

Quasirelativistic Langevin equation

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We address the problem of a microscopic derivation of the Langevin equation for a weakly relativistic Brownian particle. A noncovariant Hamiltonian model is adopted, in which the free motion of particles is described relativistically while their interaction is treated classically, i.e., by means of action-to-a-distance interaction potentials. Relativistic corrections to the classical Langevin equation emerge as nonlinear dissipation terms and originate from the nonlinear dependence of the relativistic velocity on momentum. On the other hand, similar nonlinear dissipation forces also appear as classical (nonrelativistic) corrections to the weak-coupling approximation. It is shown that these classical corrections, which are usually ignored in phenomenological models, may be of the same order of magnitude, if not larger than, relativistic ones. The interplay of relativistic corrections and classical beyond-the-weak-coupling contributions determines the sign of the leading nonlinear dissipation term in the Langevin equation and thus is qualitatively important.

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I. INTRODUCTION

Relativistic Brownian motion is the underpinning paradigm in several modern fields, including transport and thermalization processes in quark-gluon plasma, astrophysical fluids, and graphene [1]. Despite a high motivation toward the construction of a unifying approach, there is currently no consensus on the form of Langevin and master equations describing a relativistic Brownian particle. Several versions were proposed in recent years [1], but their status and validity range are often obscure. The difficulties are many and some are fundamental to relativistic many-body dynamics [1,2]. In the nonrelativistic theory, the standard equations of Brownian motion can be derived microscopically, eliminating (fast) degrees of freedom of the thermal bath with a projection operator or some other technique [3]. This is much harder to do in the relativistic domain because the Lorentz-invariant dynamics of a system of particles also involves the degrees of freedom of the field through which the particles interact. For weakly relativistic systems to second order in v/c , the elimination of field degrees of freedom is straightforward [4], but comes at the expense of the emergence of additional velocity-dependent forces, which are difficult to handle for many-particle systems within the Hamiltonian formalism [5].

Another dissonance with the standard classical approach comes from the limited validity of time-scale separation methods in the relativistic domain. For a nonrelativistic Brownian particle the mean-square momentum is linear with mass $\langle P^2 \rangle \sim k_B T M$, which implies that at any given temperature T the average thermal momentum of the heavy Brownian particle is much larger (and the velocity is much smaller) than that of particles of the bath with mass $m \ll M$. This enables one to justify the weak-coupling approximation to the lowest order in the small mass ratio parameter. In contrast, for the ultrarelativistic particle with the thermal momentum much larger than Mc , the equipartition theorem [see Eq. (14) below] takes the form that does not involve the mass of the particle $c\langle |P| \rangle = k_B T$. Clearly, conventional time-scale

separation methods cannot be applied in this case since heavy and light particles have comparable momenta. As will be shown below, a similar situation may take place also for a weakly relativistic Brownian particle when it is immersed in an ultrarelativistic bath.

Despite these difficulties (and perhaps because of them), many authors prefer to pursue an approach based on a straightforward extension of the nonrelativistic Langevin phenomenology [6–10]. In a simple version, one assumes that the dissipative force on the particle is linear in the particle's velocity V and composes the Langevin equation for the particle's momentum P in the rest frame of the bath in the form [6]

$$\frac{dP}{dt} = -\zeta V(P) + \xi(t), \quad (1)$$

where $\xi(t)$ is a stationary zero-centered delta-correlated (white) noise

$$\langle \xi(t)\xi(t') \rangle = 2D\delta(t-t'). \quad (2)$$

For the relativistic domain, Eq. (1) is nonlinear since velocity is a nonlinear function of momentum

$$V(P) = \frac{dE}{dP} = \frac{c^2 P}{E} = \frac{1}{\Gamma(P)} \frac{P}{M}, \quad (3)$$

where E is the energy of a free particle

$$E(P) = \sqrt{c^2 P^2 + M^2 c^4} = Mc^2 \Gamma(P) \quad (4)$$

and

$$\Gamma(P) = \sqrt{1 + \left(\frac{P}{Mc}\right)^2}. \quad (5)$$

Since Eq. (1) is not amenable to closed-form analytic solutions, a fluctuation-dissipation relation between the friction coefficient ζ and the strength of noise D in general cannot be established. However, further progress can be achieved under two additional assumptions. The first one is that the random force $\xi(t)$ is a Gaussian process, i.e., vectors of observed values $\{\xi(t_1), \dots, \xi(t_n)\}$ have a multivariate normal distribution. As known from the general theory [3], in this case

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the corresponding Fokker-Planck equation for the distribution function $f(P, t)$ has the form

$$\frac{\partial}{\partial t} f(P, t) = \zeta \frac{\partial}{\partial P} \{V(P) f(P, t)\} + D \frac{\partial^2}{\partial P^2} f(P, t) \quad (6)$$

for any function $V(P)$, linear or not. The second assumption is that the stationary solution of this equation $f(P) = C \exp[-\frac{\zeta}{D} E(P)]$ must coincide with the Maxwell-Jüttner distribution

$$\rho_{MJ}(P) = Z^{-1} e^{-\beta E(P)}, \quad (7)$$

where $\beta = 1/k_B T$ is the inverse temperature of the bath in the bath's reference frame and $E(P)$ is given by (4). This immediately gives the fluctuation-dissipation relation

$$\zeta = \beta D. \quad (8)$$

While attractively simple, the above phenomenological scheme suggests no clue about its range of validity. It also appears to be unnecessarily restrictive in its demand of the noise to be Gaussian. Within a nonrelativistic theory, both phenomenological and microscopic, the assumption of Gaussian noise is unnecessary to derive the fluctuation-dissipation relation. It is therefore natural to ask if, and under what conditions, the Langevin (1) and Fokker-Planck (6) equations can be derived microscopically. As mentioned above, severe difficulties of the relativistic theory of many-body interacting systems generally make such a derivation hardly possible. However, one may expect that some difficulties can be avoided for systems with contact interactions, i.e., when particles interact via pointlike binary collisions [11,12]. In this case, interactions can be fully described by conservation laws and one can avoid the infamous problem of constructing a relativistic action-at-a-distance Hamiltonian of many interacting particles.

Following this line, Dunkel and Hänggi [12] discussed the derivation of the relativistic Langevin equation for the Rayleigh model, in which a Brownian particle interacts with bath molecules via elastic instantaneous (and therefore binary) collisions. As is well known, for the classical Rayleigh model the noise is not Gaussian [13,14]. The derivation presented in [12] emphasizes the non-Gaussian nature of the noise for the relativistic domain. The authors employed a nonperturbative approach that leads to a rather complicated expression for the dissipating force F_{diss} , which is amenable only to numerical evaluation. This makes it difficult to verify the validity of the phenomenological ansatz $F_{\text{diss}} = -\zeta V(P)$ and the fluctuation-dissipation relation (8).

In this paper we address the problem of a microscopic derivation of the relativistic Langevin from different premises. Namely, we consider a Hamiltonian model in which only the free motion of particles is treated relativistically, while the interaction is described classically, i.e., by means of action-at-a-distance potentials. Such an approximation, which we refer to as quasirelativistic, was recently discussed and tested in [15]. It produces, of course, noncovariant equations of motion, yet may be acceptable for systems with very-low-density and/or short-range interactions. Numerical simulation shows that quasirelativistic many-particle system equilibrates toward the Maxwell-Jüttner distribution [15], which is similar to a fully relativistic molecular dynamics simulation [16]. Intuitively,

in the limit when the range of interaction goes to zero, one can expect to get the same results as for relativistically consistent models with contact interaction. The advantage of the quasirelativistic approach is that it enables one to apply well-developed perturbation techniques of Hamiltonian theory of nonrelativistic Brownian motion [17,18]. These methods are not easy to use within models with instantaneous binary collisions [11,12] due to the presence of singular δ -like forces.

