Connection between maximum-work and maximum-power thermal cycles

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A new connection between maximum-power Curzon-Ahlborn thermal cycles and maximum-work reversible cycles is proposed. This linkage is built through a mapping between the exponents of a class of heat transfer laws and the exponents of a family of heat capacities depending on temperature. This connection leads to the recovery of known results and to a wide and interesting set of results for a class of thermal cycles. Among other results it was found that it is possible to use analytically closed expressions for maximum-work efficiencies to calculate good approaches to maximum-power efficiencies. Behind the proposed connection is an interpretation of endoreversibility hypothesis. Additionally, we suggest that certain reversible maximum-work cycles depending on working substance can be used as reversible landmarks for FTT maximum-power cycles, which also depend on working substance properties.

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I. INTRODUCTION

As is well known, Curzon-Ahlborn (CA) [1] found an endoreversible Carnot-like thermal engine in which the isothermal branches of the cycle are not in thermal equilibrium with their corresponding heat reservoirs at absolute temperatures T_1 and $T_2 < T_1$ having an efficiency at the maximum-power (MP) regime given by

$$\eta_{\mathrm{CA}} = 1 - \sqrt{\frac{T_2}{T_1}}. \tag{1}$$

This equation was obtained by using the so-called Newton law of cooling to model the heat exchanges between the heat reservoirs and the working fluid along the isothermal branches of the cycle. In fact, Eq. (1) was obtained previously by Novikov [2] in a context very close to finite-time thermodynamics (FTT). Later, within the context of FTT some authors [3–7] demonstrated that Eq. (1) is valid only for the Newtonian heat exchanges. As a matter of fact the CA efficiency stems from taking into account constant conductances and a linear dependence between the heat fluxes and the working substance temperatures along the isothermal branches of the cycle [4–7]. Once the linearity in the heat transfer law is dropped, the square root term in the MP efficiency (η_{MP}) is lost. Very recently, this fact has been widely confirmed by many authors working on the first order irreversible thermodynamics [8–10] and on microscopic [11] and mesoscopic [12] heat engines. On the other hand, Eq. (1) also was obtained for some reversible thermal cycles performing at maximum-work (MW) regime, such as the Otto and Joule-Brayton (JB) cycles [13]. These coincident results for the CA, Otto, and JB cycles motivated Landsberg and Leff (LL) to propose that the CA efficiency possesses a nearly universal behavior for a certain class of thermodynamic cycles operating at MW. This result was achieved by means of a generalized cycle which reduces to the Otto, JB, Diesel, and some other known cycles [14]. Clearly the kind of universality of the CA efficiency claimed by LL is not of the class of the true universality of Carnot efficiency, η_{C} [15]. The square root term (SRT) observed in the CA efficiency can be found in other processes of energy conversion, such as the so-called water-powered machine, which mixes two steady streams of hot and cold water to produce an output stream of warm water at maximum kinetic energy [16]. In fact, the role of the SRT of temperatures is more general and appears also in some irreversible processes such as the irreversible cooling or heating of an ideal gas initially at temperature T_i in contact with a series of auxiliary reservoirs to reach the final temperature T_f of a main heat reservoir. The SRT appears when the generation of entropy of this process is minimized [17]. As can be seen, the SRT is found in several thermal processes (reversible or irreversible) subject to some extremal conditions. A less known result is that corresponding to the way the square root is lost in the case of reversible cycles operating at a MW regime. LL [14] first studied a cycle formed by two adiabatic processes and two paths with constant heat capacities $C > 0$ of the working fluid (see Fig. 1). This reversible cycle operating under MW conditions has an efficiency given by $\eta_{\mathrm{CA}} = 1 - \sqrt{\tau}$, where $\tau = T_-/T_+$ is the ratio between the minimum and maximum temperatures of the cycle (see Fig. 1). Actually, the first author who found this expression for a MW engine was Chambadal [18]. LL [14] generalized the model of Fig. 1, to encompass a family of symmetric and asymmetric reversible cycles with a MW efficiency that do no deviate from η_{CA} more than 14%. This behavior was referred to as a near universality property of η_{CA}. However, for the case of reversible cycles performing at MW, it will be demonstrated that the CA efficiency is lost when constant heat capacities are not used, in a similar way as occurs in FTT, where the SRT in the CA efficiency is related only to constant conductances and to a linear law of heat transfer. On the other hand, it is known that the MW-Carnot efficiency is independent of working substance [15] and as an upper bound for irreversible thermal cycles performing between extreme temperatures T_+ and T_- gives values far from their corresponding efficiencies. Even for MW-reversible thermal cycles as those of Fig. 1, η_{C} is a very distant upper bound. From this point of view, it is desirable to have at hand a MW-reversible efficiency that is dependent on the working substance. The reversible cycle of Fig. 1 depends on working fluid through its heat capacity, and in all FTT-thermal cycle models the MP efficiencies depend on working substance through the conductances and the heat

