Probability distributions extremizing the nonadditive entropy S_δ and stationary states of the corresponding nonlinear Fokker-Planck equation

Mauricio S. Ribeiro,[1,*] Constantino Tsallis,[1,2,3,†] and Fernando D. Nobre[1,2,‡]
[1]*Centro Brasileiro de Pesquisas Físicas*
[2]*National Institute of Science and Technology for Complex Systems*
Rua Xavier Sigaud 150-Urca, Rio de Janeiro, Rio de Janeiro 22290-180, Brazil
[3]*Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, New Mexico 87501, USA*
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Under the assumption that the physically appropriate entropy of generic complex systems satisfies thermodynamic extensivity, we investigate the recently introduced entropy S_δ (which recovers the usual Boltzmann-Gibbs form for $\delta = 1$) and establish the microcanonical and canonical extremizing distributions. Using a generalized version of the H theorem, we find the nonlinear Fokker-Planck equation associated with that entropic functional and calculate the stationary-state probability distributions. We demonstrate that both approaches yield one and the same equation, which in turn uniquely determines the probability distribution. We show that the equilibrium distributions asymptotically behave like stretched exponentials, and that, in appropriate probability-energy variables, an interesting return occurs at $\delta = 4/3$. As a mathematically simple illustration, we consider the one-dimensional harmonic oscillator and calculate the generalized chemical potential for different values of δ.

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I. INTRODUCTION

Since the proposal of Boltzmann-Gibbs (BG) statistical mechanics, systems with long-range interactions are considered out of the scope of this theory due to a divergent canonical partition function (see, e.g., Refs. [1–4]). Later, other restrictions, many of them appearing within the class of the so-called complex systems, like long-time memory and/or strong correlations among its particles, were also realized to yield difficulties for the BG theory. Considering that statistical mechanics may be formulated by starting from a given statistical entropy [4,5], many entropic forms were introduced in the last decades, as attempts to generalize the standard BG formulation; among those, we may mention the entropy S_q [6], associated with nonextensive statistical mechanics [5,7], as well as the proposals by Curado [8], Anteneodo-Plastino [9], and Kaniadakis [10], the recent two-parameter $S_{q,\delta}$ entropy [11], as well as the Hanel-Thurner $S_{c,d}$ entropy [12], which recovers some of the above examples as particular limits. From these, the most investigated so far has been the entropy S_q, whose associated probability distributions, known as q exponentials and q Gaussians, have emerged in many natural systems [5,7].

For an appropriate matching with thermodynamics, a given complex system should be described by means of an extensive entropy. In order to know which entropy is suitable for a given system, it is important to know how the total number of microstates W, within a microcanonical-ensemble description, scales with N, the total number of constituents. Therefore, the usual connection between the microscopic and macroscopic worlds follows from an appropriate definition of its statistical entropy. In the case of weakly interacting constituents, $W(N)$ usually increases exponentially with N, so that the BG entropy represents the correct choice,

$$W(N) \sim a\xi^N;$$
$$S_{\rm BG}(N) = k_{\rm B} \ln W(N) \sim k_{\rm B} N \ln \xi \qquad (1)$$
$$(a > 0; \xi > 1; N \to \infty),$$

yielding a linear increase with N, thus satisfying thermodynamical extensivity. Now, if we consider a system characterized by strong correlations, $W(N)$ may behave like a power of N, namely,

$$W(N) \sim bN^\rho \quad (b > 0; \rho > 0; N \to \infty), \qquad (2)$$

in such a way that the BG entropy is not extensive. In this case, one verifies that the q entropy, for $q = 1 - 1/\rho$, satisfies the desired requirement of extensivity,

$$S_q(N) = k \ln_q W(N) \propto N \quad (N \to \infty), \qquad (3)$$

where $\ln_q x = (x^{1-q} - 1)/(1 - q)$ [5].

However, intermediate situations can occur in nature; e.g., a system may present correlations among its elements that are not strong enough to be characterized by Eq. (2), neither sufficiently weak to be described by Eq. (1). As an example, one has

$$W(N) \sim c\nu^{N^\zeta} \quad (c > 0; \nu > 1; 0 < \zeta < 1; N \to \infty). \qquad (4)$$

Recently, a new entropy was proposed [5,7] as a candidate for this class of systems, namely,

$$S_\delta = k \sum_{i=1}^{W} p_i \left(\ln \frac{1}{p_i} \right)^\delta \quad (\delta > 0). \qquad (5)$$

Considering $W(N)$ as in Eq. (4), one notices that $S_\delta(N)$ becomes

$$S_\delta(N) = k(\ln W)^\delta \sim k(\ln \nu)^\delta N^{\zeta\delta}, \qquad (6)$$

leading to an extensive entropy for the choice $\delta = 1/\zeta > 1$.

A well-known and rather curious situation occurs in the case of Schwartzschild black holes, where the BG entropy scales

*Corresponding author: ribeiro@cbpf.br
†tsallis@cbpf.br
‡fdnobre@cbpf.br

as L^2, i.e., proportional to its area, and not to its volume L^3 (herein, L denotes a characteristic linear length of the system). In a similar way, a wide class of strongly quantum-entangled d-dimensional systems follows the so-called area law [13], for which S_{BG} is proportional to L^{d-1}, instead of L^d ($d > 1$). These results violate the expected extensivity requirement of the thermodynamical entropy of a d-dimensional system. The entropy S_δ of Eq. (5) has emerged recently as appropriate for reconciling this difficulty [11]; indeed, $S_{\delta=d/(d-1)} \propto L^d$. Since in most cases the comparison of theoretical results with experimental and observational verifications are carried through probability distributions, the main motivation of the present study is to analyze the distributions associated with the entropic form S_δ.

In what follows we concentrate on its associated mesoscopic dynamics, usually defined, as we shall see, by a nonlinear Fokker-Planck equation (NLFPE). In the next section we find the class of NLFPEs associated with S_δ, from which we will define the simplest NLFPE to be investigated; we show that its stationary probability distribution follows a transcendental equation. In Sec. III, using the well-known Lagrangian multipliers method, we extremize the entropic form under microcanonical and canonical constraints, obtaining in this latter case an equation for the probability distribution which, interestingly enough, can be precisely identified with the one derived from the NLFPE approach. Then we solve the transcendental equation for some typical cases and show that the probability distributions asymptotically are stretched exponentials, thus opening the door to applications in physical systems where such distributions are frequently found. In Sec. IV, we consider the Einstein's model for a solid as a mathematical illustration, compute the Lagrange multiplier μ, called herein "generalized chemical potential," that recovers the usual form for $\delta \to 1$, and suggest an unified expression for its temperature dependence. As a second illustration, we also solve the transcendental equation for an external harmonic potential and compute the associated equilibrium (stationary state, generically speaking) probability distributions for typical values of δ. Finally, in Sec. V we present our conclusions.

