Analytic expression for the mean time to absorption for a random walker on the Sierpinski fractal. III. The effect of non-nearest-neighbor jumps

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We present exact, analytic results for the mean time to trapping of a random walker on the class of deterministic Sierpinski graphs embedded in $d \geqslant 2$ Euclidean dimensions, when both nearest-neighbor (NN) and next-nearest-neighbor (NNN) jumps are included. Mean first-passage times are shown to be modified significantly as a consequence of the fact that NNN transitions connect fractals of two consecutive generations.

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I. INTRODUCTION

It is well established that a diverse set of phenomena, among them the dynamics of reaction diffusion on substrates, heterogeneous catalysis, surface diffusion of adatoms, and diffusive transport in porous and amorphous media, is modeled very effectively by random walks (RWs) on fractal lattices, and an extensive literature exists on the subject (see, e.g., Refs. [1–7] and references therein). On disordered (random) fractals, the properties of RWs are manifested as scaling relations, in accord with the statistical nature of the self-similarity in the structures. Such relations are largely based on numerical simulations, although there exist some restricted but mathematically rigorous results (see, e.g., [8–10]). On deterministic fractals, however, more detailed results can be derived, as the self-similarity is exact on all scales. The Sierpinski gasket, in particular, is an archetypal fractal structure in that it possesses a sufficient degree of connectivity, lacunarity, and ramification to make it an effective as well as tractable model from the point of view of applications. Several aspects of random walks on this fractal, its higher-dimensional generalizations, and related fractals have been studied, and the topic continues to remain an active area of current research [11–17], as results derived in these cases serve as reliable pointers to the properties of random walks on generic fractal structures.

In the context of the applications mentioned above, the aspects that are generally of interest are hitting-time distributions, mean first-passage times (MFPTs) to traps or reaction centers, and related quantities. (For recent work on a systematic procedure for the calculation of such quantities on recursively defined structures, see Ref. [18].) Now, on *non*-self-similar structures, the Central Limit Theorem guarantees that the asymptotic diffusive behavior ($\langle r^2 \rangle \sim t$) of RWs is robust, in the sense that it remains valid for both regular and disordered structures and for arbitrary step-length distributions, as long as these have finite variances, and the walk is Markovian. For instance, the inclusion of both nearest-neighbor (NN) and next-nearest-neighbor (NNN) jumps does not make a significant difference to the overall behavior of the RW. In sharp contrast, RWs on a fractal of box dimension d_f are characterized by a random-walk dimension d_w that is a measure of the subdiffusivity of the transport, according to $\langle r^2 \rangle \sim t^{2/d_w}$. Correspondingly, first-passage-time distributions and MFPTs are also modified as compared to the case of RWs on nonfractal structures ($\langle t \rangle \sim r^{d_w}$ rather than r^2). At a deeper level, properties such as the recurrent or transient nature of RWs on a fractal, the mean number of distinct sites visited in a walk of a given number of steps [19], etc. are controlled by the spectral dimension [1] $\widetilde{d} = 2d_f/d_w$. While the various scaling relations derived for RWs with NN jumps alone may continue to be valid asymptotically when NNN jumps are also permitted, we may expect significant modifications to occur in the various prefactors concerned, thereby affecting numerical values nontrivially. This pertains especially to fractal structures of *finite* generation number, which is certainly the case in practical applications as well as numerical simulations. To give a specific example, the experimental evidence for, and the physical importance of, non-nearest-neighbor jumps in the diffusion of adatoms has been pointed out in Ref. [20].