We shall assume that the rest mass M of a Brownian particle is much larger than the mass m of a bath particle so that the mass ratio parameter λ is small,

$$\lambda = \sqrt{\frac{m}{M}} \ll 1. \quad (9)$$

Also, we shall restrict the discussion to temperature regimes for which the characteristic thermal momentum of the Brownian particle P_T is much larger than that of a bath particle p_T and much smaller than Mc ,

$$p_T \ll P_T \ll Mc. \quad (10)$$

This will allow us to construct a perturbation technique similar to that for the nonrelativistic theory. As will be shown, the condition (10) is not too restrictive: While the Brownian particle is assumed to be weakly relativistic, particles of the bath may be weakly, moderately, or even ultrarelativistic.

We shall show that under the above assumptions the phenomenological Langevin equation (1) and the fluctuation-dissipation relation (8) are not valid for any regime for which nonlinearity of the function $V(P)$ is essential. The comparison of phenomenological and microscopic predictions is easier if, given the condition (10), one retains only the leading nonlinear term in the expansion of $V(P)$,

$$V(P) = \frac{1}{\Gamma(P)} \frac{P}{M} \approx \left[1 - \frac{1}{2} \left(\frac{P}{Mc} \right)^2 \right] \frac{P}{M}. \quad (11)$$

With the approximation (11) and relation (8), the phenomenological Langevin equation (1) takes the form

$$\frac{d}{dt} P(t) = -\gamma_1 P(t) - \gamma_2 P^3(t) + \xi(t), \quad (12)$$

with damping coefficients

$$\gamma_1 = \frac{\beta D}{M}, \quad \gamma_2 = -\frac{\beta D}{2M^3 c^2} < 0. \quad (13)$$

The microscopic theory developed below also leads to the Langevin equation in the form (12), but with fluctuation-dissipation relations different and more complicated than (13). Note that the nonlinear term in Eq. (12) originates from the first relativistic correction to the classical linear relation $V = P/M$. On the other hand, from the microscopic theory of nonrelativistic Brownian motion it is known that similar nonlinear dissipation terms also appear in the Langevin equation beyond the weak-coupling limit. These contributions, which are missing in phenomenological Langevin equations (1) and (12), are of classical nature and originate from higher-order terms in the expansion of the particle's propagator in powers of the mass ratio parameter λ . We shall show that these classical nonlinear corrections are of the same order

of magnitude or larger than the corresponding relativistic contributions. A consistent theory, which takes into account the interplay of both relativistic and classical contributions for the nonlinear dissipative force F_{diss} , does not support the simple ansatz $F_{\text{diss}} \sim V(P)$ adopted in the phenomenological theory.

One prediction of the presented theory is that the sign of the nonlinear damping coefficient γ_2 in Eq. (12) is not predetermined and may depend on temperature and a detailed form of the microscopic correlations. This is in contrast to the second of the phenomenological relations (13), which predicts that γ_2 is negative. For $\gamma_2 < 0$, one can show that the Langevin equation (12), as well as the corresponding Fokker-Planck equation, leads to an ill-behaved stationary distribution $f(P)$ diverging for large P . Thus, in the phenomenological theory, the approximation (11) is insufficient and one needs to retain nonlinear terms of higher orders in P/Mc . In contrast, the presented microscopic theory predicts that for certain temperature intervals γ_2 may be positive and the Langevin equation (12) has meaningful equilibrium properties. Other implications are discussed in Sec. VIII.

II. SCALING RELATIONS

We consider a Brownian particle (below referred to for short as *the particle*) that is not too far from the equilibrium in which the momentum distribution is given by the Maxwell-Jüttner distribution (7). A relativistic version of the equipartition theorem for the particle in equilibrium in one dimension has the form

$$\langle V(P)P \rangle = \left\langle \frac{1}{\Gamma(P)} \frac{P^2}{M} \right\rangle = \frac{1}{\beta}, \quad (14)$$

where $\Gamma(P)$ is given by (5) and the angular brackets mean the average with the distribution (7). Unlike its classical counterpart (when $\Gamma \rightarrow 1$), the relativistic equipartition relation (14) does not allow one to find an exact expression for the thermal momentum of the particle $P_T = \sqrt{\langle P^2 \rangle}$. Yet Eq. (14) is convenient to evaluate an approximate value of P_T as follows. Let us define the parameters

$$\epsilon = \sqrt{\frac{1}{\beta m c^2}}, \quad \delta = \lambda \epsilon = \sqrt{\frac{1}{\beta M c^2}}, \quad (15)$$

characterizing the strength of relativistic effects for the bath and the particle, respectively. Using approximations $\Gamma \sim 1$ for $\delta \lesssim 1$ and $\Gamma(P) \sim P/Mc$ for $\delta \gg 1$, from (14) one obtains

$$P_T \approx \begin{cases} \sqrt{\frac{M}{\beta}} = \delta M c & \text{for } \delta \lesssim 1 \\ \frac{1}{c\beta} = \delta^2 M c & \text{for } \delta \gg 1. \end{cases} \quad (16a)$$

$$(16b)$$

The validity of this estimation can be verified by direct evaluation of the mean-square momentum for the Maxwell-Jüttner equilibrium

$$\langle P^2 \rangle = Z^{-1} \int e^{-\beta E(P)} P^2 dP. \quad (17)$$

Indeed, in one dimension from (17) one obtains exactly

$$P_T = \sqrt{\langle P^2 \rangle} = \begin{cases} \sqrt{\frac{M}{\beta}} \phi_1(\delta) \\ \frac{1}{c\beta} \phi_2(\delta), \end{cases} \quad (18a)$$

$$(18b)$$

with dimensionless functions

$$\phi_1(\delta) = \left[\frac{K_2(1/\delta^2)}{K_1(1/\delta^2)} \right]^{1/2}, \quad \phi_2(\delta) = \frac{1}{\delta} \phi_1(\delta). \quad (19)$$

Here $K_n(x)$ are the modified Bessel functions of the second kind. As can be checked, $\phi_1(\delta) \sim 1$ for $\delta \lesssim 1$ and $\phi_2(\delta) \sim 1$ for $\delta \gg 1$, so that the exact relations (18) lead to the estimations (16). A similar consideration can be carried out to evaluate the thermal momentum $p_T = \sqrt{\langle p^2 \rangle}$ of a bath particle

$$p_T \approx \begin{cases} \sqrt{\frac{m}{\beta}} = \epsilon m c & \text{for } \epsilon \lesssim 1 \\ \frac{1}{c\beta} = \epsilon^2 m c & \text{for } \epsilon \gg 1. \end{cases} \quad (20a)$$

$$(20b)$$

In order to design an appropriate perturbation technique, we need to establish relations between P_T and p_T for different temperature regimes. We shall use the following nomenclature.