II. ASSOCIATED NONLINEAR FOKKER-PLANCK EQUATION

The H theorem represents one of the most important results of nonequilibrium statistical mechanics, guaranteeing that a system will approach an equilibrium state, after a long-time evolution. The proof of the H theorem making use of NLFPEs was developed in many works in recent years [14–21]; in the case of a system in the presence of a confining external potential, $\phi(x)$, this theorem corresponds to a well-defined sign for the time derivative of the free-energy functional,

$$F[P]=U[P]-\gamma S[P]; \quad U[P]=\int dx\, \phi(x)P(x,t), \tag{7}$$

where $P(x,t)$ is the density distribution, $U[P]$ stands for the internal-energy functional, and γ represents a positive parameter with dimensions of temperature. One should notice that the energy is defined herein in the standard (linear) way, instead of in the q-expectation form [5]. The entropy may be considered as very general,

$$S[P] = k\int dx\, g[P(x,t)]; \qquad g(0) = g(1) = 0; \quad \frac{d^2 g}{dP^2} \leqslant 0, \tag{8}$$

where k is a positive constant with dimensions of entropy and the functional $g[P(x,t)]$ should be at least twice differentiable.

Considering a form for the NLFPE [20–23],

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial\{A(x)\Psi[P(x,t)]\}}{\partial x} + \frac{\partial}{\partial x}\left\{\Omega[P(x,t)]\frac{\partial P(x,t)}{\partial x}\right\}, \tag{9}$$

where $A(x) = -d\phi(x)/dx$ represents an external force, whereas $\Psi[P]$ and $\Omega[P]$ are positive functionals of $P(x,t)$, the H theorem is satisfied, provided that the following relation holds [20,21]:

$$\frac{d^2 g[P]}{dP^2} = -\frac{1}{k\gamma}\frac{\Omega[P]}{\Psi[P]}. \tag{10}$$

The equation above relates the functionals $\Omega[P]$ and $\Psi[P]$ of the NLFPE with $g[P]$ of the entropy, associating the entropy with a mesoscopic dynamics defined by Eq. (9).

Within this framework, we are concerned with the continuous form of the entropy S_δ,

$$S_\delta[P] = k\int P(x,t)\left[\ln\frac{1}{P(x,t)}\right]^\delta dx \quad (\delta > 0), \tag{11}$$

from which the inner functional is readily identified,

$$g[P(x,t)] = P(x,t)\left[\ln\frac{1}{P(x,t)}\right]^\delta. \tag{12}$$

Substituting the above expression into Eq. (10) and taking the derivative twice, one obtains the relation

$$\frac{\Omega[P]}{\Psi[P]} = \frac{k\gamma}{P}\left[\delta\left(\ln\frac{1}{P}\right)^{\delta-1} - \delta(\delta-1)\left(\ln\frac{1}{P}\right)^{\delta-2}\right]. \tag{13}$$

It is important to notice that the ratio between the functionals $\Omega[P]$ and $\Psi[P]$ defines a family of of NLFPEs associated with the same entropy [20]. The simplest possible choice corresponds to $\Psi[P] = P(x,t)$, in which case the functional $\Omega[P]$ follows immediately from Eq. (13). Now, defining the coefficient of the diffusion term as $D = k\gamma$ (in analogy to the linear case [24]) and substituting these functionals into Eq. (9), one obtains

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial\{A(x)P(x,t)\}}{\partial x} + D\delta\frac{\partial}{\partial x}\left\{\left(\left[\ln\frac{1}{P(x,t)}\right]^{\delta-1} - (\delta-1)\left[\ln\frac{1}{P(x,t)}\right]^{\delta-2}\right)\frac{\partial P(x,t)}{\partial x}\right\}, \tag{14}$$

which represents a NLFPE associated with the entropy S_δ. Although the above equation may seem unusual, one should notice that similar NLFPEs have been introduced in the literature recently, as appropriate for a description of several natural phenomena (see, e.g., Refs. [14,15,19]. On the right-hand side of Eq. (14), the second term corresponds to a nonlinear (anomalous) diffusion contribution, whereas the first one represents a standard drift term, responsible for driving the system towards its stationary state. Therefore, after a sufficiently long time, one expects to reach a stationary state at which the probability distribution $P_{st}(x)$ does not depend on time; setting $(\partial P/\partial t) = 0$ in Eq. (14), one obtains

$$\ln^{\delta}\left(\frac{1}{P_{st}}\right) - \delta \ln^{\delta-1}\left(\frac{1}{P_{st}}\right) - \frac{1}{D}[\phi(x) - \phi_0] = 0, \quad (15)$$

where ϕ_0 is a constant representing the zero-energy scale.

The solutions of the transcendental Eq. (15) represent the stationary probability distributions, which characterize the system under study. In principle, one cannot find an explicit solution for general values of δ; however, this may be done for some special values of δ, whereas, in other cases, numerical solutions can be computed. In the next section we show that Eq. (15) can also be obtained through the maximum-entropy principle and we solve it for typical values of δ.

III. MAXIMUM-ENTROPY PRINCIPLE AND EQUILIBRIUM DISTRIBUTIONS

For a system with an entropic functional $S[P]$ and internal energy $U[P]$, the equilibrium probability distributions P_{eq} may be obtained by maximizing S under certain constraints, using the Lagrangian multipliers method [4]. For that, the concavity requirement, $(d^2 S[P]/dP^2 < 0)$, guarantees that a maximum of the entropy corresponds to an equilibrium distribution.