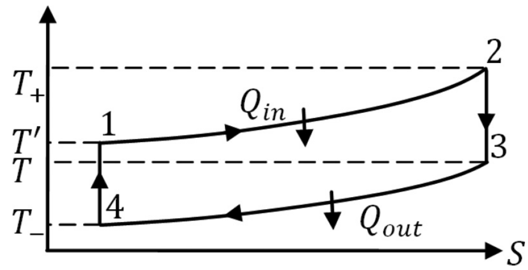

FIG. 1. T-S diagram of a cycle formed with two adiabats and two other processes with $C > 0$.

transfer law used. Thus, it is an interesting problem to search for a linkage between MW-reversible cycles and MP-FTT cycles, both depending on working substance as a kind of landmark less farther than Carnot's efficiency. The present article is organized as follows: In Sec. II we study a cycle as that of Fig. 1, where the working substance has a heat capacity dependent on temperature; in Sec. III we discuss a correspondence between MP-FTT and MW-reversible cycle efficiencies. Finally, in Sec IV we present the concluding remarks.

II. HEAT CAPACITIES OF THE FORM $C = aT^n$

Consider a heat capacity of the form $C = aT^n$ where a is a constant and n a real number. Following the cycle depicted in Fig. 1, after integrating the heat capacity over the temperature one obtains

$$Q_{\text{in}} = \begin{cases} a\,\ln\left(\frac{T_+}{T'}\right) & n=-1 \\ \frac{aT_+^{n+1}}{n+1}\left[1-\left(\frac{T'}{T_+}\right)^{n+1}\right] & n\neq -1 \end{cases}, \tag{2}$$

and

$$Q_{\text{out}} = \begin{cases} b\,ln\left(\frac{T_-}{T}\right) & n=-1 \\ \frac{bT_-^{n+1}}{n+1}\left[1-\left(\frac{T}{T_-}\right)^{n+1}\right] & n\neq -1 \end{cases}, \tag{3}$$

where b is also a constant and might be different from a. This case represents a more general scenario for cyclic processes following Fig. 1. The adjustable temperatures T and $T'$are coupled because the fluid's entropy change per reversible cycle is zero, i.e.,

$$\Delta S = \int_{T'}^{T_+} \frac{C d\mathbb{T}}{\mathbb{T}} + \int_{T}^{T_-} \frac{C d\mathbb{T}}{\mathbb{T}} = 0, \tag{4}$$

which leads to

$$T' = \begin{cases} T_+\left(\frac{T_-}{T}\right)^{\gamma} & n=0 \\ \left[T_+^n + \gamma(T_-^n - T^n)\right]^{\frac{1}{n}} & n\neq 0 \end{cases}, \tag{5}$$

where $\gamma = b/a$. Because the change in the total internal energy is zero, the reversible work done per cycle $W = |Q_{\text{in}}| - |Q_{\text{out}}|$ satisfies

$$W = \begin{cases} aT_+\left[1-\left(\frac{T_-}{T}\right)^{\gamma}\right] + b(T_- - T) & n=0 \\ a\,\ln[1+\gamma T_+(T_-^{-1} - T^{-1})] \\ \quad + b\,\ln\left(\frac{T_-}{T}\right) & n=-1 \\ \frac{aT_+^{n+1}}{n+1}\left\{1-\left[1+\gamma\left(\frac{T_-^n}{T_+^n} - \frac{T^n}{T_+^n}\right)\right]^{\frac{n+1}{n}}\right. \\ \quad \left. + \frac{bT_-^{n+1}}{n+1}\left(1-\frac{T^{n+1}}{T_-^{n+1}}\right)\right. & n\neq 0,-1 \end{cases}. \tag{6}$$

By maximizing W with respect to T it is found that T^* and T'^* are given by

$$T^* = T'^* = \begin{cases} \left(T_+T_-^{\gamma}\right)^{\frac{1}{1+\gamma}} & n=0 \\ \frac{(1+\gamma)T_+T_-}{\gamma T_+ + T_-} & n=-1 \\ \left(\frac{T_+^n+\gamma T_-^n}{1+\gamma}\right)^{\frac{1}{n}} & n\neq 0,-1 \end{cases}. \tag{7}$$

Then, as a result, the named symmetric cycles ($\gamma = 1$) and asymmetric cycles ($\gamma \neq 1$) fulfill that $T^* = T'^*$. From Eqs. (4) and (6) it can be seen that W is a convex curve with respect to any of the two intermediate temperatures (such as occurs in Fig. 2 of Ref. [13]). This curve has a maximum at $T = T' = T^*$ given by Eq. (7). This behavior is a consequence of taking into account the properties of the working fluid through its heat capacity. From Eqs. (2), (6), and (7) the efficiency $\eta = W/Q_{\text{in}}$ is immediately found under MW conditions, that is,

$$\eta_{\text{MW}} = \begin{cases} 1-\gamma\tau^{\frac{\gamma}{1+\gamma}}\left(\frac{1-\tau^{\frac{1}{1+\gamma}}}{1-\tau^{\frac{\gamma}{1+\gamma}}}\right) & n=0 \\ 1+\gamma\frac{\ln\left(\frac{\tau+\gamma}{1+\gamma}\right)}{\ln\left(\frac{1+\frac{\gamma}{\tau}}{1+\gamma}\right)} & n=-1 \\ 1-\gamma\tau^{n+1}\frac{\left(\frac{\tau^{-n}+\gamma}{1+\gamma}\right)^{\frac{n+1}{n}}-1}{1-\left(\frac{1+\gamma\tau^n}{1+\gamma}\right)^{\frac{n+1}{n}}} & n\neq 0,\ -1 \end{cases}. \tag{8}$$