Regime A is defined by relation

$$\epsilon \ll 1, \quad (21)$$

or $k_B T \ll mc^2$. Since the other relevant parameter is also small $\delta = \lambda \epsilon \ll 1$, in this regime relativistic effects are weak for both the bath and the particle. As follows from (16) and (20), the thermal momentum of a bath particle is λ times smaller than of the particle

$$p_T = \lambda P_T. \quad (22)$$

Regime B is defined by the condition

$$\epsilon \sim 1, \quad (23)$$

or $k_B T \sim mc^2$. The other relevant parameter δ is small $\delta = \lambda \epsilon \sim \lambda \ll 1$. This regime corresponds to the moderately relativistic bath and weakly relativistic particle. The relation between p_T and P_T is still given by (22). It is therefore convenient for both regimes *A* and *B* to introduce the particle's scaled momentum

$$P_* = \lambda P, \quad (24)$$

which on average is expected to be of the same order of magnitude as the thermal momentum of a bath particle.

Regime C corresponds to a subdomain of the ultrarelativistic bath defined by the relation

$$1 \ll \epsilon \ll \lambda^{-1}, \quad (25)$$

or $mc^2 \ll k_B T \ll Mc^2$. The right-hand side of the inequality (25) ensures that $\delta \ll 1$, so that the particle is still weakly relativistic and its thermal momentum P_T is given by the classical expression (16a). In contrast, since $\epsilon \gg 1$, the thermal momentum of a bath particle is given by the ultrarelativistic expression (20b). Here p_T is still smaller than P_T , but now with the scaling factor δ ,

$$p_T = \delta P_T. \quad (26)$$

For this regime we define the scaled momentum of the particle as

$$P_* = \delta P \quad (27)$$

with the expectation that on average P_* is of the same order of magnitude as momenta of bath particles.

Regime D is defined by the relation

$$\epsilon \sim \lambda^{-1}, \quad (28)$$

or $k_B T \sim M c^2$. In this case $\epsilon \gg 1$ and $\delta \sim 1$, which corresponds to the ultrarelativistic bath and the moderately relativistic particle. Thermal momenta p_T and P_T are the same as for regime *C*, related as $p_T = \delta P_T$ and, since $\delta \sim 1$, are of the same order of magnitude.

Regime E is defined by

$$\epsilon \gg \lambda^{-1}. \quad (29)$$

Since $\epsilon, \delta \gg 1$, for this regime both the particle and bath are ultrarelativistic. Thermal momenta of the particle and of the bath are given by ultrarelativistic expressions (16b) and (20b), respectively, and as for regime *D* are of the same order of magnitude.

In the next section we use the above scaling relations to formulate quasirelativistic dynamic equations in a form that explicitly involves a small parameter relevant to a given temperature regime. We shall restrict ourselves to regimes *A*, *B*, and *C* only, for which $P_T \gg p_T$. Regimes *D* and *E*, for which $P_T \sim p_T$, cannot be treated with the conventional perturbation techniques and will not be discussed further.

III. QUASIRELATIVISTIC HAMILTONIAN

Let (X, P) and $\{x_i, p_i\}$ be the sets of coordinates and momenta of the particle and particles of the thermal bath, respectively. The motion will be assumed to occur in one spatial dimension, but this assumption is not essential and is adopted merely to simplify notation. The quasirelativistic Hamiltonian [15] of the combine system of the particle and the bath is

$$H = E(P) + H_0, \quad (30)$$

where $E(P)$ is the energy of the free particle given by (4) and H_0 is the Hamiltonian of the bath interacting with the particle fixed at the position X ,

$$H_0 = \sum_i e(p_i) + U(X). \quad (31)$$

In this expression $e(p_i)$ is the energy of i th free particle of the bath

$$e(p_i) = \sqrt{c^2 p_i^2 + m^2 c^4} \quad (32)$$

and the potential $U(X) = U(X, x_i)$ describes the interaction of the particle with the bath, as well as bath particles with each other. The interaction is understood classically as action at a distance; no Darwin-like momentum-dependent corrections [4,5] are included in the potential U . Thus the only difference between our quasirelativistic Hamiltonian and that of the nonrelativistic theory is a nonquadratic dependence of free particle energy terms $E(P)$ and $e(p_i)$ on momenta. Bath particles will be assumed to have the same rest mass m ,

which is much smaller than that of the particle M , so that $\lambda = \sqrt{m/M} \ll 1$.

The Liouville operator corresponding to the Hamiltonian (30) splits naturally in two parts

$$L = L_0 + L_{\text{part}}. \quad (33)$$

The Liouville operator L_0 governs the dynamics of the bath with Hamiltonian H_0 ,

$$L_0 = \sum_i v_i \frac{\partial}{\partial x_i} + f_i \frac{\partial}{\partial p_i}. \quad (34)$$

Here $f_i = -\partial H_0 / \partial x_i$ is the force on a bath particle and a bath particle velocity as a function of momentum is

$$v_i = \frac{\partial H_0}{\partial p_i} = \frac{1}{\gamma(p_i)} \frac{p_i}{m} \quad (35)$$

with

$$\gamma(p) = \sqrt{1 + \left(\frac{p}{mc}\right)^2}. \quad (36)$$

The operator L_{part} involves derivatives with respect to the coordinate and momentum of the particle

$$L_{\text{part}} = V(P) \frac{\partial}{\partial X} + F \frac{\partial}{\partial P}. \quad (37)$$

Here $F = -\partial H / \partial X$ is the force on the particle and

$$V(P) = \frac{\partial H}{\partial P} = \frac{1}{\Gamma(P)} \frac{P}{M}, \quad (38)$$

with $\Gamma(P)$ given by (5), is the particle's velocity.

The only difference between the quasirelativistic Liouville operator L and its nonrelativistic counterpart is the presence in the above formula of dimensionless factors $\Gamma^{-1}(P)$ and $\gamma^{-1}(p)$, which makes velocities nonlinear functions of momenta.

The next step is to write the Liouville operator L in terms of the scaled momentum of the particle P_* , which would put L into a form that explicitly involves a relevant small parameter.

In regimes *A* and *B* ($\epsilon \lesssim 1$), since $\delta = \lambda \epsilon \ll 1$, the particle is weakly relativistic and

$$P_T = \sqrt{M/\beta} = \delta M c \ll M c \quad (39)$$

[see Eq. (16a)]. Then one can use the approximation

$$\frac{1}{\Gamma(P)} \approx 1 - \frac{1}{2} \left(\frac{P}{M c}\right)^2, \quad (40)$$

which also can be written as

$$\frac{1}{\Gamma(P)} \approx 1 - \frac{\delta^2}{2} \left(\frac{P_*}{p_T}\right)^2. \quad (41)$$

As discussed in the previous section, for these regimes the scaled momentum of the particle P_* and the bath's thermal momentum are defined as

$$P_* = \lambda P, \quad p_T = \sqrt{\frac{m}{\beta}} = \epsilon m c. \quad (42)$$

With the approximation (41), the particle's velocity reads

$$V = \frac{1}{\Gamma(P)} \frac{P}{M} \approx \lambda \left\{ 1 - \frac{\delta^2}{2} \left(\frac{P_*}{p_T}\right)^2 \right\} \frac{P_*}{m} \quad (43)$$

and the operator L_{part} [Eq. (37)] takes the form

$$L_{\text{part}} = \lambda L_1 + \lambda \delta^2 L_2, \quad (44)$$

with the classical part

$$L_1 = \frac{P_*}{m} \frac{\partial}{\partial X} + F \frac{\partial}{\partial P_*} \quad (45)$$

and the relativistic correction

$$L_2 = -\frac{1}{2m p_T^2} P_*^3 \frac{\partial}{\partial X}. \quad (46)$$

Thus, for regimes *A* and *B* the Liouville operator L for the total system (33) can be written as

$$L = L_0 + \lambda L_1 + \lambda \delta^2 L_2, \quad (47)$$

with L_0 , L_1 , and L_2 defined by (34), (45) and (46), respectively.