Let us then consider the entropic form of Eq. (11); first, we work in the microcanonical ensemble, where $W(E)$ represents the volume of phase space occupied by a system with energy in the interval E and $E + \Delta$. Extremizing the dimensionless entropy $(S_\delta[P]/k)$ of Eq. (11) under the probability-normalization constraint, $\int P_{eq}(x)dx = 1$, one obtains

$$\ln^{\delta}\left(\frac{1}{P_{eq}}\right) - \delta \ln^{\delta-1}\left(\frac{1}{P_{eq}}\right) - \alpha = 0, \quad (16)$$

where α is the associated Lagrangian multiplier. It is straightforward to see that the solution of Eq. (16) corresponds to $P_{eq} =$ constant and that normalization mandates equal probabilities, $P_{eq} = 1/W$ for all values of δ, as expected for a microcanonical ensemble.

In the canonical ensemble, an additional Lagrangian multiplier β related with the mean-energy constraint, $U = \int P_{eq}(x)\phi(x)dx$, is introduced. Extremizing $(S_\delta[P]/k)$ of Eq. (11) under these two constraints, one finds

$$\ln^{\delta}\left(\frac{1}{P_{eq}}\right) - \delta \ln^{\delta-1}\left(\frac{1}{P_{eq}}\right) - \alpha - \beta\phi(x) = 0. \quad (17)$$

One notices that the equation above, obtained from entropy maximization, is identical to Eq. (15), calculated for the NLFPE stationary state, if one chooses appropriately the Lagrangian multipliers, i.e., $\alpha = -\phi_0/D$ and $\beta = D^{-1} = (k\gamma)^{-1}$. From now on, we consider Eq. (17) [or equivalently, Eq. (15)] as the transcendental equation for the equilibrium distribution $P_{eq}(x)$. For computational purposes, we write this equation as

$$\ln^{\delta}\left(\frac{1}{P_{eq}}\right) - \delta \ln^{\delta-1}\left(\frac{1}{P_{eq}}\right) = h(x); \quad [h(x) \equiv \alpha + \beta\phi(x)]. \quad (18)$$

Some important points concerning the equation above are discussed next. (i) By solving Eq. (18) one will obtain $P_{eq}[h(x)]$, which is not normalized in the h variable, $\int_{-\infty}^{\infty} P_{eq}(h)dh \neq 1$. Only by specifying a physical system, i.e., the form of the potential $\phi(x)$, is that one may calculate the probability distribution $P_{eq}(x) = P_{eq}[h(x)](dh/dx)$, such that $\int_{-\infty}^{\infty} P_{eq}(x)dx = 1$. (ii) In many situations the Lagrangian parameter α [that appears in $h(x)$] cannot be directly eliminated by imposing the normalization constraint; the normalization has to be done numerically in some cases. (iii) Although Eq. (18) does not present analytical solutions for general values of δ, one can solve it in the asymptotic limit $h \gg 1$. Since $\delta > 1$, one has in this limit

$$\ln^{\delta}\left(\frac{1}{P_{eq}}\right) \gg \delta \ln^{\delta-1}\left(\frac{1}{P_{eq}}\right) \Rightarrow \ln^{\delta}\left(\frac{1}{P_{eq}}\right) \approx h, \quad (19)$$

leading to a stretched exponential, $P_{eq}(h) \sim \exp(-h^{\frac{1}{\delta}})$ $(h \gg 1)$. This result opens the possibility of applications for the entropy S_δ, since stretched-exponential distributions are currently found in nature. (iv) Equation (18) can be solved analytically in some particular limits. By defining the variable $y = \ln^{\delta-1}(1/P_{eq})$, one can solve the cases $\delta = \{1,(4/3),(3/2),2,3,4\}$; some of these solutions are discussed below.

(i) $\delta = 1$. Considering $\delta = 1$ in Eq. (17), one finds the Boltzmann weight as expected,

$$P_{eq}(x) = \exp[-(1+\alpha) - \beta\phi(x)] = \frac{\exp[-\beta\phi(x)]}{\int dx \exp[-\beta\phi(x)]}, \quad (20)$$

where, as usual, the Lagrangian parameter α was chosen by imposing probability normalization.

(ii) $\delta = 2$. In this case, Eq. (18) becomes a quadratic equation,

$$\ln^{2}\left(\frac{1}{P_{eq}}\right) - 2\ln\left(\frac{1}{P_{eq}}\right) = h, \quad (21)$$

that presents the following solution:

$$P_{eq}(h) = \exp\{-[1 + (1+h)^{\frac{1}{2}}]\}. \quad (22)$$

This illustrates a peculiarity which occurs indeed for all $\delta > 1$, namely the fact that no function of the Lagrangian parameter α [that appears in $h(x)$] can be factorized by imposing the normalization constraint. This is precisely what happens, as well known, with the chemical potential in Fermi-Dirac and Bose-Einstein statistics. In the present case, such a difficulty comes as a direct consequence of the nonlinear aspect of the distribution $P_{eq}(h)$ above, and as usually happens in nonlinear equations, one has generically to make use of numerical procedures. This restriction is often found in generalized-entropy formalisms (see, e.g., Ref. [25]), where one needs to compute numerically the value of α in order to normalize