In earlier work [11,12,15], we have studied the first-passage-time problem and certain related aspects on the Sierpinski gasket and its higher-dimensional counterparts. In this paper, we quantify the effects of the inclusion of NNN jumps by presenting an exact result for the mean time to trapping on the class of Sierpinski fractals, in the case when both NN and NNN jumps are included in the random walk. What makes the incorporation of NNN jumps significant is the fact that these jumps link fractals of two successive generations, as will be seen below. The main effect is the occurrence of a dimension-dependent modulating factor that multiplies the part of the MFPT that scales from one generation to the next. In addition, there is a small correction to the scaling part, which is also determined exactly. We believe that our exact and explicit results for the Sierpinski gasket and its higher-dimensional counterparts, i.e., the Sierpinski towers, will serve as useful and reliable markers against which the results of numerical simulations and approximations in the modeling of diffusive transport in hierarchical media can be compared and tested—in particular, with reference to the dependence on parameters such as the system size (represented

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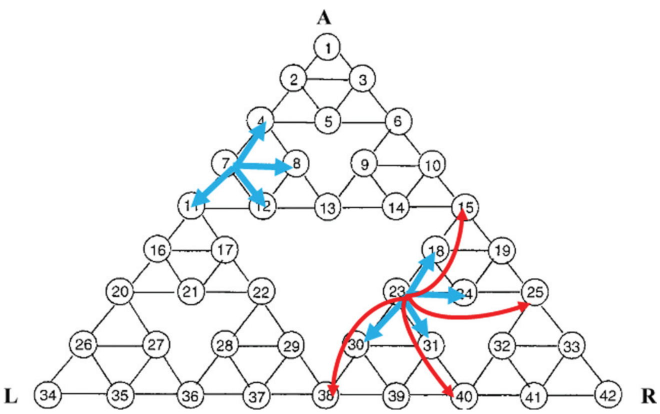

FIG. 1. (Color online) The gasket G_3, showing NN and NNN jumps from typical sites.

here by the generation number of the fractal structure) and associated dimensions ($d_f, d_w, \widetilde{d}$, and the Euclidean dimension d in which the fractal is embedded).

II. MEAN FIRST-PASSAGE TIME ON THE SIERPINSKI GASKET

The geometry of the Sierpinski gasket embedded in $d = 2$ Euclidean dimensions is well known. For our purposes, it is convenient to describe the nth-generation Sierpinski gasket (more accurately, the Sierpinski graph) G_n as follows. We start with an equilateral triangle (the zeroth-generation gasket G_0) with vertices denoted by A (the apex), L (the lower left corner), and R (the lower right corner), and decorate the centers of its sides with the vertices of the "lacunary triangle" that is removed to obtain the first-generation gasket G_1. Repetition of this process of decoration and excision leads to G_n. It has 3^{n-r} lacunary triangles of "size" r, where r runs from 1 for the smallest such triangle to n for the largest. The total number of sites on G_n is $N_n = \frac{1}{2}(3^{n+1} + 3)$, labeled by the site index α. The sites are labeled sequentially from the apex site A ($\alpha = 1$) downwards, from the left to the right in each row. L and R correspond, respectively, to $\alpha = N_n - 2^n$ and $\alpha = N_n$. Figure 1 shows the $n = 3$ gasket. Traps are located at any or all of the outermost vertices A, L, and R. Without loss of generality, we take the trap site(s) to be (i) A if there is a single trap, (ii) L and R if there are two traps, and, of course, (iii) A, L, and R if there are three traps. A simple random walk on G_n is ergodic (it is an irreducible, aperiodic Markov chain) and first passage to one of the traps from any starting site α is a sure event, with a mean first-passage time (MFPT) $T_\alpha^{(n)}$. ($T_\alpha^{(n)} \equiv 0$ if α is a trap site.) We seek the site-averaged MFPT,

$$T^{(n)} = \frac{1}{(N_n - N_t)} \sum_{\alpha=1}^{N_n} T_\alpha^{(n)}, \tag{1}$$

where N_t (=1, 2, or 3) is the number of traps, for a random walk with both NN and NNN jumps. In particular, we wish to determine the precise manner in which the inclusion of NNN jumps affects the MFPT as a function of the NNN jump probability q.

We begin by noting that it is natural and consistent to define an NNN jump on G_n as follows: it is an NN jump on G_{n-1}.