Clearly Eq. (8) reproduces the η_{CA} efficiency only for a few combinations of γ and n. Figure 2 depicts the plot of η_{MW} versus the exponent n considering $\gamma = 1$ (symmetric scenario). In this figure it can be observed that η_{CA} is obtained only for two values of n, the known case $n = 0$ (constant heat capacity) and at $n = -1/2$, which is a novel case with MW efficiency given by η_{CA}. Another relevant point is $n = -1/4$, where the MW efficiency reaches its maximum value.

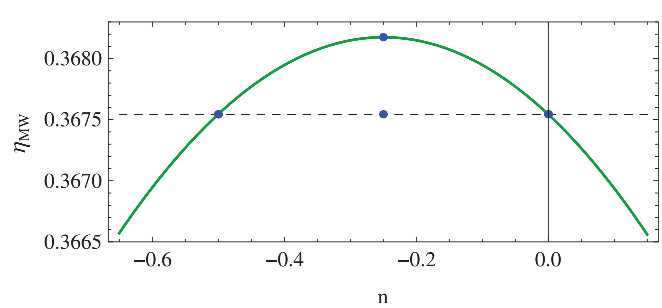

FIG. 2. (Color online) η_{MW} vs n with $\gamma = 1$ and $\tau = 2/5$. The cases $n = 0, -1/2$ reproduce the well-known CA efficiency. Note that there is a maximum at $n = -1/4$.

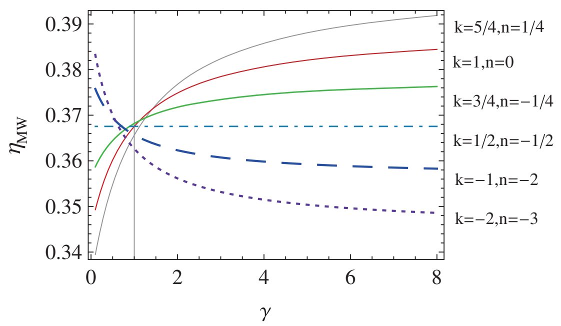

FIG. 3. (Color online) η_{MW} vs γ with $\tau = 2/5$. The case $n = 0$ reproduces the CA efficiency only when $\gamma = 1$ as is known, but the case $n = -1/2$ reproduces the same result for any γ.

By fixing the value of τ it is possible to find (Fig. 3) that the CA efficiency does not have a special behavior with respect to other efficiencies, in the sense that it does not represent the maximum value for an efficiency at MW. In Fig. 3 it is shown how the case $n = 0$ reaches the value η_{CA} only for $\gamma = 1$, that is, the case of symmetric cycles with constant heat capacities performing at MW [13,14]. In addition, η_{CA} is also obtained for MW-efficiency asymmetric cases ($\gamma \neq 1$) and $n \neq 0$. It is quite remarkable that the unique case independent of γ is that with $n = -1/2$; that is, η_{MW} ($n = -1/2$) $= \eta_{CA}$ for any value of γ. In any case, a cycle as shown in Fig. 1 with $C = aT^{-1/2}$ in the process $1 \to 2$ and $C = bT^{-1/2}$ in the process $3 \to 4$ is the true cycle characterized by the η_{CA} at MW for any value of γ. However, within the context of FTT, the cycle with MP efficiency given by η_{CA} and independent of $\gamma' = \beta/\alpha$ [α and β being heat conductances; see Eqs. (11) and (12)] is indeed the Curzon and Ahlborn cycle.

Through their papers on reversible cycles performing at MW, LL [13,14] established a bridge with FTT-MP cycles, basically by means of the CA efficiency. In recent years several authors [8–10,19–23] have renewed interest in the CA efficiency. Discussion of this famous formula has been mainly concerned with its possible universal nature within the context of finite-time cycles. Practically since the beginning of FTT as an active discipline, the limited validity of the CA efficiency has been established [3–7]. Nonetheless, at the end of the 1980s the CA efficiency was found linked to MW reversible cycles [13,14] and recently with its role in microscopic [11], mesoscopic [12], and macroscopic thermal cycles [8,9] the CA efficiency has gained new insights, which have to do with a possible universality at least at the first few terms in the Taylor expansion of the efficiency as a function of $\eta_C = 1 - \tau$ [12,19,21,23]. Clearly Eq. (8) admits this treatment, leading to a series in terms of η_C as functions of γ and n at any level of approximation. The series for the efficiencies in Eq. (8) are given by the following expressions:

$$\eta^* = \begin{cases} \frac{\eta_C}{2} + \frac{(1+2\gamma)\eta_C^2}{12(1+\gamma)} + \frac{(1+2\gamma)\eta_C^3}{24(1+\gamma)} + O[\eta_C]^4 & n = 0 \\ \frac{\eta_C}{2} + \frac{(2+\gamma)\eta_C^2}{12(1+\gamma)} + \frac{(2+2\gamma+\gamma^2)\eta_C^3}{24(1+\gamma)^2} + O[\eta_C]^4 & n = -1 \\ \frac{\eta_C}{2} + \frac{(1-n+2\gamma+n\gamma)\eta_C^2}{12(1+\gamma)} + \\ \frac{(1-n+3\gamma-n\gamma-2n^2\gamma+2\gamma^2+n\gamma^2)\eta_C^3}{24(1+\gamma)^2} + O[\eta_C]^4 & n \neq 0, \ -1 \end{cases} . \tag{9}$$

The linear term is really the same in every case. This fact strengthen the idea that this is a characteristic for cycles operating in the maximum work regime. When γ is restricted to the symmetric case, $\gamma = 1$, the efficiencies are

$$\eta^* = \begin{cases} \frac{\eta_C}{2} + \frac{\eta_C^2}{8} + \frac{\eta_C^3}{16} + O[\eta_C]^4 & n = 0 \\ \frac{\eta_C}{2} + \frac{\eta_C^2}{8} + \frac{5\eta_C^3}{96} + O[\eta_C]^4 & n = -1 \\ \frac{\eta_C}{2} + \frac{\eta_C^2}{8} + \frac{(6-n-2n^2)\eta_C^3}{96} + O[\eta_C]^4 & n \neq 0, \ -1 \end{cases} . \tag{10}$$

For $n = 0$ and $n = -1/2$ the coincidence with η_{CA} extends to any order of approximation and for the rest of the cases only up to quadratic order in η_C, such as occurs for MP strong coupling models that possess a left-right symmetry [21,24].

III. A CORRESPONDENCE BETWEEN EFFICIENCIES IN FTT AND REVERSIBLE CYCLES

Following the spirit of the articles of LL [13,14], one may wonder: Could it be possible to link the results obtained with heat capacities depending on temperature given by Eq. (8) with finite-time cycles of the CA type? It is suggested that this connection is indeed possible. As is well known, in reversible cycles of the Otto and JB type, Q_{in} and Q_{out} correspond to quasistatic processes accomplished by means of an infinite set of auxiliary heat reservoirs that lead the working substance temperature from T to T_+ and from T' to T_-, respectively. These heat quantities are calculated by means of integrals that lead to Eqs. (2) and (3). In the case of FTT cycles as the CA engine, $Q_{in/out}$ per cycle are given by irreversible heat transfer laws depending on the conductances (α, β) and the temperatures of the corresponding heat reservoirs and the working substance. As asserted by Wang and Tu [22], for the CA cycle, along both "isothermal" branches, the effective temperature of the working substance can vary. Thus, it can be proposed that in a T-S plane, the CA cycle follows a topologically equivalent diagram as that of Fig. 1. Behind this equivalence is the fact that in both cases the unavoidable role of the working substance is taken into account. For FTT cycles, heat transfer laws of the form

$$\dot{Q}_{in} = \alpha T_1^k \left[1 - \left(\frac{T_{1W}}{T_1} \right)^k \right], \tag{11}$$

$$\dot{Q}_{out} = \beta T_2^k \left[\left(\frac{T_{2W}}{T_2} \right)^k - 1 \right], \tag{12}$$

are typically used, where $T_{1W/2W}$ are the working substance temperatures, $T_{1/2}$ are the heat reservoir temperatures, and k is a real number. Although evidently the conceptual meaning of the heat quantities ($Q_{in/out}$) is different within the framework of reversible cycles and FTT cycles, respectively, it is quite remarkable how their corresponding efficiencies have a good agreement for nonarbitrary couples of n and k values. As is well known, $\dot{Q}_{in/out}$, power output, and MP efficiency for CA engines with heat transfer laws given by Eqs. (11) and (12) are numerically calculated in an easy and direct manner by maximizing the power output P given by the following

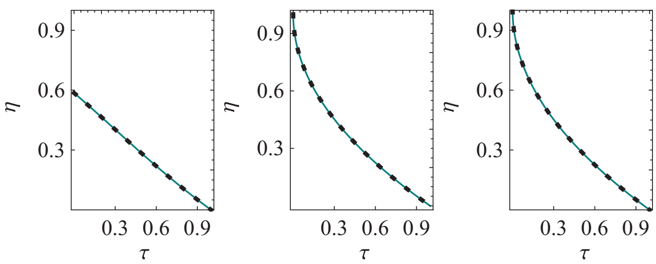

FIG. 4. (Color online) Comparison between the MW (thick line) and their corresponding MP (dotted line) efficiencies. The depicted efficiencies are for symmetric cases $\gamma = 1$: (a) $n = -2$ in Eqs. (2) and (3) and $k = -1$ in Eqs. (11) and (12); (b) $n = -1/2$ and $k = 1/2$; (c) $n = 0$ and $k = 1$.