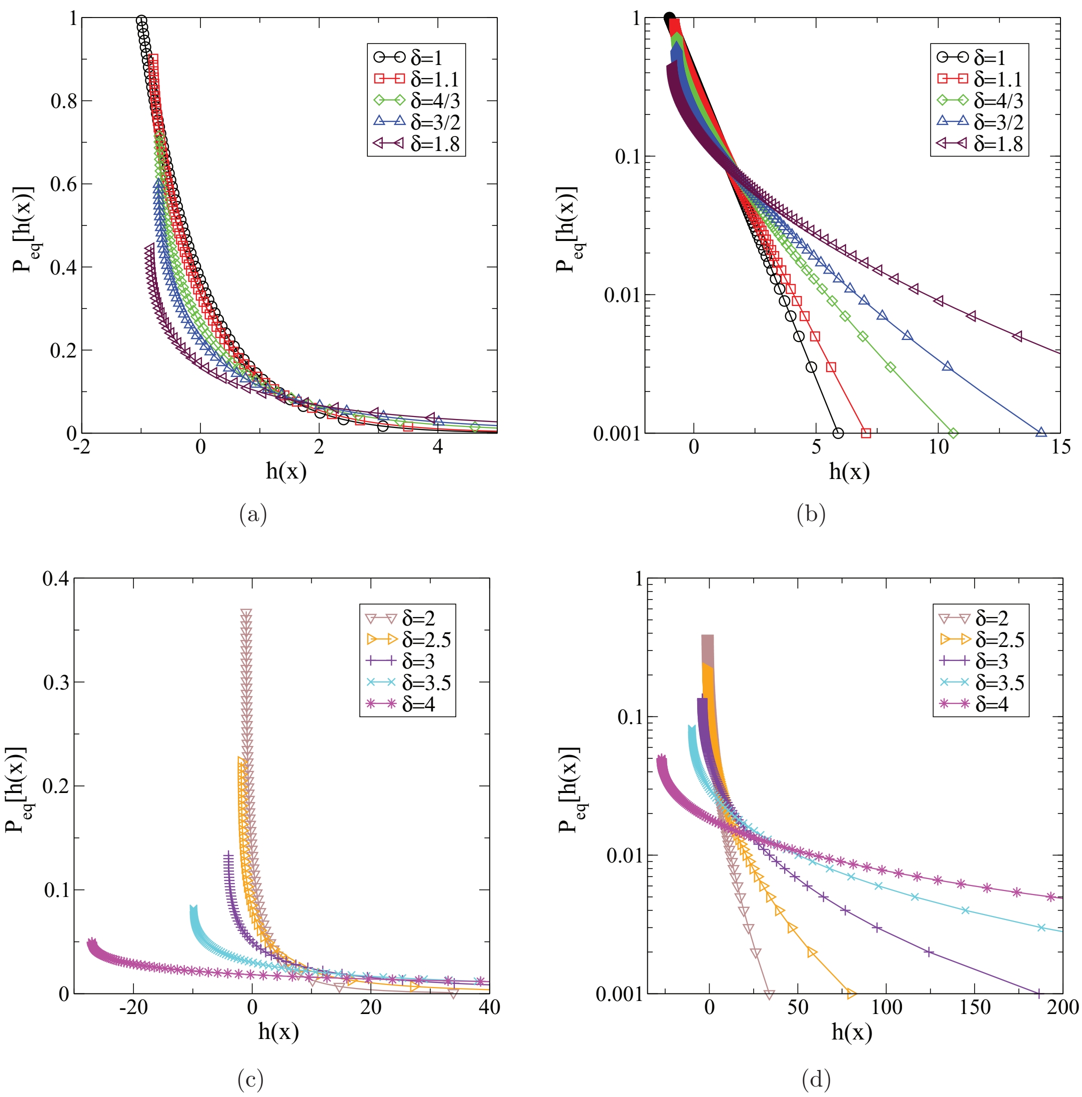


FIG. 1. (Color online) Equilibrium distributions $P_{\text{eq}}[h(x)]$ versus $h(x)$, as computed from Eq. (18), are presented in the linear [panels (a) and (c)] and log-linear [panels (b) and (d)] representations, for typical values of δ in the range $1 \leqslant \delta \leqslant 4$. The full lines are guides for the eye.

the distribution. As expected, in the asymptotic limit ($h \gg 1$), one has a stretched exponential $P_{\text{eq}}(h) \sim \exp(-h^{\frac{1}{2}})$.

(iii) $\delta = 3$. The case $\delta = 3$ yields the cubic equation

$$\ln^3\left(\frac{1}{P_{\text{eq}}}\right) - 3\ln^2\left(\frac{1}{P_{\text{eq}}}\right) = h, \tag{23}$$

from which just one of its roots presents the expected requirements for a probability distribution. This solution is given by

$$P_{\text{eq}}(h) = \exp\left\{-\left[1 + \left(\frac{2}{2+h+\sqrt{h^2+4h}}\right)^{1/3} + \left(\frac{2+h+\sqrt{h^2+4h}}{2}\right)^{1/3}\right]\right\}, \tag{24}$$

yielding in the asymptotic limit ($h \gg 1$) the stretched exponential, $P_{\text{eq}}(h) \sim \exp(-h^{\frac{1}{3}})$.

In Fig. 1 we present probability distributions $P_{\text{eq}}(h)$, obtained from Eq. (18), in both linear [panels (a) and (c)] and log-linear [panels (b) and (d)] representations, for typical values of δ in the range $1 \leqslant \delta \leqslant 4$. In the cases $\delta = \{1,(4/3),(3/2),2,3,4\}$ the distributions were calculated analytically, whereas those for $\delta = \{1.1,1.8,2.5,3.5\}$ were computed numerically. In the log-linear plots, the curves approach straight lines as $\delta \to 1$, up to the exponential limit ($\delta = 1$), where the distribution becomes effectively a straight line in this representation. Additionally, in the log-linear plots one notices that the significant weight of the tails appears clearly for increasing values of δ, whereas the asymptotic stretched exponential predicted in Eq. (19) is also suggested; strong numerical evidence of this latter behavior is provided below in Fig. 3.

Except for the case $\delta = 1$, one sees cutoffs h^* in the curves presented in Fig. 1. These cutoffs, which appear always for $h(x) < 0$, represent lower-limit points up to which the distributions $P_{\text{eq}}(h)$ remain real and positive. From Figs. 1(a) and 1(c) one notices that $h^* \to -\infty$, whereas $P_{\text{eq}}(h^*) \to 0$ as $\delta \to \infty$. Moreover, by increasing δ a rapid decay of $P_{\text{eq}}(h)$ for $h \gtrsim h^*$ is verified, followed by significant weights in the tails. The coordinates of these cutoffs are plotted in Fig. 2, where we represent $P_{\text{eq}}(h^*)$ versus $-h^*$. One notices a curious return for

