Second-harmonic generation for dispersive elastic waves in a discrete granular chain

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The propagation of nonlinear compressional waves in a one-dimensional granular chain driven at one end by a harmonic excitation is studied. The chain is described by a Fermi-Pasta-Ulam (FPU) lattice model with quadratic nonlinearity (α-FPU model), valid for strong initial compression of the chain by an external static force. A successive approximations method is used to obtain the analytical expressions for the amplitudes of the static displacement field and of the fundamental and second harmonics propagating through the lattice. Both propagating and evanescent second harmonics are shown to influence the nonlinear propagation characteristics of the fundamental frequency. The propagating regime is characterized by a periodic energy transfer between first and second harmonics, resulting from dispersion, which disappears when the second harmonic becomes evanescent.

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I. INTRODUCTION

The acoustic propagation in three-dimensional granular media is, in general, an extremely complex problem, since the number of parameters involved in the description is usually large and the parameters themselves are difficult to connect to the acoustic properties (the statistical distributions of shape, size, and constitutive materials of the grain assembly, the dynamical behavior of individual contacts, or the geometry of the contact lattice, to cite some). For that reason, one often relies on simplified configurations that are more suitable for analysis but still capture some of the basic features of nonlinear wave propagation in granular materials. The simplest and most widely used model of granular material consists of a one-dimensional chain of identical spherical beads maintained in contact at rest. The study of nonlinear wave propagation in such granular chains has been, since the pioneering works of Nesterenko [1], an active field of both fundamental and applied research. The nonlinearity results from the specific type of restoring force between two adjacent elastic spheres, which is governed by the Hertz law [2] and is located at the contacts, where the deformation occurs. Often an external static force is applied, resulting in a precompression of the chain. In this way, the properties of the chain and in particular the strength of the nonlinearity can be controlled. Depending on the magnitude of the applied force, two limit situations have been investigated, namely the strongly and the weakly compressed chains. In the weak compression limit where the imposed dynamic deformation can be larger than the deformation associated with the static force the chain behavior is highly nonlinear and supports the propagation of solitary waves with compact spatial extension [3–6] or discrete breathers [7,8] for example. The opposite limit, the strongly compressed chain, has been much less explored. In this case, the model reduces under some assumptions to a Fermi-Pasta-Ulam equation with quadratic nonlinearity (also known as the α-FPU problem in the literature [9]). As is well known, for disturbances with a characteristic scale much larger than the particle diameter (the so-called long-wavelength approximation) one can ignore discreteness effects and apply a continuum approach, which leads to the Korteweg-de Vries (KdV) equation. This equation is a canonical model for weakly nonlinear, weakly dispersive systems and is completely integrable.

Although the literature on the Fermi-Pasta-Ulam (FPU) problem is extremely vast (see a recent review in Ref. [9]), most of the studies have been performed for initial value conditions, where one considers the evolution of an initial distribution of the particle displacements along the chain. This is not the typical situation with granular chains, where the propagation of a signal injected at one end is of interest. Thus, for granular chains the problem must be formulated as a boundary value problem.

An approach to this problem was presented in Ref. [10], where the excitation of the chain with two different but close high frequencies was considered. There, the self-demodulation phenomenon (generation and propagation of the difference frequency mode) was examined. In particular, the propagative low frequency component was shown to carry information about the high frequencies (nonpropagative or evanescent) lying above the cutoff frequency. There are also few recent studies on the propagation of harmonic signals, submitted at one boundary, in nonlinear chains or lattices. In the long-wavelength limit, experimental results on the generation of harmonics are found to be consistent with the Hertz theory for precompressed granular chains [11,12]. A widely studied

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discrete system, both theoretically and experimentally, is the electrical transmission line [13]. Some remarkable results of this system are the existence of envelope (bright) and hole (dark) solitons related to modulationally unstable wave solutions [13], the propagation of signals even when the chain is driven above the cutoff frequency (supratransmission phenomenon) [14,15], or bistable transmission regimes [16]. Recently, harmonic generation and self-action in a weakly nonlinear diatomic granular chain have been studied experimentally and theoretically [17]. The second-harmonic generation is described with a weakly nonlinear approach considering only the quadratic nonlinearity. The strong self-action effects, leading to an amplitude dependent band structure are explained via the role of the hysteretic nonlinearity of normal contacts. In Ref. [18], the harmonic balance method has been applied to the prediction of amplitude dependent effects on the dispersion curves in monoatomic and diatomic Hertzian granular chains, as well as in two-dimensional granular lattices.

