $r_i = 0.1 \pm 0.016$ m in (ii) and $c = 0.015$ m in (iv). The results are show in Fig. 7. As can be seen, all the above predictions are confirmed numerically.

VI. DISCUSSION AND SUMMARY

From the simplified model and its theoretical analysis, we conclude that the faster-is-slower effect comes from the competition between the driving force and the nonlinear friction. The result is consistent with a previous study, which dealt with this phenomenon numerically [19–21]. The nonlinearity is the result of the coupling of the repulsive force with the linear friction, which is based on the effect of the exclusive volume and the surface friction.

This phenomenon would be observed not only in pedestrian flow but also in any system that realizes such a mechanism. The origin of the repulsive force seems less critical. The effect of collisions is also not important, at least qualitatively.

Of course, our model is so simple that some issues remain. Though we introduced many assumptions in the derivation of the model, further investigation is needed to obtain more evidence that supports the assumptions used in our model. Another issue is that the results presented here are qualitative because of the simplicity of the model. Note that the flow velocity obtained from the model is overestimated compared to that of the SF simulation due to nonstrict treatment of the collision effect.

We have provided an analytical investigation of the faster-is-slower effect by using a simple phenomenological model. We have shown that our analytical model is capable of describing the essentials of the phenomenon and that the competition between the driving force and the nonlinear friction is crucial for this phenomenon. Furthermore, the effects of each parameter in the original many-particle system were predicted analytically by our model and confirmed numerically by using the SF model. The simplified technique presented here may be useful for understanding the other phenomena originating from competition of flows in dissipative self-driven particle systems, such as lane formation, flow oscillations at a bottleneck, and freezing by heating [1].

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