TABLE I. Site-averaged mean time to trapping at one of the outermost vertices of a Sierpinski gasket of generation n. The argument of $T^{(n)}$ is the next-nearest-neighbor jump probability q.

n	$T^{(n)}(0)$	$T^{(n)}(\frac{1}{6})$	$T^{(n)}(\frac{1}{2})$
1	$\frac{46}{5}$	$\frac{38}{5}$	$\frac{26}{5}$
2	$\frac{608}{14}$	$\frac{490}{14}$	$\frac{313}{14}$
3	$\frac{8674}{41}$	$\frac{6950}{41}$	$\frac{4364}{41}$
4	$\frac{127772}{122}$	$\frac{102250}{122}$	$\frac{63967}{122}$
5	$\frac{1904566}{365}$	$\frac{1523750}{365}$	$\frac{952526}{365}$
6	$\frac{28507448}{1094}$	$\frac{22806250}{1094}$	$\frac{14254453}{1094}$

On G_n, therefore, we have the following two types of sites. (i) $\frac{1}{2}(3^n + 3)$ sites that belong to both G_{n-1} as well as G_n. Both NN and NNN jumps can occur from any of these sites, with respective probabilities p and $q = 1 - p$. (ii) A fresh set of 3^n sites that decorate the centers of the edges of G_{n-1} to yield G_n. These sites are the vertices of the smallest ($r = 1$) lacunary triangles in G_n. Only NN jumps occur from these sites. (See Fig. 1.) The coupled linear equations for the set of MFPTs $\{T_\alpha^{(n)}(q)\}$ that follow from the Laplace transform of the backward Kolmogorov equation are given by

$$T_\alpha^{(n)} - (p/\nu_\alpha) \sum_{\beta \in \{\mathrm{NN}\}} T_\beta^{(n)} - (q/\nu_\alpha) \sum_{\gamma \in \{\mathrm{NNN}\}} T_\gamma^{(n)} = 1, \tag{2}$$

where α runs over all nontrap sites, ν_α is the coordination number of the site α ($\nu_\alpha = 2$ for the sites A, L, and R; 4 for all other sites), and β, γ are summed, respectively, over the NN and NNN sites of α. We note in passing that, although we have set $p + q = 1$, it is quite straightforward to allow for a nonzero *sojourn* probability $1 - (p + q)$ at each site, should the diffusive transport being modeled warrant it in any specific application. In principle, the set of Eqs. (2) can be solved for any given values of n and q, to find explicitly the MFPT for every starting site α on each G_n, and hence the site-averaged mean time to trapping.

We have carried out extensive numerical calculations to find each $T_\alpha^{(n)}$ up to generation $n = 6$, for the representative values $q = \frac{1}{6}$ and $q = \frac{1}{2}$ of the NNN jump probability. As a sample of these results, we list in Table I the values of the site-averaged MFPT $T^{(n)}(q)$ in the case of a single trap, for these values of q. For ready reference and comparison, we have included the values of $T^{(n)}(0)$, corresponding to random walks with only NN jumps, and written all the values with the same denominator [$=N_n - 1 = \frac{1}{2}(3^{n+1} + 1)$], for each n. These direct calculations rapidly become very tedious, however, because $N_n \sim 3^n$, and the inversion of an ($N_n \times N_n$) matrix is involved. Moreover, the inclusion of NNN jumps doubles the number of nonzero off-diagonal elements of each row from four to eight. However, the scale invariance of G_n comes to our aid, and an exact real-space renormalization of the MFPTs can be carried out to determine $T^{(n)}(q)$ for arbitrary n and q in closed, analytic form, as we now proceed to show.