The problem considered in this paper is the propagation of an initially harmonic signal through a strongly compressed granular chain. The chain dynamics is described here by a discrete FPU equation with quadratic nonlinearity. Particular attention is paid to the dispersive regime, where the input frequency is of the same order of magnitude as the cutoff frequency (or band edge frequency) of the chain, where the continuous (long-wavelength) limit fails and discreteness effects become important [19]. The main motivation of this work is the study of the second-harmonic generation process in boundary-driven nonlinear chains, realistic in experiments [17,19]. The precise analytical description of nonlinear acoustic effects in nonlinear discrete lattices is intended to be implemented in the future studies on nonlinear phononic processes in acoustics. Applications could be the design of materials able to suppress nonlinear wave distortion, able to maximize energy conversion between harmonics, or more generally to conceive bricks of more advanced wave tailoring devices, rectifiers, for instance [20,21]. Although boundary harmonic driving has been considered in FPU lattices with cubic nonlinearity (the β-FPU model) in Ref. [22], the nonlinearly generated higher harmonics were neglected. In this work we observe both analytically and numerically the presence of propagating and evanescent second harmonics depending on the driving frequency, the generation of a constant displacement, and the associated effects on the fundamental wave. We note that, although the physical system modeled in this work is a granular lattice, the results may found applicability for other systems described by the FPU equation.

The outline of the paper is as follows. In Sec. II the model for a granular chain with power law (Hertz) nonlinearity is presented, and the equation of motion of the chain in the quadratic approximation and its dispersion relation are derived. Next, a successive approximations technique is introduced to analyze both the nondispersive (Sec. III) and the dispersive (Sec. IV) limits of the chain, and a hierarchy of linear equations is presented. In the dispersive case, analytical solutions are presented, for both propagative and evanescent second harmonics. The results are compared with numerical simulations. Finally, in Sec. V the concluding remarks are presented.

II. THE α-FPU LIMIT OF THE GRANULAR CHAIN

Consider a homogeneous one-dimensional chain of spherical beads in contact, each with a mass m and a radius R, as shown in Fig. 1. In the absence of external loading [Fig. 1(a)] the distance between centers is $a = 2R$. Under the effect of an external constant force F_0 [Fig. 1(b)] the chain is compressed, and the distance between centers is reduced by an amount δ_0 resulting in $a = 2R - \delta_0$. Denoting by u_n the displacement of the nth bead from its equilibrium position and assuming that the beads repel upon Hertz-type potential $V(\delta) \propto \delta^{3/2}$, with $\delta = \delta_0 - (u_n - u_{n-1})$ being the bead-bead overlap, the dynamics of the chain is described by a system of coupled differential equations [2],

$$m\frac{d^2u_n(t)}{dt^2} = A[(\delta_0 - u_n + u_{n-1})^{3/2} - (\delta_0 - u_{n+1} + u_n)^{3/2}], \quad (1)$$

where $A = Y\sqrt{2R}/[3(1-\sigma^2)]$, with Y and σ denoting, respectively, the Young's modulus and the Poisson's ratio of the grain material. Impulse propagation in Eq. (1) has been extensively studied [2]. Most of the previous work was conducted in the limit of weakly compressed chain $\delta_0 \ll |u_n - u_{n-1}|$, where highly nonlinear solitary structures were predicted and observed. The opposite limit $\delta_0 \gg |u_n - u_{n-1}|$, corresponding to a strongly compressed chain [3], has received much less attention. Under such a condition, the power terms in Eq. (1) can be expanded in series, and in the quadratic approximation this results in

$$m\frac{d^2u_n(t)}{dt^2} = \alpha(u_{n+1} - 2u_n + u_{n-1}) - \frac{\beta}{2}[(u_{n+1} - 2u_n + u_{n-1})(u_{n+1} - u_{n-1})], \quad (2)$$

where $\alpha = (3A/2)\delta_0^{1/2}$ and $\beta = (3A/4)\delta_0^{-1/2}$ are the linear and nonlinear (quadratic) coupling coefficients, respectively. Although in Eq. (2) the position is identified by the bead index n, it is useful to define a discrete spatial coordinate $x_n = na$. As stated in the Introduction, Eq. (2) is equivalent, after the proper scalings, to the α-FPU lattice model [9].