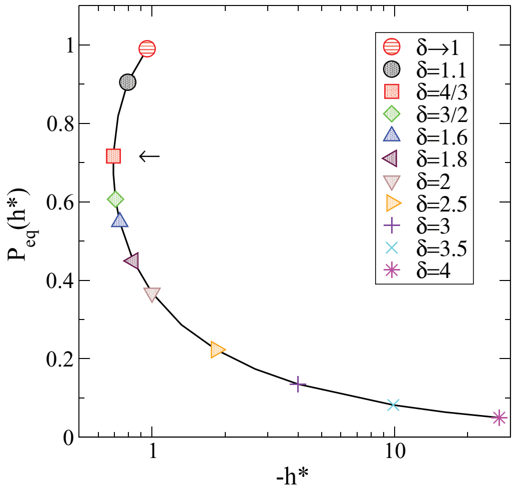
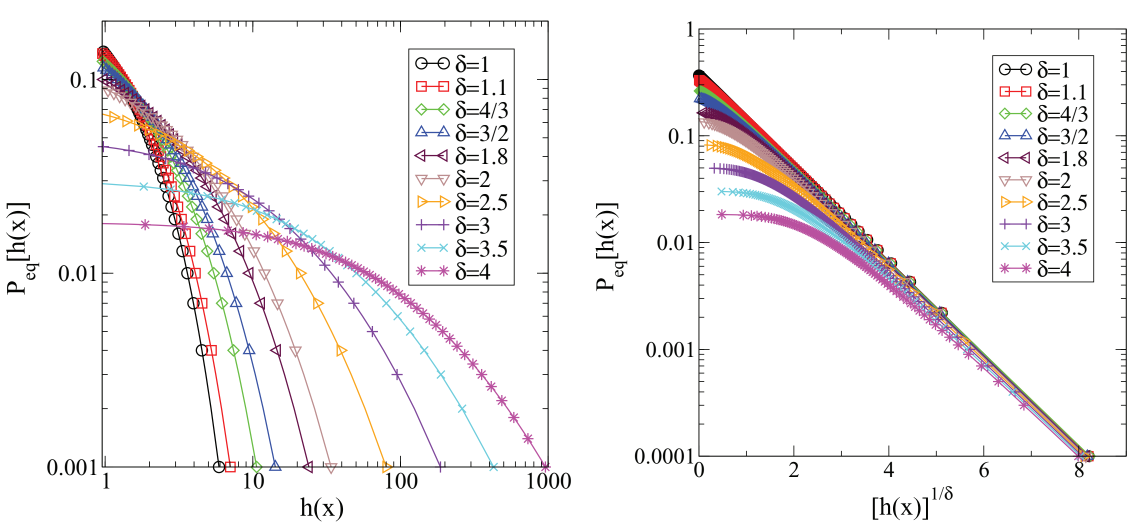


FIG. 2. (Color online) Coordinates of the lower-limit points of the distributions $P_{\mathrm{eq}}(h)$ presented in Fig. 1 are exhibited as $P_{\mathrm{eq}}(h^*)$ versus $-h^*$, for typical values of δ. One notices an interesting return occurring for $\delta = 4/3$ (see arrow). At the BG limit ($\delta \to 1$), we verify $[P_{\mathrm{eq}}(h^*),h^*] = (1,-1)$. The full line is a guide for the eye.

$\delta = 4/3$; if one considers the identification done in Ref. [11] (related to the area law in strongly quantum-entangled d-dimensional systems), namely $\delta = d/(d-1)$, this return point occurs for dimension $d = 4$. According to this identification, the BG limit ($\delta \to 1$) occurs for $d \to \infty$; this is analogous to what happens in conformal quantum-field theory, where the central charge $c \to \infty$ corresponds to the BG limit [26]. Notice also that, by indefinitely increasing δ, one approaches one-dimensional quantum-entangled systems. Since h^* represents a lower limit for the variable $h(x) = \alpha + \beta\phi(x)$ up to which the distribution $P_{\mathrm{eq}}(h)$ is defined, the plot of Fig. 2 shows that the Lagrange parameters α and β should be defined together with the external potential $\phi(x)$ to yield appropriate probability values of $P_{\mathrm{eq}}(h)$ for any given value of δ. Since $0 \leqslant P_{\mathrm{eq}}(h) \leqslant 1$ and $h > h^*$ define the allowed region for the probabilities, the line shown in Fig. 2 separates the physically accessible region (to the left of the curve) from the physically inaccessible one (to the right of the curve). One sees that the cutoffs h^* tend to disappear for $d \to 1$, which corresponds to the dimension where S_{BG} changes its scaling from L^{d-1} to $\ln L$. However, in the interval $1 \leqslant \delta \leqslant 2$, i.e., $2 \leqslant d < \infty$, there are two different values of δ with the same cutoff h^* (which varies slightly around $h^* = -1$), presenting an effect very similar to a reentrance found in some phase diagrams, e.g., in disordered magnetic systems. Within this interval, only at the turn point $\delta = 4/3$ ($d = 4$) does that one have a single value for h^*; this curious effect may hide some elusive physical property associated with $d = 4$.

The approximation carried in Eq. (19) applies for $h \gg 1$, but fails for $h \approx 0$, or $h < 0$. In Fig. 3 we reinforce the validity of the assumption considered in Eq. (19), by presenting numerical evidence that the stretched-exponential behavior always occurs for any $\delta > 1$. In Fig. 3(a) we present log-log plots of the distributions $P_{\mathrm{eq}}(h)$ versus h, particularly exhibiting the tails of the distributions. One sees that the distribution decay becomes weaker, by increasing δ; i.e., one gets fat tails in this limit. In Fig. 3(b) we represent $P_{\mathrm{eq}}(h)$ versus $[h(x)]^{1/\delta}$, where the stretched exponentials should appear as straight lines; one verifies such a behavior in the large h limit, as expected. This result suggests that the entropy S_δ might be particularly relevant, since stretched-exponential distributions have been observed in many natural phenomena (see, e.g., Ref. [27]), such as luminescence decays [28], anomalous diffusion associated with fractional-diffusion equations [29], and turbulent flows [30], among others. Moreover, it has also been associated with the entropy of Anteneodo and Plastino [9] and found within the context of superstatistics [31].

In the next section we study in further detail the Lagrangian parameter α associated with the normalization of the distribution, by relating it to a chemical potential μ, similar to what is done in BG statistical mechanics.



(a) (b)

FIG. 3. (Color online) Probability distributions are shown for different entropic indexes δ, emphasizing the large h limit. (a) Log-log plots of the distributions $P_{\mathrm{eq}}(h)$ versus h. (b) Probability distributions exhibited in the representation $P_{\mathrm{eq}}(h)$ versus $[h(x)]^{1/\delta}$, particularly useful for identifying the asymptotic stretched-exponential behavior. The full lines are guides for the eye.

IV. NORMALIZATION AND CHEMICAL POTENTIAL ANALYSIS

In the previous section we studied equilibrium probability distributions $P_{\rm eq}(h)$ for typical values of $\delta > 1$, by introducing a variable $h(x) = \alpha + \beta\phi(x)$, related to the two Lagrangian parameters α and β, as well as to the external confining potential $\phi(x)$ acting on the system [cf. Eq. (18)]. Due to its generality, the distributions presented were not normalized; for that, one needs to specify the form of $\phi(x)$, in such a way that the normalization comes from a particular choice of the Lagrangian parameter α.