In regime *C* ($1 \ll \epsilon \ll \lambda^{-1}$), since $\delta = \lambda \epsilon \ll 1$, the particle is still weakly relativistic $p_T \ll M c$ and the approximation (40) for Γ^{-1} is meaningful. One can check that the expression (41) retains its form, although now the scaled momentum of the particle P_* and the bath's thermal momentum are defined as

$$P_* = \delta P, \quad p_T = \frac{1}{c\beta} = \epsilon^2 m c, \quad (48)$$

as prescribed by Eqs. (26) and (27) in the previous section. The particle velocity now has the form

$$V = \frac{1}{\Gamma} \frac{P}{M} \approx \delta \left\{ 1 - \frac{\delta^2}{2} \left(\frac{P_*}{p_T} \right)^2 \right\} \frac{P_*}{p_T} c \quad (49)$$

and the operator L_{part} [Eq. (37)] reads

$$L_{\text{part}} = \delta L_1 + \delta^3 L_2, \quad (50)$$

where

$$L_1 = c \frac{P_*}{p_T} \frac{\partial}{\partial X} + F \frac{\partial}{\partial P_*} \quad (51)$$

and

$$L_2 = -\frac{c}{2} \left(\frac{P_*}{p_T} \right)^3 \frac{\partial}{\partial X}. \quad (52)$$

Thus, for regime *C* the Liouville operator L for the whole system (33) takes the form

$$L = L_0 + \delta L_1 + \delta^3 L_2, \quad (53)$$

where L_0 , L_1 , and L_2 are given by (34), (51), and (52), respectively.

It is worthwhile to observe that the above relations can be obtained from the corresponding expressions for regimes *A* and *B* by making the replacements

$$\lambda \rightarrow \delta, \quad m \rightarrow \frac{p_T}{c}. \quad (54)$$

In Eqs. (47) and (53), the dependence of the Liouville operator L on small parameters is explicit, which makes these expressions convenient for developing a perturbation technique, as discussed in the following sections.

IV. PRE-LANGEVIN EQUATION

In this section we apply the Mazur-Oppenheim projection operator technique [17] to modify the exact equation of motion for the scaled momentum of the particle P_* into a form convenient for the subsequent derivation of the Langevin equation with a perturbation method.

In regimes *A* and *B*,

$P_* = \lambda P$ and the equation of motion is

$$\frac{d}{dt} P_*(t) = \lambda F(t) = \lambda e^{Lt} F, \quad (55)$$

where L is given by (47) and $F = F(t=0)$. The propagator e^{Lt} can be decomposed as

$$e^{Lt} = e^{Qt} + \int_0^t e^{L(t-\tau)} \mathcal{P} L e^{QL\tau} d\tau, \quad (56)$$

where $Q = 1 - \mathcal{P}$ and \mathcal{P} is an arbitrary operator. This follows from the operator identity

$$e^{(A+B)t} = e^{At} + \int_0^t e^{A(t-\tau)} B e^{(A+B)\tau} d\tau, \quad (57)$$

with $A = L$ and $B = -\mathcal{P}L$. Inserting (56) into (55) yields

$$\frac{d}{dt} P_*(t) = \lambda F^\dagger(t) + \lambda \int_0^t e^{L(t-\tau)} \mathcal{P} L F^\dagger(\tau) d\tau, \quad (58)$$

where the projected force is

$$F^\dagger(t) = e^{QLt} F. \quad (59)$$

We shall assume that the initial distribution for bath degrees of freedom is

$$\rho_0 = Z^{-1} e^{-\beta H_0}. \quad (60)$$

Hereafter the angular brackets will denote the average with the distribution ρ_0 . We define the operator \mathcal{P} to be the projection operator ($\mathcal{P}^2 = \mathcal{P}$) that averages over the initial degrees of freedom of the bath

$$\mathcal{P}(\dots) = \int \rho_0(\dots) \prod_i dx_i dp_i = \langle \dots \rangle. \quad (61)$$

The major benefit of this choice for \mathcal{P} is the orthogonality relation

$$\mathcal{P} L_0 = 0, \quad (62)$$

which makes in the equation of motion (58) a crucial reduction

$$\frac{d}{dt} P_*(t) = \lambda F^\dagger(t) + \lambda^2 \int_0^t e^{L(t-\tau)} \mathcal{P} (L_1 + \delta^2 L_2) F^\dagger(\tau) d\tau. \quad (63)$$

Now the integral term in this equation does not involve derivatives with respect to the bath degrees of freedom (except for in the propagator e^{Lt}).

The next step is to take into account the explicit expressions for L_1 and L_2 given by (45) and (46) and also the relation

$$\mathcal{P} \frac{\partial}{\partial X}(\dots) = \left\langle \frac{\partial}{\partial X}(\dots) \right\rangle = -\beta \langle F \dots \rangle, \quad (64)$$

which can be proved by integration by parts. This puts Eq. (63) into the “pre-Langevin” form

$$\begin{aligned} \frac{d}{dt} P_*(t) &= \lambda F^\dagger(t) + \lambda^2 \int_0^t d\tau e^{L(t-\tau)} \\ &\times \left\{ -\frac{1}{p_T^2} P_* + \frac{\partial}{\partial P_*} + \delta^2 \frac{1}{2p_T^4} P_*^3 \right\} \langle F F^\dagger(t) \rangle. \end{aligned} \quad (65)$$

Recall that for the given regimes $p_T = \sqrt{m/\beta}$. Since $\mathcal{P}\mathcal{Q} = 0$ and $\langle F \rangle = 0$, the projected force $F^\dagger(t)$ is zero centered $\langle F^\dagger(t) \rangle = \mathcal{P}e^{\mathcal{Q}L} F = 0$.

In regime *C*, the scaled momentum is $P_* = \delta P$ and the equation of motion is

$$\frac{d}{dt} P_*(t) = \delta F(t) = \delta e^{Lt} F, \quad (66)$$

where L is now given by (53). Then the same procedure as the one described above for regimes *A* and *B* puts the equation of motion into the form

$$\begin{aligned} \frac{d}{dt} P_*(t) &= \delta F^\dagger(t) + \delta^2 \int_0^t d\tau e^{L(t-\tau)} \\ &\times \left\{ -\frac{1}{p_T^2} P_* + \frac{\partial}{\partial P_*} + \delta^2 \frac{1}{2p_T^4} P_*^3 \right\} \langle F F^\dagger(t) \rangle. \end{aligned} \quad (67)$$

This equation is similar to Eq. (65) for regimes *A* and *B* except that λ is now replaced by δ and the thermal momentum of a bath particle is $p_T = 1/c\beta$ instead of $p_T = \sqrt{m/\beta}$ for regimes *A* and *B*.