It is possible, and convenient for the subsequent analysis, to reduce the number of parameters in Eq. (2). We introduce the normalized displacement $u_n = u_n/u_0$, where u_0 is a characteristic amplitude. We also define a dimensionless

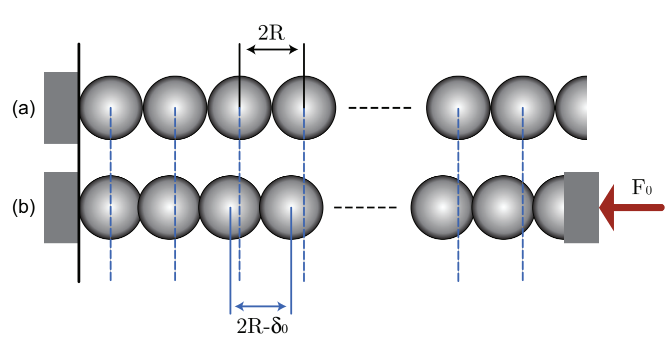

FIG. 1. (Color online) Chain of spherical beads in contact (a) and compressed chain under the action of an external static force F_0 (b).

time $t' = \omega_c t$, with $\omega_c = 2\sqrt{\alpha/m}$ being the cutoff frequency, and a dimensionless spatial coordinate $x'_n = k_c x_n$, where $k_c = 1/a$. Finally, the dimensionless nonlinearity parameter is defined as $\varepsilon = u_0\beta/\alpha = u_0/2\delta_0$. These definitions lead to the dimensionless equation

$$\frac{d^2u_n(t)}{dt^2} = \frac{1}{4}(u_{n+1} - 2u_n + u_{n-1}) - \frac{\varepsilon}{8}[(u_{n+1} - 2u_n + u_{n-1})(u_{n+1} - u_{n-1})], \quad (3)$$

where the primes have been removed. We consider a driven lattice, subjected to the boundary condition $u_0(t) = \cos(\Omega t)$, where $\Omega = \omega/\omega_c$. In this way, we are considering that the characteristic amplitude u_0 is the amplitude of the harmonic displacement externally imposed to the first element in the chain. Note that, in Eq. (3), the forcing amplitude appears implicitly in the nonlinearity coefficient ε. We assume that the waves propagate only in one direction, i.e., the chain is considered to be semi-infinite, or alternatively that it possesses a nonreflecting (absorbing) termination.

In the linear limit ($\varepsilon \to 0$) the solutions have the form $u_n = \exp[i(\Omega t - kn)]$, with k being the wave number normalized to $k_c = 1/a$, which leads to the lattice linear dispersion relation

$$\Omega = |\sin(k/2)|. \quad (4)$$

Note that, in terms of normalized magnitudes, the band edge corresponds to the values $\Omega_B = 1$, $k_B = \pi$.

In the next two sections we consider the chain behavior in two different cases: the nondispersive regime corresponding to low frequencies $\Omega \ll 1$ (long-wavelength limit), where the dispersion relation takes the form $\Omega \simeq k/2$, and the dispersive regime valid for $\Omega = O(1)$, where discreteness effects must be taken into account.