III. DERIVATION OF THE GENERAL FORMULA FOR THE MFPT

Two essential ingredients are involved in this process. The first is the identification of the basic rescaling of MFPTs on G_n occasioned by the inclusion of NNN jumps. This factor may be deduced in a number of different ways, all of which lead to the same result (as they ought to). The simplest of these is as follows. Consider, *for the moment*, a random walk starting at the vertex site 1 on a trapless gasket. The walker reaches one of the NN sites 2 and 3 in one time step. What is the MFPT "to go twice as far," i.e., to reach one of the NNN sites 4 and 6? Let $t_\alpha(q)$ be the MFPT from the site α (=1,2,3,5) to hit any of these sites, for a given value of the NNN jump probability q. From the site 1, both NN and NNN jumps are allowed in a single time step, with respective probabilities p and $q = 1 - p$. From the sites 2, 3, and 5, only NN jumps are allowed. Then, using the obvious symmetry property $t_2 = t_3$, we obtain the coupled equations

$$t_1 = p\,t_2 + 1, \quad 3t_2 = t_1 + t_5 + 4, \quad 2t_5 = t_2 + 2. \tag{3}$$

Eliminating t_2 and t_5, we get

$$t_1(q) = 5\,\frac{(3-2q)}{(3+2q)}. \tag{4}$$

Thus, the value $t_1(0) = 5$ for a simple random walk on the Sierpinski gasket (which leads to the value $d_w = \ln 5/\ln 2$ for the random-walk dimension [1] on this fractal) is modulated, when NNN jumps are permitted, by the factor

$$\lambda(q) = \frac{(3-2q)}{(3+2q)}. \tag{5}$$

This observation is crucial for what follows. In the absence of NNN jumps, $\lambda(0) = 1$, as required.

The second key step is a sum rule for the MFPTs from the vertices of any triangle in G_n. Let (i_r, j_r, k_r) denote the vertices of a lacunary triangle of size r, where $1 \leqslant r \leqslant n$. Let (I_r, J_r, K_r) denote the vertices of the triangle of which (i_r, j_r, k_r) is the central lacunary triangle. The coupled equations satisfied by the set $\{T_\alpha^{(n)}\}$ of MFTPs yield the following fundamental sum rules among these MFPTs: For $r = 1$, we have

$$T_{i_1}^{(n)} + T_{j_1}^{(n)} + T_{k_1}^{(n)} = T_{I_1}^{(n)} + T_{J_1}^{(n)} + T_{K_1}^{(n)} + 6, \tag{6}$$

while for $2 \leqslant r \leqslant n$, we find

$$T_{i_r}^{(n)} + T_{j_r}^{(n)} + T_{k_r}^{(n)} = T_{I_r}^{(n)} + T_{J_r}^{(n)} + T_{K_r}^{(n)} + (6\lambda)\,5^{r-1}. \tag{7}$$

Using these relations and the enumeration of triangles of different sizes on G_n, we can reduce the sum of MFPTs on the structure to combinatorial factors and the sum

$$\tau(q) \equiv T_A^{(n)}(q) + T_L^{(n)}(q) + T_R^{(n)}(q) \tag{8}$$

of the MFPTs from the outer vertices of G_n, as will be seen shortly. Not unexpectedly, it turns out that $\tau(q) = \lambda(q)\tau(0)$. The evaluation of $\tau(0)$ then enables us to express $\sum_\alpha T_\alpha^{(n)}(q)$ in closed form, as we now show.

Had the final term in Eq. (6) been 6λ [in conformity with the general relation (7)] rather than 6, we would have had a straightforward rescaling of MFPTs, leading to the simple relation $T^{(n)}(q) = \lambda(q)\,T^{(n)}(0)$ for the site-averaged MFPT. Then, once the combinatorics required to determine the site-averaged MFPT $T^{(n)}(0)$ in the absence of NNN jumps was worked out, the identification of the correct modulating factor $\lambda(q)$ would essentially have completed the solution. As it stands, however, it is clear that $T^{(n)}(q)$ must exceed $\lambda(q)\,T^{(n)}(0)$ because of the absence of NNN jumps from the vertices of the smallest ($r = 1$) lacunary triangles. The excess, representing a correction to exact rescaling of MFPTs in the presence of NNN jumps, is conveniently deduced by writing $6 = 6\lambda + 6(1-\lambda)$ in Eq. (6), and adding the contributions due the term $6(1-\lambda)$ from each lacunary triangle of size 1. As there are 3^{n-1} such triangles, we have