equation [25–27]:

$$P(\eta) = \frac{\alpha T_1 \gamma' \eta[(1-\eta)^k - \tau^k]}{1 - \eta + \gamma'(1-\eta)^k}, \tag{13}$$

with respect to the efficiency η. In Fig. 4 is observed the excellent fitting between three MW efficiencies and their corresponding FTT-MP cases. Clearly the mapping between n and k is given by $k \to n + 1$, which stems from associating the exponents of Eqs. (2) and (3) with those of Eqs. (11) and (12). This rule excludes the case $n = -1$, because it corresponds to $k = 0$, which is incompatible with Eqs. (11) and (12). Later it will be seen how this particular case is consistently treated within the context of the problem of relating reversible-MW and FTT-MP thermal cycles.

The fitting between $\eta_{\rm MW}$ and $\eta_{\rm MP}$ goes from excellent (symmetric cases) to very good (asymmetric cases) (see Figs. 4 and 7), in such a way that the analytical closed expressions of $\eta_{\rm MW}$ given by Eq. (8) can be used as reliable first approximations for the FTT corresponding cycles, which commonly have to be calculated by means of numerical methods.

Interestingly, the MP-FTT efficiencies in Fig. 4 and 7 were calculated within the context of endoreversible thermodynamics, that is, by means of a Carnotian endoreversibility hypothesis (the internal reversible cycle is taken as a rectangle-shaped Carnot cycle in a T-S diagram). The question is: Why is the fitting in Fig. 4 so good despite the fact that there is not a rectangle in the reversible case of Fig. 1? A clue may be glimpsed in the article by Anacleto and Ferreira (AF) [17], where it was shown that a process as $1 \to 2$ in Fig. 1, but under minimum entropy production conditions (low dissipation, $\Delta S_{1,2} > 0$ for a finite number N of equilibrium states), has a temperature distribution path along the N auxiliary reservoirs given by $T_j = \sqrt{T_{j-1}T_{j+1}}$ (for constant heat capacity) with $j = 1, \ldots, N$. This means that the minimum entropy trajectory is not arbitrary but is given by a geometric mean distribution. This can be used to construct an irreversible first approximation of reversible paths such as those shown in Fig. 1, considering a finite number of auxiliary reservoirs instead of an infinite number of them in the processes $1 \to 2$ and $3 \to 4$ in Fig. 1. As a matter of fact, the "isotherms" of a Carnot cycle in an endoreversible construction are obtained after the power maximization [4]. Their respective temperatures (T^*_{1W} and T^*_{2W}) do not correspond necessarily to actual temperatures of the working substance, but are a kind of representative temperatures that help to express the irreversible heat fluxes which lead to the unavoidable entropy production, which permits a step towards more realistic models of finite-time cycles.

From this point of view, one can propose another suitable form of the endoreversibility hypothesis based on an internal cycle of the JB type, for example, where the working fluid temperatures can vary along the "isotherms," that is, an internal cycle such as that of Fig. 1 performing between the extreme temperatures T_+ and T_-. Once this cycle is constructed one could return to a rectangle-shaped endoreversible cycle in a T-S diagram by means of the following procedure: First, geometric means ($T_{\rm eff1}$ and $T_{\rm eff2}$) between T' and T_+, and T_- and T are calculated, approximating reversible nonadiabatic processes as those of Fig. 1 to minimum entropy irreversible ones. Then, by means of these mean effective temperatures $T_{\rm eff1}$ and $T_{\rm eff2}$ an equivalent Carnot rectangle is constructed (see Fig. 5). As was established by Wang and Tu [22], for a CA-type engine, the operation of the maximum power regime is equivalent to minimum irreversible entropy production in each finite-time "isothermal" process. If in addition the MW condition given by Eq. (7) is used, then

$$\eta_{\rm MW} \approx \eta_{\rm MP} = 1 - \frac{T_{\rm eff2}}{T_{\rm eff1}} = 1 - \frac{\sqrt{T'\,T_-}}{\sqrt{T\,T_+}} = 1 - \sqrt{\frac{T_-}{T_+}}, \tag{14}$$

that is, the CA efficiency (for the case of constant heat capacity). The same procedure can be generalized for heat capacities of the form $C = aT^n$. In Fig. 6 the cases $n = -3/2$ and $n = 2$ are shown, presenting three curves for the