III. NONDISPERSIVE CASE

Let us adopt the standard continuum approximation, assuming that displacement variations occurring between two neighboring beads are much smaller than the initial separation between their centers. Then, the discrete variable can be substituted by a continuous one, $u_n(t) = u(x,t)$, and the displacement of the nearest-neighbor beads can be expressed, using Taylor expansion, as

$$u_{n\pm 1} = u \pm \frac{\partial u}{\partial x} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \cdots, \quad (5)$$

where $|\partial^2 u/\partial x^2| \ll |\partial u/\partial x| \ll |u|$. Inserting Eq. (5) into Eq. (3) leads, to the leading order, to the nonlinear wave equation,

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{4}\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[-\frac{\varepsilon}{8}\left(\frac{\partial u}{\partial x}\right)^2\right] \equiv \frac{\partial \sigma_{NL}}{\partial x}, \quad (6)$$

where

$$\sigma_{NL} = -\frac{\varepsilon}{8}\left(\frac{\partial u}{\partial x}\right)^2 \quad (7)$$

can be interpreted, using the analogy with wave propagation in solids, as a nonlinear stress term.

Equation (6) is of the form of Boussinesq equation and has been studied for a long time in the context of nonlinear plane acoustic waves in fluids. It is known that Eq. (6) can develop shock waves from initially harmonic waves [23,24]. Approximate solutions of Eq. (6) can be obtained by perturbative methods, in particular by a successive approximations approach. The use of perturbative methods is based on the existence of a small parameter in the original model. Under the assumptions used in the derivation of Eq. (3), it follows that the nonlinearity parameter $\varepsilon \simeq u_0/\delta_0 \ll 1$. On the other hand, the displacement u_n is on the order 1. Then we can express the displacement as

$$u(x,t) = u^{(0)} + \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \cdots, \quad (8)$$

which substitution in (6) leads to a hierarchy of equations at different orders in ε. At order $O(\varepsilon^{(0)})$ one retrieves the linear wave equation

$$\frac{\partial^2 u^{(0)}}{\partial t^2} - \frac{1}{4}\frac{\partial^2 u^{(0)}}{\partial x^2} = 0, \quad (9)$$

which can be solved as $u^{(0)} = \cos(\Omega t - kx)$. At the next order $O(\varepsilon^{(1)})$ we find

$$\frac{\partial^2 u^{(1)}}{\partial t^2} - \frac{1}{4}\frac{\partial^2 u^{(1)}}{\partial x^2} = \frac{\partial}{\partial x}\left[-\frac{1}{8}\left(\frac{\partial u^{(0)}}{\partial x}\right)^2\right], \quad (10)$$

where the normalized nonlinear stress (the term between square brackets) is

$$\sigma_{NL} = -\frac{\Omega^2}{4}\left[1 - \cos(2\Omega t - 2kx)\right], \quad (11)$$

which contains both stationary (nonoscillating) and oscillating contributions.

Let us consider the stationary part of the stress. It must be accompanied by a stationary contribution to the strain, and therefore the offset displacement (the equilibrium position) of any bead in the chain, obtained by integration of the strain, must increase linearly with distance,

$$\langle u^{(1)} \rangle \propto x, \quad (12)$$

where $\langle\cdot\rangle$ denotes temporal average. In other words, the presence of a nonlinear static stress results in an expansion of the chain in the positive direction of the x axis. This effect is similar, but in the opposite sense, to that created by the initial static force F_0 applied to the chain, which also results in an offset displacement as shown in Fig. 1. On the other hand, if we directly substitute Eq. (11) into (10) we get

$$\frac{\partial^2 u^{(1)}}{\partial t^2} - \frac{1}{4}\frac{\partial^2 u^{(1)}}{\partial x^2} = \Omega^3 \sin(2\Omega t - 2kx), \quad (13)$$

where the oscillating source term in the right-hand side indicates that only the second harmonic is generated; i.e., only the oscillating part of $u^{(1)}$ appears. This apparent contradiction is, however, mathematical and can be resolved by such physical arguments as those mentioned above. In fact, Eq. (12) was obtained by implicitly allowing for the material to expand only in the positive direction of x axis, while in accordance with Eq. (5) any stress homogeneous in space (without gradient) produces zero acceleration and cannot displace the material.

To get Eq. (12) it is necessary to assume that there exists a boundary at the right of the material (say at infinity). The points of this boundary are not in symmetrical situation; all the others in the chain are. The nonlinear forces are acting on them only from the left, they start to move, and finally all the material $0 < x < +\infty$ is expanded as $t \to +\infty$.