The only approximation made so far is the truncated expansion (40) of $\Gamma^{-1}(P)$ for a weakly relativistic particle. Otherwise, the equations of motion (65) and (67) are exact. Compared to the corresponding nonrelativistic equations, they contain an additional nonlinear term cubic in P_* . In order to make further progress and to put these equations into the Langevin form one needs to expand $F^\dagger(t)$ in powers of a relevant small parameter. As can be observed from (65) and (67), higher-order terms of this expansion must be taken into account in order to consistently retain the leading nonlinear relativistic correction.

V. LANGEVIN EQUATION: REGIMES A AND B

Let us find the perturbation expansion of the projected force $F^\dagger(t)$ for regimes *A* and *B*, when the Liouville operator L is given by (47), $L = L_0 + \lambda L_1 + \lambda \delta^2 L_2$. Since $\mathcal{P}L_0 = 0$ and $\mathcal{Q}L_0 = L_0$, we can write

$$F^\dagger(t) = e^{\mathcal{Q}L} F = e^{(L_0 + \lambda \mathcal{Q}L_1 + \lambda \delta^2 \mathcal{Q}L_2)t} F. \quad (68)$$

Next, as follows from the operator identity (57), the part of the propagator involving L_2 gives a contribution of order $\lambda \delta^2$,

$$F^\dagger(t) = e^{(L_0 + \lambda \mathcal{Q}L_1)t} F + O(\lambda \delta^2). \quad (69)$$

In what follows we shall retain in the expansion of $F^\dagger(t)$ only terms up to second order in λ ,

$$F^\dagger(t) \approx F_0(t) + \lambda F_1(t) + \lambda^2 F_2(t). \quad (70)$$

The term $O(\lambda \delta^2)$ in (69) does not contribute to this approximation because $\lambda \delta^2 = \lambda^3 \epsilon^2$ is of order λ^3 for regime *B* ($\epsilon \sim 1$)

or less for regime *A* ($\epsilon \ll 1$). Applying the identity (57) to the operator $\exp[(L_0 + \lambda \mathcal{Q}L_1)t]$ in a recurrent manner, one obtains

$$F_0(t) = e^{L_0 t} F, \quad (71)$$

$$F_1(t) = \int_0^t d\tau e^{L_0(t-\tau)} \mathcal{Q}L_1 F_0(\tau), \quad (72)$$

$$F_2(t) = \int_0^t d\tau e^{L_0(t-\tau)} \mathcal{Q}L_1 F_1(\tau). \quad (73)$$

The term $F_0(t)$ is the pressure force, i.e., the force exerted by the bath on the fixed particle. Terms $F_1(t)$ and $F_2(t)$ have no direct physical meaning and depend on the particle momentum. This dependence is to be explicitly extracted.

To the lowest order, one substitutes $F^\dagger(t) \approx F_0(t)$ into the pre-Langevin equation of motion (65) and retains terms up to order λ^2 . The relativistic nonlinear term cubic in P_* is of order $\lambda^2 \delta^2 = \lambda^4 \epsilon^2 \lesssim \lambda^4$ and does not show up in this approximation. As a result, one obtains the linear generalized Langevin equation

$$\frac{d}{dt} P_*(t) = \lambda F_0(t) - \frac{\lambda^2}{p_T^2} \int_0^t d\tau P_*(\tau) C_0(t - \tau), \quad (74)$$

with the memory kernel

$$C_0(t) = \langle F F_0(t) \rangle. \quad (75)$$

As discussed in the Introduction, the quasirelativistic description is expected to be asymptotically valid only in the limit of instantaneous point interactions. Therefore, the above equation must be taken in the Markovian limit

$$\frac{d}{dt} P_*(t) = \lambda F_0(t) - \lambda^2 \alpha_0 P_*(t), \quad (76)$$

with

$$\alpha_0 = \frac{1}{p_T^2} \int_0^\infty C_0(t) dt = \frac{\beta}{m} \int_0^\infty \langle F F_0(t) \rangle dt. \quad (77)$$

The equation for the true momentum $P = \lambda^{-1} P_*$ reads

$$\frac{d}{dt} P(t) = F_0(t) - \gamma_0 P(t), \quad (78)$$

with

$$\gamma_0 = \frac{\beta D_0}{M}, \quad D_0 = \int_0^\infty \langle F F_0(t) \rangle dt. \quad (79)$$

Thus, for regimes *A* and *B* in the lowest order in λ one obtains the Langevin equation of the same form as for the nonrelativistic domain with the standard fluctuation-dissipation relation. Although the fluctuating force $F_0(t)$ is governed by the relativistic Liouville operator L_0 for the bath, this only modifies the values of the damping coefficient γ_0 and the effective strength of the noise D_0 . Otherwise, relaxation properties of the particle remain indistinguishable from that for the nonrelativistic domain.

In order to take into account nontrivial relativistic effects, we must retain in the expansion for $F^\dagger(t)$ the higher-order terms. Let us adopt the λ^2 -order approximation (70) and evaluate the correlation $\langle F^\dagger(t) F \rangle$ in the pre-Langevin equation (65), extracting explicitly the dependence on P_* . After some algebra

the result can be presented in the form

$$\begin{aligned} \langle FF^\dagger(t) \rangle &= \langle FF_0(t) \rangle + \lambda^2 \langle FF_2(t) \rangle \\ &= C_0(t) + \lambda^2 \left[\left(\frac{P_*}{m} \right)^2 C_1(t) + \frac{1}{m} C_2(t) \right]. \end{aligned} \quad (80)$$

Here $C_0(t)$ is the correlation function of the pressure force (75), while functions $C_1(t)$ and $C_2(t)$ are expressed in terms of more complicated correlations

$$\begin{aligned} C_1(t) &= \int_0^t dt_1 \int_0^{t_1} dt_2 \langle\langle G_0 G_2(t, t_1, t_2) \rangle\rangle, \\ C_2(t) &= \int_0^t dt_1 \int_0^{t_1} dt_2 \langle\langle G_0 G_0(t-t_1) G_1(t, t_2) \rangle\rangle. \end{aligned} \quad (81)$$

Here we use the notation

$$\begin{aligned} G_0(t) &= F_0(t), \\ G_1(t, t_1) &= S(t-t_1)F_0(t_1), \\ G_2(t, t_1, t_2) &= S(t-t_1)S(t_1-t_2)F_0(t_2), \end{aligned} \quad (82)$$

with the operator $S(t) = e^{L_0 t} \frac{\partial}{\partial X}$ and $G_0 = G_0(0)$. The double angular brackets stand for cumulants $\langle\langle A_1 A_2 \rangle\rangle = \langle A_1 A_2 \rangle - \langle A_1 \rangle \langle A_2 \rangle$ and $\langle\langle A_1 A_2 A_3 \rangle\rangle = \langle A_1 A_2 A_3 \rangle - \langle A_1 \rangle \langle A_2 \rangle \langle A_3 \rangle - \langle A_1 \rangle \langle\langle A_2 A_3 \rangle\rangle - \langle A_2 \rangle \langle\langle A_1 A_3 \rangle\rangle - \langle A_3 \rangle \langle\langle A_1 A_2 \rangle\rangle$.