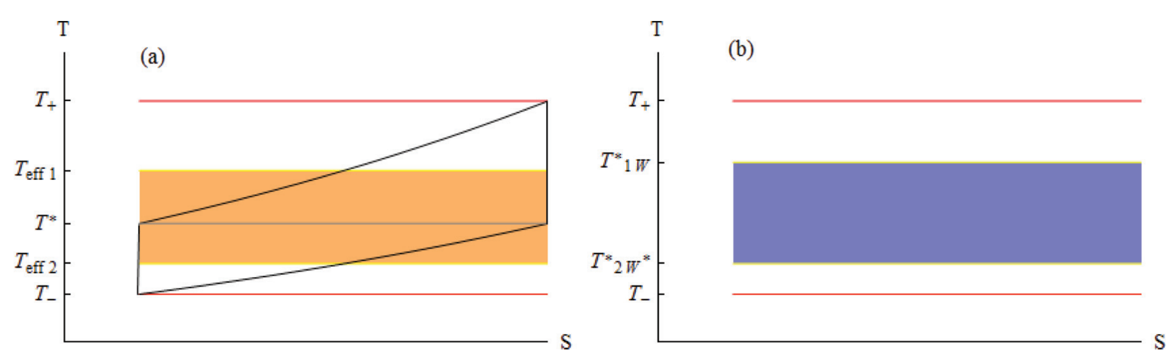

FIG. 5. (Color online) A comparison between two Carnotian endoreversible constructions: (a) by means of the effective temperature procedure and (b) by means of a standard CA procedure. Both diagrams were calculated for $T_+ = 600$ K and $T_- = 200$ K, which lead to $T_{\rm eff1} = 455.901$ K, $T_{\rm eff2} = 263.215$ K, $T^*_{1W} = 473.205$ K, and $T^*_{2W} = 273.205$ K. Remarkably, $T_{\rm eff2}/T_{\rm eff1} = T^*_{2W}/T^*_{1W} = \sqrt{T_-/T_+}$.

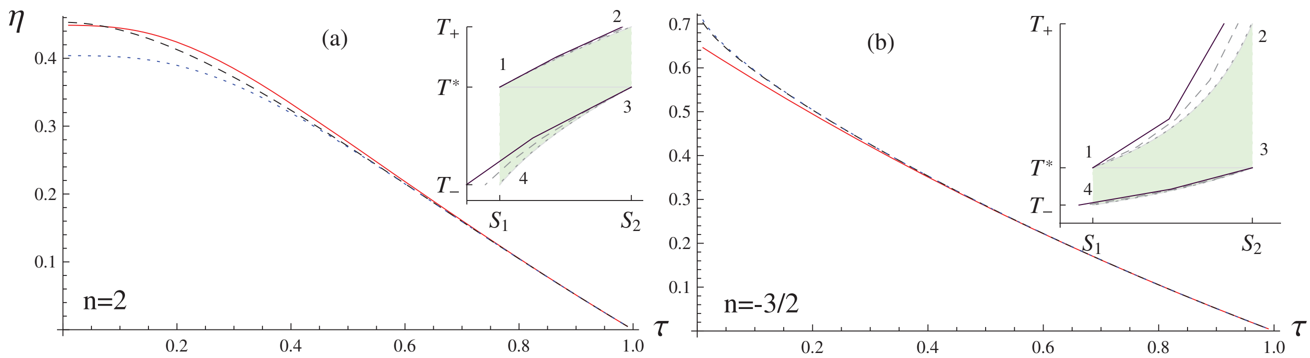

FIG. 6. (Color online) Comparison between three efficiencies: the reversible JB-type cycle efficiency (dashed line), its irreversible approximation obtained from the procedure described in the text (continuous line), and that corresponding to the FTT case (dotted line) for (a) $n = 2$ and (b) $n = -3/2$. In each inset there is a JB cycle (colored region) with the condition of maximum work [T^* given by Eq. (7)]. Three AF paths are shown for the processes $1 \to 2$ and $3 \to 4$: one built with one auxiliary reservoir (thin line), another with three auxiliary reservoirs, and the third one with 62 auxiliary reservoirs. Notice that the more auxiliary reservoirs, the better the approximation to the reversible process.

efficiencies: one for the reversible JB-MW efficiency [Eq. (8)], another for the standard FTT-MP efficiency by maximizing Eq. (13), and the third for the procedure based on effective temperatures. As can be seen, in Fig. 6 the three curves are very close to each other, especially for τ in the interval $[0.3,1]$. In summary, the similarity between $\eta_{\rm MW}$ and $\eta_{\rm MP}$ has to do with the proper flexibility of the endoreversibility hypothesis.