In this section we work with normalized probability distributions by introducing a quantity μ related to the parameter α. Moreover, we restrict the present analysis to two cases for which the equilibrium distribution is solvable analytically from Eq. (18), namely, $\delta = 2$ and $\delta = 3$. As shown next, the quantity μ plays a role similar to the chemical potential of BG statistical mechanics; for this purpose, we postulate the relation

$$(\delta - 1) + \alpha = -\beta\mu, \tag{25}$$

which recovers the standard definition in the case $\delta = 1$.

First, we consider Einstein's model for a solid, described in terms of noninteracting one-dimensional harmonic oscillators, each with energy spectrum $E_i = n_i\hbar\omega$, where $\{n_i\}$ represent the usual quantum numbers, and we have subtracted off the zero-point energy. This interesting mathematical exercise provides some insight on $\mu(\beta)$ for values of $\delta \neq 1$. Besides that, we are motivated by previous investigations on systems of noninteracting particles using the q entropy, which by adjusting a value $q \neq 1$ were able to describe complex behavior of long-range-interacting physical systems in agreement with experiments, e.g., those found in the study of manganites [32].

In a second example we use an external harmonic confining potential, which is commonly considered in the study of probability distributions [24]. In some cases, one can approach a given complex system in the presence of a confining harmonic potential $\phi(x)$ by means of a coarse-graining approximation on its equations of motion, leading to a NLFPE [like the one in Eq. (14)]. In this equation, the contribution due to the interactions among particles result in the anomalous-diffusion term, particularly in its diffusion coefficient D [33,34]; therefore, the harmonic potential $\phi(x)$ that appears in Eq. (15) plays a crucial role in the associated probability distributions.

Let us then consider the first example, namely, the energy spectrum $E_i = n_i\hbar\omega$. For that, one needs to extremize (S_δ/k) in its discrete form [cf. Eq. (5)]; considering the constraints of normalization, $\sum_{i=1}^{W} p_i = 1$, and mean energy, $\bar{E} = \sum_{i=1}^{W} p_i E_i$, one obtains

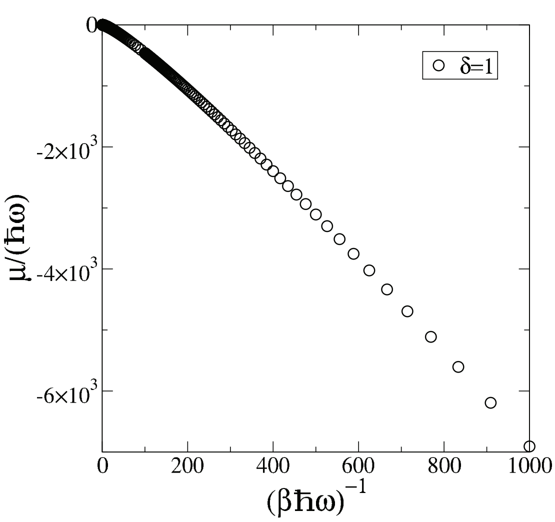
$$\ln^{\delta}\left(\frac{1}{p_i}\right) - \delta\ln^{\delta-1}\left(\frac{1}{p_i}\right) - \alpha - \beta E_i = 0, \tag{26}$$

which coincides with Eqs. (17) and (18) if one replaces $p_i \to P_{\rm eq}$ and $E_i \to \phi(x)$. Therefore, solving analytically Eq. (26) one obtains discrete sets of probabilities $\{p_i\}$ presenting solutions similar to those of Sec. III, which in the limit of sufficiently large sets should approach a continuous probability distribution. Then one calculates numerically the value of μ such as to normalize these distributions for a given value of β. It is important to mention that the normalization procedure depends strongly on the value of β. As β gets smaller the associated distribution exhibits heavier tails, so that a larger number of points is necessary to ensure probability normalization; we have generated sets $\{p_i\}$ from Eq. (26), with typically $(300/\beta)$ probabilities for each value of β.

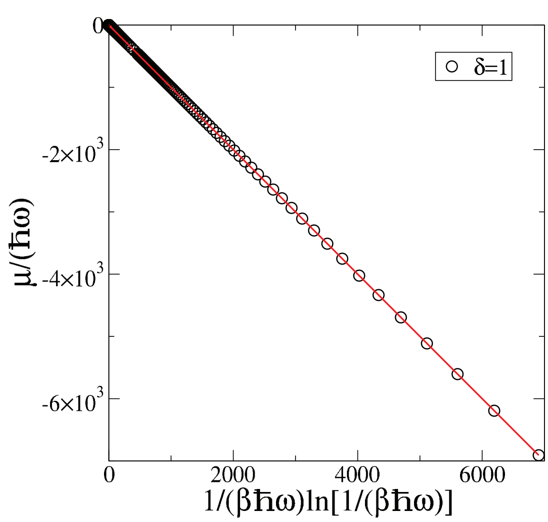
For completeness, we considered the case $\delta = 1$ as well, for which the analytical result is known exactly [35],

$$\begin{aligned}\mu &= (1/\beta)\ln\{1 - \exp[-(\beta\hbar\omega)]\}\\ &\propto -(1/\beta)\ln\{1/(\beta\hbar\omega)\} \quad (\beta \to 0),\end{aligned} \tag{27}$$

and we have written above the high-temperature limit ($\beta \to 0$) of the chemical potential explicitly. This chemical potential is presented in Fig. 4 versus $(\beta\hbar\omega)^{-1}$ [Fig. 4(a)] and



(a)



(b)

FIG. 4. (Color online) The chemical potential μ is represented versus convenient variables in the case $\delta = 1$. (a) Dimensionless quantity $\mu/(\hbar\omega)$ versus $(\beta\hbar\omega)^{-1}$; (b) dimensionless quantity $\mu/(\hbar\omega)$ versus $[1/(\beta\hbar\omega)]\ln[1/(\beta\hbar\omega)]$. The red curve is a linear regression, with $\mu/(\hbar\omega) = [1/(\beta\hbar\omega)]\ln[1/(\beta\hbar\omega)] - 0.49$.