Note that the result (80) for $\langle F^\dagger(t)F \rangle$ does not involve a contribution of the first order in λ . One can show that this contribution $\lambda \langle FF_1(t) \rangle$ is proportional to the correlation $\langle FG_1(t, t_1) \rangle$, which vanishes for the homogeneous bath. Note also that expressions (80) and (81) are the same as the corresponding results for the nonrelativistic theory [19], except that the bath dynamics propagator L_0 now is of the quasirelativistic form (34). Let us stress that functions $C_i(t)$ do not depend on P_* , so the expression (80) presents the explicit dependence of the kernel $\langle FF^\dagger(t) \rangle$ on P_* to order λ^2 . Substitution of (80) into the pre-Langevin equation (65) and retaining terms up to order λ^4 (neglecting terms of order $\lambda^4 \delta^2$) produces the generalized (non-Markovian) nonlinear Langevin equation

$$\begin{aligned} \frac{d}{dt} P_*(t) &= \lambda F^\dagger(t) - \lambda^2 \int_0^t d\tau M_1(\tau) P_*(t-\tau) \\ &\quad - \lambda^4 \int_0^t d\tau M_2(\tau) P_*^3(t-\tau), \end{aligned} \quad (83)$$

with the memory kernels

$$\begin{aligned} M_1(t) &= \frac{1}{p_T^2} C_0(t) - \frac{2\lambda^2}{m^2} C_1(t) + \frac{\lambda^2}{mp_T^2} C_2(t), \\ M_2(t) &= \frac{1}{m^2 p_T^2} C_1(t) - \frac{\epsilon^2}{2p_T^4} C_0(t), \end{aligned} \quad (84)$$

where correlations $C_i(t)$ are given by (75) and (81).

As discussed above, there is no reason to believe that the quasirelativistic approach is satisfactory for any systems but with short-range binary collisions. In such cases memory effects are negligible and one can apply the Markovian ansatz

$$M_i(t) \rightarrow \delta(t)\alpha_i, \quad \alpha_i = \int_0^\infty M_i(t) dt. \quad (85)$$

This puts the above generalized Langevin equation into the local form

$$\frac{d}{dt} P_*(t) = \lambda F^\dagger(t) - \lambda^2 \alpha_1 P_*(t) - \lambda^4 \alpha_2 P_*^3(t), \quad (86)$$

with the damping coefficients

$$\alpha_1 = \frac{1}{p_T^2} D_0 - \frac{2\lambda^2}{m^2} D_1 + \frac{\lambda^2}{mp_T^2} D_2, \quad (87)$$

$$\alpha_2 = \frac{1}{m^2 p_T^2} D_1 - \frac{\epsilon^2}{2p_T^4} D_0, \quad (88)$$

where $p_T = \sqrt{m/\beta}$ and

$$D_i = \int_0^\infty C_i(t) dt, \quad i = 0, 1, 2. \quad (89)$$

Compared to the λ^2 -order Langevin equation (76), two new features appear in Eq. (86) of order λ^4 . First, as one can see from (87), there are λ^2 -order corrections to the linear damping coefficient $\alpha_0 = p_T^{-2} D_0$. These corrections do not involve the relativistic parameter ϵ and therefore are purely classical. Second, and more interesting, a nonlinear dissipation term emerges with the damping coefficient α_2 given by (88). The first term on the right-hand side of Eq. (88) is classical and the second one is relativistic.

Note that the nonlinear classical and relativistic contributions in Eq. (86) are of order λ^4 and $\lambda^4 \epsilon^2$, respectively. Therefore, this equation is perturbatively consistent in general only for regime *B* when $\epsilon \sim 1$. For regime *A* ($\epsilon \ll 1$) the λ^2 -order approximation (70) for $F^\dagger(t)$ may be insufficient. For instance, if $\epsilon \sim \lambda$ then relativistic nonlinear corrections are of order λ^6 . This would require the expansion of $F^\dagger(t)$ up to order λ^4 and dealing with more complicated correlation functions.

Recall that Eq. (86) is for the scaled momentum $P_* = \lambda P$. The Langevin equation for the particle's true momentum P reads

$$\frac{d}{dt} P(t) = F^\dagger(t) - \gamma_1 P(t) - \gamma_2 P^3(t), \quad (90)$$

with damping coefficients

$$\begin{aligned} \gamma_1 &= \lambda^2 \alpha_1 = \frac{\beta}{M} D_0 - \frac{2}{M^2} D_1 + \frac{\beta}{M^2} D_2, \\ \gamma_2 &= \lambda^6 \alpha_2 = \frac{\beta}{M^3} D_1 - \frac{\beta}{2M^3 c^2} D_0. \end{aligned} \quad (91)$$

Comparing these results with phenomenological fluctuation-dissipation relations (13), one observes that the latter are recovered if D_0 is identified as the total noise strength D , while D_1 and D_2 both vanish or negligible,

$$D_0 \rightarrow D, \quad D_1 \rightarrow 0, \quad D_2 \rightarrow 0. \quad (92)$$

Needless to say, neither of these conditions is satisfied in general.

A qualitatively new feature is the presence of the new term involving $D_1 = \int_0^\infty C_1(t) dt$ in the expression for the nonlinear damping coefficient γ_2 . As a result, the sign of γ_2 is not necessarily negative, as in the phenomenological theory, but depends on relative values of D_0 and D_1 and therefore on temperature. For a classical model it was found

that $D_1/D_0 = m\beta/6$ [19]. Using this as a rough estimation, one would get from (91) or (88) the expression

$$\gamma_2 = \frac{D_0}{2} \left(\frac{\beta}{m} \right)^2 \left(\frac{1}{3} - \epsilon^2 \right), \quad (93)$$

which is positive for regime A, $\epsilon \ll 1$, and also for a subdomain $\epsilon < 1/\sqrt{3}$ of regime B.

VI. LANGEVIN EQUATION: REGIME C

One can show that the Langevin equation and fluctuation-dissipation relations derived in the previous section retain their forms for regime C also. The relevant small parameter now is $\delta = \lambda\epsilon$, the Liouville operator is given by (53), $L = L_0 + \delta L_1 + \delta^3 L_2$, and the pre-Langevin equation has the form (67) with $p_T = 1/c\beta$. Otherwise the derivation is similar to that for regimes A and B.

Substitution of the lowest-order approximation for the projected force $F^\dagger(t) \approx F_0(t)$ into the pre-Langevin equation of motion (67) yields, in the Markovian limit, the linear Langevin equation and the fluctuation-dissipation relation, both in standard forms (78) and (79). As for regimes A and B, no relativistic effects show up in this lowest approximation except for the modified value of the damping parameter γ_0 .

The higher-order approximation corresponds to the expansion

$$F^\dagger(t) \approx F_0(t) + \delta F_1(t) + \delta^2 F_2(t), \quad (94)$$

with $F_i(t)$ still given by expressions (71)–(73), but now with the operator L_1 defined by (51). As we already noted, the results for regime C can be obtained from those for regimes A and B by making the substitutions in (54), $\lambda \rightarrow \delta$ and $m \rightarrow p_T/c$. In particular, for the correlation $\langle FF^\dagger(t) \rangle$, instead of (80) one obtains

$$\langle FF^\dagger(t) \rangle = C_0(t) + \delta^2 \left[\left(\frac{cP_*}{p_T} \right)^2 C_1(t) + \frac{c}{p_T} C_2(t) \right], \quad (95)$$

with the same functions $C_i(t)$. Substitution of this into the pre-Langevin equation (67) and taking the Markovian limit leads to the nonlinear Langevin equation for the scaled momentum

$$\frac{d}{dt} P_*(t) = \lambda F^\dagger(t) - \delta^2 \alpha_1 P_*(t) - \delta^4 \alpha_2 P_*^3(t), \quad (96)$$

with

$$\alpha_1 = \frac{1}{p_T^2} D_0 - 2 \left(\frac{c\delta}{p_T} \right)^2 D_1 + \frac{c\delta^2}{p_T^3} D_2, \quad (97)$$

$$\alpha_2 = \frac{c^2}{p_T^4} D_1 - \frac{1}{2p_T^4} D_0, \quad (98)$$

and $p_T = 1/c\beta$. Then, as is easy to check, the equation for the true momentum $P = \lambda^{-1} P_*$ has the same form (90) as for regime B with the same fluctuation-dissipation relations (91).