Another remarkable fact found in Fig. 7 is that the MW efficiency when $n = -2$ and the corresponding MP efficiency at $k = -1$ are exactly the same for any value of γ (henceforth $\gamma = \gamma'$), that is,

$$\eta_{\rm MW}(n=-2) = \eta_{\rm MP}(k=-1) = \frac{\eta_C}{2-\left(1-\sqrt{\frac{\gamma+\tau^2}{1+\gamma}}\right)}, \quad (15)$$

which has two limits: $\gamma \to 0$ and $\gamma \to \infty$ bounding the possible values of the efficiency for a given τ, at $n = -2$ ($k = -1$):

$$\lim_{\gamma\to\infty}(\eta_{\rm MW}) = \frac{\eta_{\rm C}}{2} < \eta_{\rm MW/MP} < \frac{\eta_{\rm C}}{2-\eta_{\rm C}} = \lim_{\gamma\to 0}(\eta_{\rm MW}). \quad (16)$$

Recently some authors have underscored the importance of these limits (first found in Ref. [5]), which have been reported within different contexts, such as a stochastic heat engine [11], a low-dissipation Carnot engine [20], and for a linear irreversible Carnot-like heat engine [22]. However, it will be seen below that these limits are of a particular validity only among a numerous set of limits for different values of k (or n). On the basis of Eq. (8) the limits of $\eta_{\rm MW}$ for $\gamma \to 0$ and $\gamma \to \infty$ are obtained which bound the values of $\eta_{\rm MW}$. These χ-shaped curves (continuous curves) are depicted in Fig. 7 where the corresponding $\eta_{\rm MP}$ curves (large dashed) are also shown along with the symmetric cases for both efficiencies. Well-known FTT numerical methods have been used to plot the $\eta_{\rm MP}$ curves [25–27].

Some interesting facts can be remarked from this figure: as mentioned before, for $n = -2$ ($k = -1$) both efficiencies $\eta_{\rm MW}$ and $\eta_{\rm MP}$ have the same limits when $\gamma \to 0$ and $\gamma \to \infty$. Interestingly, the χ-shaped curve corresponding to the $\eta_{\rm MW}$ efficiencies has an exact specular symmetry with respect to the value $n = -1/2$ ($k = 1/2$). At this point, the upper and lower limits of the efficiency are the same because $\eta_{\rm MW}$ ($n = -1/2$) does not depend on γ. This specular symmetry has as a consequence that both limits given by Eq. (16) are also found in the MW efficiency for $n = 1$ ($k = 2$; see Fig. 7). On the other hand, the χ-shaped curve for the $\eta_{\rm MP}$ efficiencies does not have a specular symmetry with respect to the crossing point at $k = 1$, where both limits ($\gamma \to 0$ and $\gamma \to \infty$) are the same because for $k = 1$, $\eta_{\rm MP} = \eta_{\rm CA}$ is independent of γ. This lack of symmetry precludes that the limits given by Eq. (16) appear for any other $k \neq -1$. It should be noted that at the left side and at the right side of the crossing points over the MW and MP χ-shaped curves, the lower and upper limits are interchanged. For the specular symmetric MW case, when exchanging $n \to -(n+1)$ [$k \to -(k-1)$], both limits have the same value, but they are inverted ($\gamma \to 0$ is replaced by $\gamma \to \infty$ and vice versa). There is another fact of great interest about the χ-shaped curves. For heat transfer laws with approximately $k \in (-10,10)$, in some regions $\eta_{\rm MW} < \eta_{\rm MP}$. This inequality is not an artifact of numerical solutions for $\eta_{\rm MP}$, because in that region exist some cases where the inequality is an exact analytical result, for example, for $k = 1/2$ ($n = -1/2$). However, clearly, if in addition to the heat fluxes many other irreversibilities are considered, the above-mentioned inequality should be inverted. For the Stefan-Boltzmann case, that is, $k = 4$ ($n = 3$), the upper and lower limits, for both η_{MW} and $\eta_{\rm MP}$, are not the limits given by Eq. (16), and additionally they are inverted, in such a way that for $\gamma' \to \infty$ a Müser-type engine ($\beta \gg \alpha$) is obtained [28] and $\eta_{\rm MW} > \eta_{\rm MP}$.

In Fig. 7 a large range of values of k (or n) is considered, showing that at the same limits $\gamma \to 0$, $\gamma \to \infty$, and $\gamma \to 1$, the values of the MP and MW efficiencies are very similar to each other, strengthening the idea that the analytical forms of the MW efficiencies are a good approximation to the corresponding MP efficiencies. The matching is improved for values of $\tau > 0.4$. In addition, for all symmetric cases $\eta_{\rm MW} \geqslant \eta_{\rm MP}$ and both convex curves ($\gamma = 1$) tend to zero when $|n|,|k| \to \infty$. It is noteworthy that the superior branches of the χ-*shaped* curves for both $\eta_{\rm MW}$ and $\eta_{\rm MP}$ tend asymptotically to $\eta_{\rm C}$, and both inferior branches tend to zero when $|n|,|k| \to \infty$.