$$\begin{aligned}\sum_\alpha T_\alpha^{(n)}(q) &= \lambda(q)\sum_\alpha T_\alpha^{(n)}(0) + 6[1-\lambda(q)] \times 3^{n-1}\\ &= \lambda(q)\sum_\alpha T_\alpha^{(n)}(0) + \frac{(8\times 3^n)q}{(3+2q)}\end{aligned} \tag{9}$$

on using Eq. (5) for $\lambda(q)$ in the correction term. The sum $\sum_\alpha T_\alpha^{(n)}(0)$ is found by a repeated application of Eqs. (6) and (7), with λ set equal to unity, as follows.

Let σ_r denote the sum of the MFPTs from the vertices of *all* lacunary triangles of size r [we drop the superscript (n) for the moment, for simplicity of notation]. We observe that (i) every $T_\alpha^{(n)}$, where α is a site of the type I_r, J_r, or K_r, appears exactly twice in such sums, and (ii) every site on G_n, other than the outermost vertices A, L, and R, is *uniquely* a vertex of a lacunary triangle of some size r (where $1 \leqslant r \leqslant n$). The outcome of these observations is a set of linear equations for the partial sums σ_r, given by

$$\sigma_r = \tau(0) + 2\sum_{k=r+1}^{n}\sigma_k + (2\times 5^{r-1}3^{n-r+1}). \tag{10}$$

After some algebra, the solution to this set of equations is found to be

$$\sigma_r = 3^{n-r}[\tau(0) + 5^n + 5^{r-1}], \quad 1 \leqslant r \leqslant n. \tag{11}$$

Hence $\sum_\alpha T_\alpha^{(n)}(0)$, which can be written as $\tau(0) + \sum_{r=1}^n \sigma_r$, is given by

$$\sum_\alpha T_\alpha^{(n)}(0) = \frac{1}{2}\,[(3^n+1)\tau(0) + 3^n(5^n-1)]. \tag{12}$$

It remains to determine $\tau(0)$. This is very simply done by writing down the answer for the triangle G_0 and scaling up the result by the factor 5^n.

(i) If there is a single trap (at A, say), then $\tau(0) = T_L^{(n)}(0) + T_R^{(n)}(0) = 2T_L^{(n)}(0) = 2T_L^{(0)}(0)\times 5^n = 4\times 5^n$.

(ii) If there are two traps (at L and R, say), then $\tau(0) = T_A^{(n)}(0) = 1\times 5^n$.

(iii) If there are three traps (at A, L, and R), then $\tau(0) = 0$.

Using these values in Eq. (12) and substituting the result in Eq. (9), we arrive at the following exact solutions for the site-averaged mean time to trapping in each of the three cases.

(i) Trap at any one of the outermost vertices:

$$T^{(n)}(q) = \frac{(3-2q)[5\,(15^n) + 4\,(5^n) - 3^n] + 16q\,(3^n)}{(3+2q)(3^{n+1}+1)}. \tag{13}$$

(ii) Traps at any two of the outermost vertices:

$$T^{(n)}(q) = \frac{(3-2q)[2\,(15^n)+5^n-3^n]+16q\,(3^n)}{(3+2q)(3^{n+1}-1)}. \quad (14)$$

(iii) Traps at all three outermost vertices:

$$T^{(n)}(q) = \frac{(3-2q)[5\,(15^{n-1})-3^{n-1}]+16q\,(3^{n-1})}{(3+2q)(3^n-1)}. \quad (15)$$

We note that $T^{(n)}(q)$ is a rational number when q is rational. Equations (13)–(15) [as well as Eq. (20) below] represent the principal results of this paper. The numerical results mentioned in Sec. II and partially listed in Table I have been verified to be in complete and precise agreement with the general formulas for $T^{(n)}(q)$ in all cases.