VII. MOMENTS AND THERMALIZATION

Although the nonlinear Langevin equation (90) cannot be integrated in an analytical form, the presented method is convenient to describe relaxation processes perturbatively. As

the first example consider the relaxation of the first moment $\langle P(t) \rangle$ for, say, regime B. From (86) one gets

$$\frac{d}{dt} \langle P_*(t) \rangle = -\lambda^2 \alpha_1 \langle P_*(t) \rangle - \lambda^4 \alpha_2 \langle P_*^3(t) \rangle. \quad (99)$$

Since the third moment $\langle P_*^3(t) \rangle$ enters this equation multiplied by λ^4 , it is sufficient to describe its dynamics in the lowest order in λ ,

$$\frac{d}{dt} \langle P_*^3(t) \rangle = -\lambda^2 \alpha_3 \langle P_*^3(t) \rangle + \lambda^2 \alpha_4 \langle P_*(t) \rangle, \quad (100)$$

where $\alpha_3 = 3D_0/p_T^2$ and $\alpha_4 = 6D_0$. In the phenomenological theory this equation is derived from the linear Langevin equation under the assumption of the Gaussian random force [20], but it also can be derived microscopically without this assumption [see Eq. (110) below]. The closed system (99) and (100) is perturbatively consistent and describes the relaxation of $\langle P(t) \rangle$ with nonexponential corrections of order λ^4 .

In order to describe λ^4 -order dynamics of higher moments $\langle P_*^n(t) \rangle$, $n > 1$, without the assumption of Gaussian noise one needs the Langevin equations for powers $P_*^n(t)$. The derivation of these equations, first discussed for the nonrelativistic domain in [18] and recently in [21], is a straightforward generalization of the method described above. The equations for $\langle P_*^n(t) \rangle$ can be used, in particular, to prove the particle's thermalization towards the Maxwell-Jüttner distribution $\rho_{MJ}(P)$ [Eq. (7)] for which the equilibrium moments in one dimension are

$$\begin{aligned} \langle P^{2n} \rangle_{eq} &= \int_{-\infty}^{\infty} \rho_{MJ}(P) P^{2n} dP \\ &= (2n-1)!! \left(\frac{M}{\beta} \right)^n \frac{K_{1+n}(\delta^{-2})}{K_1(\delta^{-2})}, \end{aligned} \quad (101)$$

where $K_i(x)$ is the modified Bessel function of the second kind. To the leading order in δ^2 this expression reads

$$\langle P^{2n} \rangle_{eq} \approx (2n-1)!! \left(\frac{M}{\beta} \right)^n \left[1 + \left(n + \frac{n^2}{2} \right) \delta^2 \right]. \quad (102)$$

In what follows we derive the equations for the moments $\langle P^n(t) \rangle$ of a weakly relativistic particle ($\delta \ll 1$) and show explicitly that they converge to the equilibrium values (102).

We shall assume that the temperature corresponds to regime B; the consideration for regimes A and C is similar. Starting with the exact equation of motion for the powers of the scaled momentum

$$\frac{d}{dt} P_*^n(t) = e^{Lt} L P_*^n \quad (103)$$

and using the operator identity (56) for the propagator e^{Lt} , one gets

$$\frac{d}{dt} P_*^n(t) = \lambda R(t) + \lambda \int_0^t e^{L(t-\tau)} \mathcal{P} L R(\tau), \quad (104)$$

with the zero-centered projected force

$$R(t) = \lambda^{-1} e^{QLt} L P_*^n = n e^{QLt} F P_*^{n-1}. \quad (105)$$

Proceeding with steps similar to those in Sec. IV, one obtains the pre-Langevin equation

$$\begin{aligned} \frac{d}{dt} P_*^n(t) &= \lambda R(t) + \lambda^2 \int_0^t d\tau e^{L(t-\tau)} \\ &\times \left\{ -\frac{1}{p_T^2} P_* + \frac{\partial}{\partial P_*} + \delta^2 \frac{1}{2p_T^4} P_*^3 \right\} \langle FR(t) \rangle. \end{aligned} \quad (106)$$

To the lowest order in λ the nonlinear term in this equation should be omitted and the projected force is

$$R(t) \approx n F_0(t) P_*^{n-1} \equiv R_0(t), \quad (107)$$

where, recall, $F_0(t) = e^{L_0 t} F$. Substitution of this approximation into the above pre-Langevin equation and taking the Markovian limit yields the Langevin equations for P_*^n

$$\frac{d}{dt} P_*^n(t) = \lambda R_0(t) + k_1 P_*^{n-2}(t) + k_2 P_*^n(t), \quad (108)$$

with coefficients

$$k_1 = \lambda^2 n(n-1) D_0, \quad k_2 = -\lambda^2 n D_0 p_T^{-2}, \quad (109)$$

where $D_0 = \int_0^\infty \langle F F_0(\tau) \rangle$ and $p_T = \sqrt{m/\beta}$. As expected, no relativistic corrections appear to the lowest perturbation order. The moments are governed by the equation

$$\frac{d}{dt} \langle P_*^n(t) \rangle = k_1 \langle P_*^{n-2}(t) \rangle + k_2 \langle P_*^n(t) \rangle \quad (110)$$

and relax in the long time limit to the equilibrium Maxwellian values. In particular

$$\langle P_*^2(t) \rangle \rightarrow p_T^2, \quad \langle P_*^4(t) \rangle \rightarrow 3p_T^4. \quad (111)$$

Consider now the expansion of the projected force to the second order in λ ,

$$R(t) \approx R_0(t) + \lambda R_1(t) + \lambda^2 R_2(t), \quad (112)$$

where $R_0(t)$ is given by (107) and

$$\begin{aligned} R_1(t) &= \int_0^t e^{L_0(t-\tau)} \mathcal{Q} L_1 R_0(\tau), \\ R_2(t) &= \int_0^t e^{L_0(t-\tau)} \mathcal{Q} L_1 R_1(\tau). \end{aligned} \quad (113)$$

We need to evaluate the explicit dependence on P_* of the correlation

$$\langle FR(t) \rangle = \langle FR_0(t) \rangle + \lambda \langle FR_1(t) \rangle + \lambda^2 \langle FR_2(t) \rangle \quad (114)$$

in the pre-Langevin equation (106). According to (107), the first term is of the form

$$\langle FR_0(t) \rangle = c_0(t) P_*^{n-1}. \quad (115)$$

The explicit evaluation shows that the second term vanishes identically $\langle FR_1(t) \rangle = 0$ due to symmetry and the third term can be written as

$$\begin{aligned} \langle FR_2(t) \rangle &= c_1(t) P_*^{n+1} + c_2(t) P_*^{n-1} \\ &\quad + (n-1)(n-2) c_3(t) P_*^{n-3}. \end{aligned} \quad (116)$$

In these expressions $c_0(t) = n \langle F F_0(t) \rangle$ and the other functions $c_i(t)$ are expressed in terms of more complicated correlation

functions and do not depend on P_* . Remarkably, as shown below, neither the explicit form of functions $c_i(t)$ nor their possible relations are needed to prove the convergence of the moments to the equilibrium values (101). [In the last term of Eq. (116) we extracted explicitly the factors $(n-1)(n-2)$ to make it clear that this term vanishes for the first and second moments.]