The contradiction can be resolved by a proper mathematical formulation. Instead of the harmonic solution for the linear problem, we consider a weakly damped solution by adding a small attenuation to the linear wave,

$$u^{(0)} = \cos(\Omega t - kx)e^{-\alpha x}. \tag{14}$$

Substituting (14) into (10), performing a time averaging, and keeping the leading contribution of nonlinear nonoscillating stress, we obtain

$$\frac{\partial \langle u^{(1)} \rangle}{\partial x} = \Omega^2 e^{-2\alpha x}. \tag{15}$$

Finally, the offset displacement can be found integrating Eq. (15), and taking into account the boundary condition $\langle u^{(1)}(x=0)\rangle = 0$ to evaluate the integration constant. We get,

$$\langle u^{(1)} \rangle = \frac{\Omega^2}{2\alpha}(1 - e^{-2\alpha x}), \tag{16}$$

which in the limit $\alpha \to 0$, and turning into the original variables, results in

$$\langle u(x,t) \rangle = \varepsilon \Omega^2 x. \tag{17}$$

Considering the oscillating terms, and proceeding further with the perturbation analysis, one can find the amplitudes of the higher harmonics. The result is known and is not discussed here: The synchronous propagation of the nonlinearly generated higher harmonics leads to shock formation in a finite time. Especially in a nondispersive medium, all the harmonics of the initial monochromatic wave propagate synchronously; therefore, the spectrum broadens during propagation and the energy is continuously pumped into the higher harmonics, leading to wave-form distortion and eventually to the formation of shock fronts [23,24].

IV. DISPERSIVE CASE

The results of the previous section are valid in the small frequency limit, where the continuous approach finds applicability. However, for arbitrary frequencies the behavior of the solutions is substantially different, and discreteness and dispersive effects play an important role. The successive approximations approach of the previous section can be again used, but taking into account the discrete and dispersive character of the system. We start assuming that the displacement can be expressed as a power series in terms of ε, as

$$u_n = u_n^{(0)} + \varepsilon u_n^{(1)} + \varepsilon^2 u_n^{(2)} + \cdots. \tag{18}$$

After substituting Eq. (18) in (3), and collecting terms at each order in ε, we obtain the following system of equations:

$$L\big(u_n^{(0)}\big) = 0, \tag{19a}$$

$$L\big(u_n^{(1)}\big) = -\tfrac{1}{8}\big(u_{n+1}^{(0)} - 2u_n^{(0)} - u_{n-1}^{(0)}\big)\big(u_{n+1}^{(0)} - u_{n-1}^{(0)}\big), \tag{19b}$$

$$L\big(u_n^{(2)}\big) = -\tfrac{1}{8}\big[\big(u_{n+1}^{(0)} - 2u_n^{(0)} - u_{n-1}^{(0)}\big)\big(u_{n+1}^{(1)} - u_{n-1}^{(1)}\big) + \big(u_{n+1}^{(0)} - u_{n-1}^{(0)}\big)\big(u_{n+1}^{(1)} - 2u_n^{(1)} - u_{n-1}^{(1)}\big)\big], \tag{19c}$$

$$L\big(u_n^{(3)}\big) = \cdots, \tag{19d}$$

with the linear operator defined as

$$L\big(u_n^{(j)}\big) = \frac{d^2 u_n^{(j)}}{dt^2} - \frac{1}{4}\big(u_{n+1}^{(j)} - 2u_n^{(j)} - u_{n-1}^{(j)}\big). \tag{20}$$

The above equations can be solved recursively, as in the previous section. We separate the analysis in two cases, corresponding to two different regimes depending on whether the frequency of the second harmonic belongs to the propagation band, i.e., propagative regime $(0 < \Omega < 1/2)$ or it is above the cutoff value, i.e., evanescent regime $(1/2 < \Omega \leqslant 1)$.