It is evident that the inclusion of NNN jumps significantly alters the MFPT, essentially halving it as q approaches the value $\frac{1}{2}$. The scale factor $\lambda(q)$ decreases monotonically from 1 to $\frac{1}{5}$ as q increases from 0 to 1. It is also clear that the contribution from the second term on the right in Eq. (9), which (as argued in the foregoing) may be regarded as a correction to the exact scaling contribution $\lambda(q)\sum_\alpha T_\alpha^{(n)}(0)$, is quite small in all cases. In all three cases above, it leads to a contribution to $T^{(n)}$ that tends, as $n \to \infty$, to the value $16q/(9+6q)$. The latter merely increases from 0 to $\frac{16}{15}$ as q increases from 0 to 1.

As n becomes very large, the leading asymptotic behavior of $T^{(n)}(q)$ in all three cases is $\sim 5^n \sim N_n^{2/\widetilde{d}}$, where $\widetilde{d} = \ln 9/\ln 5$ is the spectral dimension of the Sierpinski gasket, as in the absence of NNN jumps, but the prefactor is modulated by the factor $\lambda(q)$, as expected.

IV. EXTENSION TO THE SIERPINSKI TOWER IN *d* DIMENSIONS

We turn now to the extension of the results obtained in the foregoing to the case of a random walk with NN and NNN jumps on the Sierpinski tower embedded in d-dimensional Euclidean space, where $d \geqslant 3$. The case $d = 3$ is of obvious physical interest, as it is pertinent to diffusive transport in porous media modeled by a fractal. On the other hand, the problem turns out to be exactly solvable for general d. The availability of such an analytic solution enables us to examine the d dependence of the MFPT explicitly, and thus adds to our insight into random walks on fractals. The combinatorics for $d \geqslant 3$ are naturally more involved than those in the case of the $d = 2$ gasket. We shall skip the details and present only the salient points in a form that enables ready comparison with their $d = 2$ counterparts given in the foregoing.

As in the case of the gasket, the construction of the tower is described as follows, for our purposes. We start with the generation-0 tower, which is a hypertetrahedron (a simplex) with $(d+1)$ vertices, and decorate the midpoint of each of its $d(d+1)/2$ edges with a site. When a central lacunary region is removed, we are left with a generation-1 tower, which comprises a set of $(d+1)$ hypertetrahedra, each with $(d+1)$ vertices and $d(d+1)/2$ sides of unit length, sharing vertices such that the total number of vertices of the structure is $(d+1)(d+2)/2$. Figure 2, depicting the generation-1 tower in $d = 3$, helps one visualize the case $d > 3$ as well. The

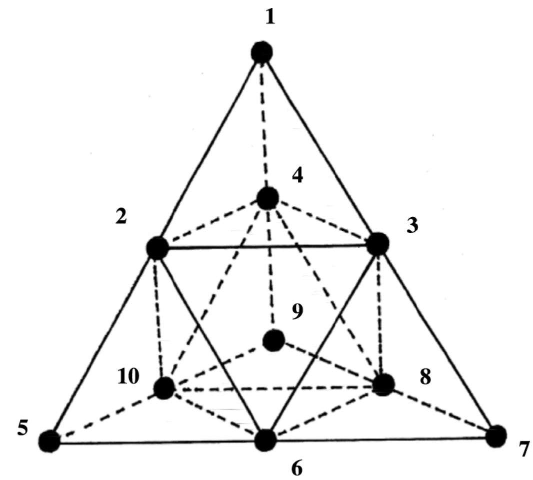

FIG. 2. The $n = 1$ Sierpinski tower in $d = 3$.

nth-generation tower has

$$N_n = \tfrac{1}{2}(d+1)[(d+1)^n + 1] \quad (16)$$

sites. The coordination number of each site is $2d$, except for the $(d+1)$ outermost vertices, each of which has a coordination number d. The fractal dimensionalities of the structure [1] are as follows: the box-counting dimension is $d_f = \ln(d+1)/\ln 2$ and the random-walk dimension is $d_w = \ln(d+3)/\ln 2$, so that the spectral dimension is $\widetilde{d} = 2\ln(d+1)/\ln(d+3)$.