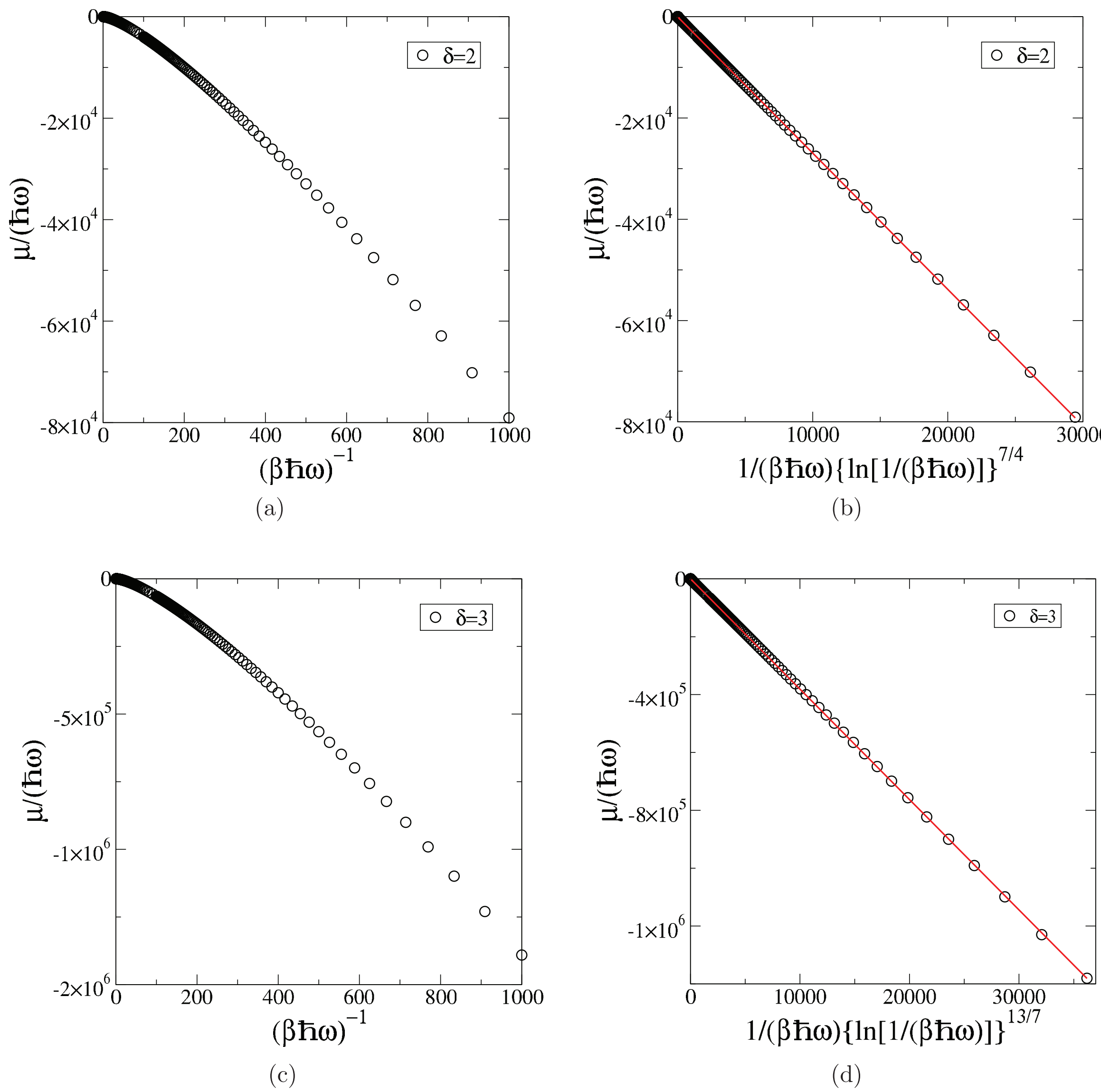


FIG. 5. (Color online) The chemical potential μ is exhibited versus convenient variables for the cases $\delta = 2$ [panels (a) and (b)] and $\delta = 3$ [panels (c) and (d)]. (a) Dimensionless quantity $\mu/(\hbar\omega)$ versus $(\beta\hbar\omega)^{-1}$; (b) dimensionless quantity $\mu/(\hbar\omega)$ versus $[1/(\beta\hbar\omega)]\{\ln[1/(\beta\hbar\omega)]\}^{7/4}$. (c) Dimensionless quantity $\mu/(\hbar\omega)$ versus $(\beta\hbar\omega)^{-1}$; (d) dimensionless quantity $\mu/(\hbar\omega)$ versus $[1/(\beta\hbar\omega)]\{\ln[1/(\beta\hbar\omega)]\}^{13/7}$. The red lines are linear regressions.

$[1/(\beta\hbar\omega)]\ln[1/(\beta\hbar\omega)]$ [Fig. 4(b)]. In the latter case the straight-line behavior confirms the high-temperature behavior of Eq. (27).

We have carried similar procedures for the variable μ defined according to Eq. (25), in the cases $\delta = 2$ and $\delta = 3$, for which the equilibrium distributions were calculated exactly for a general $\phi(x)$ in the previous section [cf. Eqs. (22) and (24), respectively]. Then, imposing normalization of the distributions for each value of β, one obtains $\mu(\beta)$ for $\delta = 2$ and $\delta = 3$. The corresponding results are exhibited in Fig. 5, where in panels (a) and (c) one sees that the plots $\mu/(\hbar\omega)$ versus $(\beta\hbar\omega)^{-1}$ show a qualitative behavior similar to the case $\delta = 1$ [cf. Fig. 4(a)]. Actually, in Figs. 5(b) and 5(d) one verifies that $\mu/(\hbar\omega)$ scale as powers of $\ln[1/(\beta\hbar\omega)]$, where red lines represent linear regressions leading to

$$\mu/(\hbar\omega) \propto (\beta\hbar\omega)^{-1}\{\ln[1/(\beta\hbar\omega)]\}^{7/4} \quad (\delta = 2), \tag{28}$$

$$\mu/(\hbar\omega) \propto (\beta\hbar\omega)^{-1}\{\ln[1/(\beta\hbar\omega)]\}^{13/7} \quad (\delta = 3). \tag{29}$$