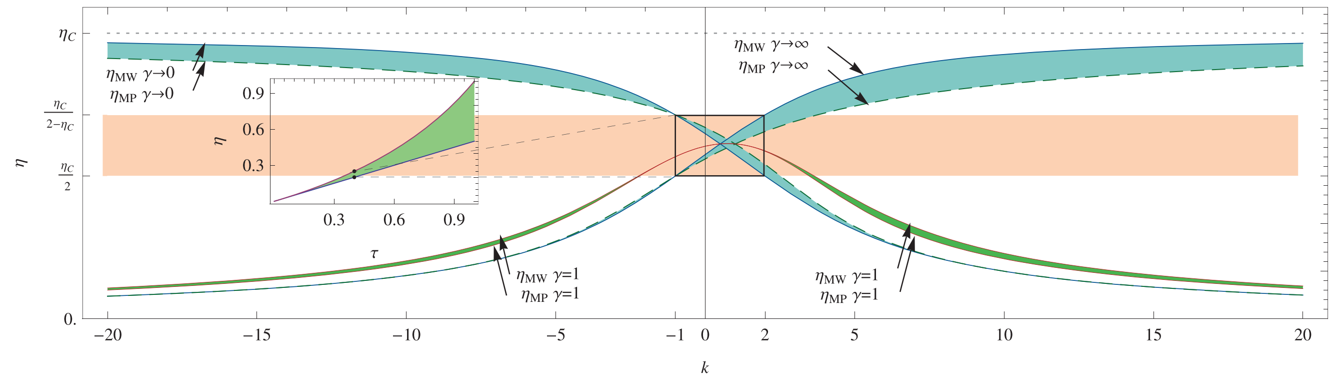

FIG. 7. (Color online) The bounds for the η_{MW} and η_{MP} are given by the χ-shaped curves for $\gamma \to 0$ and $\gamma \to \infty$, with $\tau = 2/5$. For $k = -1$ ($n = -2$) the well-known limits reported by Esposito *et al.* [20] are reproduced for both η_{MW} and η_{MP} (see inset). These limits are also reproduced by η_{MW} at $n = 1$. For the symmetric cases ($\gamma = 1$) $\eta_{MW} \geqslant \eta_{MP}$ with a maximum at $k = 3/4$ ($n = -1/4$; see also Fig. 2). Over the χ-shaped curves some regions can be observed where $\eta_{MW} \geqslant \eta_{MP}$. Notice that the upper asymptote is the value of η_C and the lower one is zero. Numerical calculations for η_{MP} have been used.

These results are a consequence of Eq. (8) and the numerical solutions of FTT-MP efficiencies. The physical consistency of all these asymptotic limits stems from the restrictions imposed by Eqs. (5) and (7) and the first and second laws of thermodynamics.

The asymptotic behavior towards η_C shown in Fig. 7 suggests that through the thermal properties of materials or metamaterials [29] it is possible to approach to η_C. Concerning the values of exponents k (or n), there are cases beyond conventional values, such as the scaling of photon bremsstrahlung emissivity in the optical thin limit [30], which correspond to a case with $k = 1/2$ in Eqs. (11) and (12), and even processes with k in the range from six to nine may be found in astronomical cooling mechanisms [30,31]. The fact that in Fig. 7 there are some intervals of γ and k (or n) values that approximate the efficiencies to η_C may shed some light on the searching of paths towards an improvement of energetic properties of working substances and cycles.

As mentioned before, the case $n = -1$ deserves a special treatment. The reversible heats Q_{in} and Q_{out} are given by Eqs. (2) and (3), which coincide with the heat laws given by (see Eq. (33) of Ref. [32])

$$\dot{Q}_{in} = \alpha \ln \frac{T_1}{T_{1w}} \tag{17}$$

and

$$\dot{Q}_{out} = \beta \ln \frac{T_{2w}}{T_2}, \tag{18}$$

which were tested as feasible heat laws in Ref. [32]. For the symmetric case ($\gamma = 1$), and also for the limits $\gamma \to 0$ and $\gamma \to \infty$ the matching between η_{MW} and η_{MP} is excellent. Thus, in the reversible-MW cases, the logarithmic expression appearing in the heats $n = -1$ goes to a logarithmic expression for the FTT-MP heats. That is, the mapping $k \to n + 1$ works very well for all cases except for the logarithmic one ($n = -1$). However, this special case also has a corresponding FTT-MP heat law that completes the agreement between η_{MW} and η_{MP}.

IV. CONCLUDING REMARKS

In summary, in this article it has been demonstrated that there exists a strong and rich relationship between a class of reversible MW cycles and finite-time MP cycles of the CA type. This connection opens a wide spectrum of interesting results containing known facts and some new findings. All results suggest that behind the remarkable agreement of the mentioned connection there is a kind of extended endoreversibility contained in Figs. (1), (6), and (7) for very low-dissipation cycles, as those of very large compression ratios [3,20]. This extended endoreversibility has to do with the procedure based in effective temperatures described in Sec. III. On the other hand, we have proposed that MW reversible cycles such as that of Fig. 1 can function as a more suitable reversible benchmark depending on working substance for MP-FTT cycles depending on working fluid. Finally, we suggest that through the behavior shown in Fig. 7 one has an asymptotic path towards Carnot efficiency by using the thermal properties of the working substances.

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