As before, an NNN jump on the nth-generation tower is defined as an NN jump on the $(n-1)$th-generation tower. The NN and NNN jump probabilities are, as before, p and $q = 1-p$, respectively. A key step is the identification of the corresponding scale factor $\lambda_d(q)$, i.e., the counterpart of $\lambda(q)$ in Eq. (5), that multiplies $\sum_\alpha T_\alpha^{(n)}(0)$ to yield the dominant contribution to $\sum_\alpha T_\alpha^{(n)}(q)$, apart from a small correction to exact scaling. The latter correction is the counterpart of the term $(8 \times 3^n)q/(3+2q)$ in Eq. (9). It decreases rapidly with increasing generation number n, and becomes quite negligible even for relatively small values of n. We shall therefore disregard it, and present results in the scaling approximation in which the site-averaged MFPT is given by

$$T^{(n)}(q) \simeq \lambda_d(q)\, T^{(n)}(0), \quad (17)$$

to a very high degree of accuracy.

Once again, the factor $\lambda_d(q)$ may be determined by considering a generation-1 tower with no traps, as follows. Starting at the apex site, labeled 1 as usual, the walker jumps with a probability p/d to any one of the d NN sites which we denote by the generic label a (exemplified for $d = 3$ by the sites 2, 3, and 4 in Fig. 2), and with a probability q/d to any one of the d NNN sites which we denote by the generic label c (exemplified by the sites 5, 7, and 9 in this figure). We seek the MFPT $t_1(q)$ to reach one of the sites of type c from the initial site 1, allowing for all possible excursions between the initial site, the set of NN (or type-a) sites, and the set of "intermediate" sites which we denote by the generic label b (for $d = 3$, the sites 6, 8, and 10 in Fig. 2). This last set reduces to a single site in the gasket in $d = 2$, labeled site 5 in Fig. 1. By an obvious symmetry, the MFPTs from all the type-a sites are equal to each other, as are the MFPTs from all the type-b sites. Each a site has one bond to site 1, one bond to a type-c site, $(d-1)$ bonds to other type-a sites, and $(d-1)$ bonds to type-b sites. Each b site has two bonds to type-a sites, two bonds to type-c sites, and $2(d-2)$ bonds to other type-b sites. Writing down the coupled equations for the set of MFPTs and imposing the symmetry conditions, we arrive at the following reduced set of equations for the MFPTs:

$$t_1 = p\,t_a + 1, \quad (d+1)\,t_a = t_1 + (d-1)\,t_b + 2d,$$
$$2t_b = t_a + d. \tag{18}$$

These are the d-dimensional generalizations of Eqs. (3). Eliminating t_a and t_b to solve for t_1, and setting $p = 1-q$, we get

$$t_1(q) = (d+3)\,\frac{(d+1-dq)}{(d+1+2q)}. \tag{19}$$

In the absence of NNN jumps ($q=0$), this yields $t_1(0) = d+3$ as the mean time "to go twice as far" on the tower, leading immediately to the known result $d_w = \ln(d+3)/\ln 2$ for the random-walk dimension on this fractal. When NNN jumps are included, this MFPT is modulated by the scale factor

$$\lambda_d(q) = \frac{(d+1-dq)}{(d+1+2q)}. \tag{20}$$

This is the generalization of Eq. (5) for the scale factor $\lambda(q)$ corresponding to the gasket in $d=2$. In the case of the $d=3$ Sierpinski tower, in particular, the scale factor is $\lambda_3(q) = (4-3q)/(4+2q)$. As q increases from 0 to 1, MFPTs on the tower in d dimensions are essentially reduced by a multiplicative factor ranging from 1 to $1/(d+3)$. When $p = q = \frac{1}{2}$, i.e., when NN and NNN jumps are equally probable, $\lambda_d(\frac{1}{2}) = \frac{1}{2}$ for all values of d. Further, as $d \to \infty$, $\lambda_d(q) \to 1-q = p$.