Substitution of (114)–(116) into (106), applying the Markovian limit and taking the average leads to the following equation for the moments to order λ^4 :

$$\frac{d \langle P_*^n \rangle}{dt} = r_1 \langle P_*^{n-2} \rangle + r_2 \langle P_*^n \rangle + r_3 \langle P_*^{n+2} \rangle + r_4 \langle P_*^{n-4} \rangle, \quad (117)$$

with coefficients

$$\begin{aligned} r_1 &= \lambda^2(n-1)b_0 + \lambda^4(n-1)b_2 - \lambda^4 p_T^{-2}(n-1)(n-2)b_3, \\ r_2 &= -\lambda^2 p_T^{-2} b_0 + \lambda^4(n+1)b_1 - \lambda^4 p_T^{-2} b_2, \\ r_3 &= \lambda^2 \delta^2 (2p_T)^{-4} b_0 - \lambda^4 p_T^{-2} b_1, \\ r_4 &= \lambda^4(n-3)(n-2)(n-1)b_3, \end{aligned} \quad (118)$$

where $b_i = \int_0^\infty c_i(t) dt$. Compared to the λ^2 -order Eqs. (110) and (117) shows that to order λ^4 , the moment $\langle P_*^n \rangle$ is coupled in general not only with $\langle P_*^{n-2} \rangle$, but also with $\langle P_*^{n+2} \rangle$ and $\langle P_*^{n-4} \rangle$. Note that in Eqs. (118) the only relativistic correction is the first term ($\sim \delta^2$) in the expression for r_3 .

Let us focus on the equation for the second moment

$$\frac{d \langle P_*^2(t) \rangle}{dt} = r_1 + r_2 \langle P_*^2(t) \rangle + r_3 \langle P_*^4(t) \rangle, \quad (119)$$

with

$$\begin{aligned} r_1 &= \lambda^2 b_0 + \lambda^4 b_2, \\ r_2 &= -\lambda^2 p_T^{-2} b_0 + 3\lambda^4 b_1 - \lambda^4 p_T^{-2} b_2, \\ r_3 &= \lambda^2 \delta^2 (2p_T)^{-4} b_0 - \lambda^4 p_T^{-2} b_1. \end{aligned} \quad (120)$$

Using Laplace transformations

$$A_n(s) = \int_0^\infty e^{-st} \langle P_*^n(t) \rangle dt, \quad (121)$$

the stationary value of the second moment can be written as

$$\begin{aligned} \lim_{t \rightarrow \infty} \langle P_*^2(t) \rangle &= \lim_{s \rightarrow 0} s A_2(s) \\ &= -\frac{1}{r_2} [r_1 + r_3 \lim_{s \rightarrow 0} s A_4(s)]. \end{aligned} \quad (122)$$

The stationary value for the fourth moment $\lim_{s \rightarrow 0} s A_4(s)$ appears here multiplied by $r_3 \sim \lambda^4$. Then one should assign to it the equilibrium value found above in the lowest perturbation order [Eq. (111)]

$$\langle P_*^4 \rangle_{eq} = \lim_{s \rightarrow 0} s A_4(s) = 3p_T^4. \quad (123)$$

Then, from (122) and (120) one obtains to order δ^2

$$\lim_{t \rightarrow \infty} \langle P_*^2(t) \rangle = p_T^2 \left(1 + \frac{3}{2} \delta^2 \right), \quad (124)$$

which is consistent with the prediction (102) of the equilibrium theory with the Maxwell-Jüttner distribution.

Thermalization of the moments of higher orders can be considered in a similar way. In particular, one can show that

for the fourth moment equation (117) leads to the asymptotic result

$$\lim_{t \rightarrow \infty} \langle P_*^4(t) \rangle = 3p_T^4(1 + 4\delta^2), \quad (125)$$

which is the correct δ^2 -order approximation for the equilibrium value $\langle P_*^4 \rangle_{eq}$ given by Eq. (102).

VIII. CONCLUSION

In this paper we argue that the conventional Langevin phenomenology, with a single fluctuation-dissipation relation, cannot be extended to the relativistic domain. For a non-relativistic Brownian particle the Langevin equation can be recovered from microscopic dynamics in the weak-coupling limit, i.e., in the leading order in λ . For a relativistic Brownian particle such a procedure is inconsistent because nonlinear relativistic corrections are of the same order of magnitude (or even smaller, for regime A) as classical corrections to the weak-coupling approximation. We believe that this conclusion is to be valid in general, even though the presented theory employs the quasirelativistic approximation.

The necessity to go beyond the weak-coupling limit leads to more than one and more complicated fluctuation-dissipation relations (91). One interesting consequence is that the damping coefficient γ_2 of the nonlinear dissipation term in the Langevin equation (90) may change its sign with temperature. This may lead to qualitatively different relaxation behavior for different temperature intervals. For example, consider the ensemble of Brownian particles for which the initial first moment $\langle P(0) \rangle$ is zero, but the third moment $\langle P^3(0) \rangle$ is not. Then it can be shown [22] that for $t > 0$ the average momentum of the ensemble is temporarily nonzero and its direction is determined by the sign of the dissipation coefficient γ_2 .

In contrast to phenomenological models, the presented approach does not assume that the fluctuating force in the Langevin equation is a Gaussian process. Fluctuation-dissipation relations involve cumulants of orders higher than 2, which in general do not vanish. With a non-Gaussian noise

many conventional methods of the phenomenological theory, for instance, the evaluation of higher moments, cannot be applied. Yet the perturbational approach developed in this paper provides a systematic method to solve the equations of stochastic dynamics analytically to any given order of a relevant small parameter. As an example, we discussed in Sec. VII the thermalization problem and showed that the moments $\langle P^n(t) \rangle$ relax towards the equilibrium values prescribed by the Maxwell-Jüttner distribution, provided this distribution holds for particles of the bath. Previously, the validity of the Maxwell-Jüttner distribution was questioned in a number of papers [11,23–25], but was supported by numerical simulations [16].

The presented procedure can also be applied to derive the Fokker-Planck equation for the distribution function $f(P, t)$. As is well known [13,14], beyond the weak-coupling limit this equation in general contains derivatives with respect to P of orders higher than 2 and therefore is not of the form (6) implied in phenomenological models with Gaussian noise.

The quasirelativistic approach adopted in this paper treats systems with finite-range interactions only approximately and contains no parameter that would describe qualitatively the validity of this approximation. This obliges one to restrict the application to systems with collisionlike interactions, which can be described in the Markovian limit. A systematic incorporation of non-Markovian effects requires a much more elaborate theory that would explicitly take into account the fields' degrees of freedom.

As a final comment let us note that while the presented theory provides explicit microscopic expressions for the damping coefficients γ_1 and γ_2 , it is not clear if there is a general relation between the two quantities. Remarkably, such a relation is not required to prove thermalization of the particle towards the Maxwell-Jüttner equilibrium distribution.

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