From Eqs. (27)–(29) we propose the following generalization for the chemical potential μ,

$$\mu = a(1/\beta)(\ln\{1 - \exp[-(\beta\hbar\omega)]\})^{b} + c, \tag{30}$$

where a, b, and c are all δ dependent. However, according to Figs. 5(b) and 5(d), in the limit $\beta \to 0$, one should have

$$\mu \propto -(1/\beta)(\ln\{1/(\beta\hbar\omega)\})^{b} \quad (\beta \to 0), \tag{31}$$

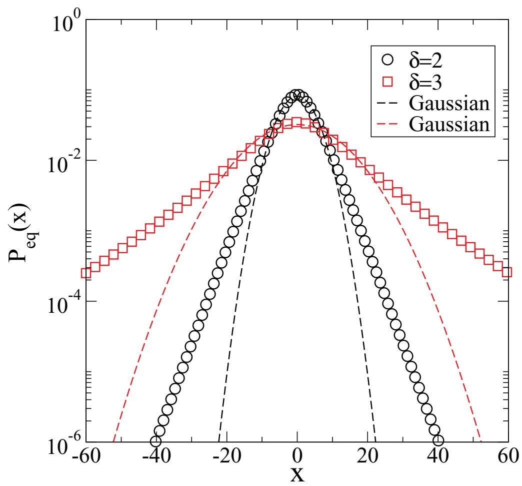


FIG. 6. (Color online) Normalized probability distributions $P_{eq}(x)$, considering a harmonic potential $\phi(x) = -x^2/2$, for $\delta = 2$ [from Eq. (22)] and $\delta = 3$ [from Eq. (24)]. In each case, a Gaussian regression is exhibited, by adjusting the central region, for comparison.

where we conjecture $b(\delta) = (6\delta - 5)/(3\delta - 2)$. These results show that μ plays a role very similar to the chemical potential in BG statistical mechanics.

Another illustrative example concerns a complex system subjected to an external harmonic potential, $\phi(x) = x^2$. Then, substituting the definition of Eq. (25) in $h(x)$ as introduced in Eq. (18), one obtains

$$\begin{aligned} h(x) &= \beta[\phi(x) - \mu] - (\delta - 1) \\ &= \beta(x^2 - \mu) - (\delta - 1). \end{aligned} \quad (32)$$

In the following analysis we use the expression above in Eqs. (22) and (24) in order to obtain normalized distributions for the cases $\delta = 2$ and $\delta = 3$, respectively. In the first case one has

$$P_{eq}(x) = \exp(-\{1 + [\beta(x^2 - \mu)]^{\frac{1}{2}}\}). \quad (33)$$

In Fig. 6 we present the distribution of Eq. (33), called herein a δ Gaussian (characterized by an exponent $1/\delta = 1/2$), where we have used $\mu = -21.25$ and $\beta = 0.1$. Additionally, the $\delta = 3$ distribution of Eq. (24) is also exhibited and normalized by using the parameters $\mu = -41.65$ and $\beta = 0.1$. In each case, Gaussian fits are presented for comparison, showing a good agreement in the central region, but diverging strongly for large values of x. In agreement with the results of the previous section, namely Eq. (19) and Fig. 3(b), both δ-Gaussian distributions shown in Fig. 6 approach stretched exponentials, $P_{eq}(x) \sim \exp[-(\beta x^2)^{1/\delta}]$, in the asymptotic limit. Since these distributions are Gaussian-like in their central region, with long tails following stretched-exponential behavior, they appear as candidates for anomalous-diffusion processes described in terms of fractal-derivative equations, where both behaviors have been found [29].

V. CONCLUSIONS

We have investigated properties of the probability distribution associated with the recently proposed entropic form S_δ (which recovers the usual BG form for $\delta = 1$). By considering a generalized version of the H theorem, we have found a NLFPE related to the S_δ entropy, whose stationary-state solution follows a transcendental equation. This later equation was precisely the same obtained by extremizing the entropy, a procedure which yields equilibrium distributions. Such a transcendental equation was solved for typical values of $\delta > 1$, analytically for special values of δ, as well as numerically in other cases.

First, we have analyzed the equilibrium distributions $P_{eq}(h)$ in terms of a general variable, $h(x) = \alpha + \beta\phi(x)$, where α and β are Lagrange multipliers and $\phi(x)$ represents a general confining potential. In this case, we have shown that the equilibrium distributions approach stretched exponentials in the asymptotic regime for all $\delta > 1$. Moreover, cutoffs h^* were introduced, given by lower-limit points up to which the distributions $P_{eq}(h)$ remain real and positive. A plot of the coordinates of such points, $P_{eq}(h^*)$ versus $-h^*$ for several values of δ, has shown a curious effect at the value $\delta = 4/3$, characterizing a return point of the associated curve.

As an illustration, we have solved the transcendental equation by considering the energy spectrum of a quantum one-dimensional harmonic oscillator. In this case, by imposing probability normalization, we have found a generalized chemical potential $\mu(\beta)$ in terms of the Lagrange parameter β, considering values of δ for which the transcendental equation is solved analytically, namely $\delta = 1$, 2, and 3. The qualitative behavior of $\mu(\beta)$ in the cases $\delta = 2$ and 3 is very similar to the one in the BG particular case.

Recently, the entropy S_δ was shown to be an appropriate choice for the thermodynamics of black holes, as well as for strongly quantum-entangled systems [11]. In these systems the BG entropy S_{BG} does not present the scaling expected for an appropriate thermodynamical framework. Considering L as a characteristic linear length of the system, S_{BG} scales as L^2 (and not as L^3) in the case of a black hole, whereas it scales as L^{d-1} (and not as L^d) for d-dimensional strongly quantum-entangled systems ($d > 1$). In the present study we have defined its associated mesoscopic dynamics, following a NLFPE, and have analyzed its stationary-state distributions. Since in most cases the comparison of theoretical results with experimental and observational verifications are done by means of probability distributions, the present analysis is expected to be useful for this purpose. Additionally, the fact that these distributions decay asymptotically as stretched exponentials suggests its applicability for many other phenomena in which such a behavior has already been verified, like luminescence decays, anomalous diffusion associated with fractional-diffusion equations, and turbulent flows. Moreover, very recently it has been advanced [36] that S_δ appears to be relevant to the discussion of the dark energy of the universe in connection with the observational luminosity-redshift data available today.

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