It remains to compute $T^{(n)}(0)$. This is done by an extension [12] of the procedure outlined above in the case $d=2$. The sum rule that replaces Eq. (6) is found to be

$$\sum_{i=1}^{d(d+1)/2} T_i^{(n)} = \frac{1}{2}d\sum_{I=1}^{d+1} T_I^{(n)} + \frac{1}{2}d^2(d+1). \tag{21}$$

Here i labels the vertices of the lacunary region of smallest size ($r=1$), while I labels the vertices of the hypertetrahedron "circumscribing" this lacunary region. Omitting the rest of the details, the final relation that represents the counterpart of Eq. (12) is

$$\sum_{\alpha=1}^{N_n} T_\alpha^{(n)}(0) = \frac{1}{2}[(d+1)^n + 1]\tau(0)$$
$$+ \frac{1}{2}[d^2/(d+2)]\,(d+1)^n[(d+3)^n - 1], \tag{22}$$

where $\tau(0)$ is again the sum of the MFPTs from the $(d+1)$ vertices of the outermost hypertetrahedron to the trap(s) located at one or more of these vertices. Once again, the manner in which the expression in Eq. (22) is a generalization of that for $d=2$ [Eq. (12)] is manifest. The sum $\tau(0)$ depends, of course, on the configuration of traps at the $(d+1)$ outermost corners of the tower. It is easily obtained for traps located at an arbitrary number N_t of these vertices as follows. Consider the generation-0 tower. First passage from any one of the $(d+1-N_t)$ nontrap sites to a trap site may be regarded as a Bernoulli trial with a "success" probability equal to N_t/d (since the starting site has d neighbors). Hence the mean time to trapping is simply d/N_t. Adding up the mean times from all the nontrap sites and scaling up the result by $(d+3)^n$, we obtain the counterpart of this sum on the nth-generation tower, namely,

$$\tau(0) = [d(d+1-N_t)/N_t](d+3)^n. \tag{23}$$

In particular, in the analogs of the situations considered in Sec. III in the case of the gasket, we obtain the following results:

(i) When there is a single trap at just one of these corners ($N_t = 1$), $\tau(0) = d^2(d+3)^n$. We then find

$$T^{(n)}(0) = \frac{d^2\,[(d+1)^n\,(d+3)^{n+1} + (d+2)(d+3)^n - (d+1)^n]}{(d+2)\,[(d+1)^{n+1} + d - 1]}. \tag{24}$$

(ii) If there are traps at all of the corners except one ($N_t = d$), $\tau(0) = (d+3)^n$. Then

$$T^{(n)}(0) = \frac{(d^2+d+2)(d+1)^n(d+3)^n + (d+2)(d+3)^n - d^2(d+1)^n}{(d+2)\,[(d+1)^{n+1} - d + 1]}. \tag{25}$$

(iii) If there are traps at all of the outermost corners ($N_t = d+1$), $\tau(0) \equiv 0$. We then get

$$T^{(n)}(0) = \frac{d^2(d+1)^{n-1}\,[(d+3)^n - 1]}{(d+2)\,[(d+1)^n - 1]}. \tag{26}$$

As already mentioned, the site-averaged MFPTs in the presence of NNN jumps [see Eq. (17)] are essentially these expressions multiplied by the scale factor $\lambda_d(q)$.

Finally, we mention that the inclusion of NNN jumps, when extended to the scaling relation or renormalization of waiting-time distributions for general continuous-time random walks on the Sierpinski gasket [3], leads to an interesting and significant modification of this relation. Likewise, it is of interest to deduce the exact recurrence relations among the eigenvalues of the transition matrix [15] occurring in the master equation for the random walk, as functions of q. These and other results will be reported elsewhere.

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