A. Propagative second harmonic

The solution to the $O(\varepsilon^0)$ problem, Eq. (19a), compatible with the boundary condition, is $u_n^{(0)} = \cos\theta_n$, with the phase term defined as $\theta_n = \Omega t - k(\Omega)n$. However, as stated in the previous section, such a solution fails in predicting a constant strain component in the lattice. Instead, we consider a damped solution,

$$u_n^{(0)} = \cos\theta_n e^{-\alpha n}, \tag{21}$$

which introduces the necessary lattice asymmetry and gives the correct results for $\alpha \to 0$.

The solution to the $O(\varepsilon^1)$ problem can be sought as a combination of stationary and oscillating terms. Since the right-hand side of Eq. (19b) contains terms with frequencies 0 and 2Ω, the general solution is of the form

$$u_n^{(1)} = A_n + \tfrac{1}{2}B_{2\Omega}e^{i2\theta_n} + \tfrac{1}{2}B'_{2\Omega}e^{i2\varphi_n} + \text{c.c.}, \tag{22}$$

where $A_n = \langle u_n^{(1)} \rangle$. The solution contains two kinds of oscillating terms for the second harmonics: a free oscillation, satisfying the dispersion relation [with phase $2\varphi_n = 2\Omega t - k(2\Omega)n$], corresponding to the kernel of the linear operator $L^{(1)}$, and a forced one with phase $2\theta_n = 2\Omega t - 2k(\Omega)n$. Note that $B_{2\Omega} = -B'_{2\Omega}$, in order to fulfill the boundary conditions (absence of frequency 2Ω at the boundary).

Substituting Eqs. (21) and (22) into Eq. (19b), and after time averaging we get the equation for the amplitude of the stationary mode,

$$A_{n+1} - 2A_n + A_{n-1} = \sinh\alpha\,(\cos k - \cosh\alpha)e^{-2\alpha n}. \tag{23}$$

The solution of Eq. (23) can be readily found as

$$A_n = \frac{1}{2\sinh\alpha}(\cosh\alpha - \cos k)(1 - e^{-2\alpha n}). \tag{24}$$

It is time to recover the real situation where dissipation is absent (or negligible). Then, taking the limit $\alpha \to 0$ in Eq. (24) and recovering the original variables, it follows that

$$\langle u_n \rangle = \varepsilon \sin^2\left(\frac{k}{2}\right) n = \varepsilon \Omega^2 n, \tag{25}$$

which corresponds to a constant strain along the chain and is the discrete version of Eq. (17).

To find the amplitudes of the oscillating terms we can neglect the decay term (the artifact is needed only to account for the correct solution for the stationary mode). Equating the terms with the same phase $2\theta_n$ we get the amplitude of the second harmonic,

$$B_{2\Omega} = \frac{i}{4}\cot\left(\frac{k}{2}\right). \tag{26}$$

At the next order, Eq. (19c), with the solution given by Eq. (22), generates terms with the phases $\pm\theta_n$, $\pm(2\varphi_n \pm \theta_n)$, $\pm 3\theta_n$, so the solution can be written as

$$u_n^{(2)} = \tfrac{1}{2}C_\Omega(e^{i\theta_n} - e^{i(2\varphi_n-\theta_n)}) + \tfrac{1}{2}C_{3\Omega}(e^{i3\theta_n} - e^{i(2\varphi_n+\theta_n)}) + \text{c.c.} \tag{27}$$

In particular, C_Ω results in contributions to the amplitude of the first harmonic obtained at previous orders. Substituting Eq. (27) into (19c) and equating terms with the phase $2\varphi_n - \theta_n$, we obtain

$$C_\Omega = \frac{1}{8}\left\{\frac{\sin[k(2\Omega)/2]}{\sin[\Delta k(\Omega)/2]} - 1\right\}. \tag{28}$$

where $\Delta k(\Omega) = 2k(\Omega) - k(2\Omega)$ is the wave number mismatch due to dispersion. Finally, the solution including up to the second harmonic reads

$$u_n = \varepsilon\Omega^2 n + \frac{1}{2}\left[1 - i2C_\Omega\varepsilon^2\sin\left(\frac{\Delta k}{2}n\right)e^{i\frac{\Delta k}{2}n}\right]e^{i\theta_n} + \frac{\varepsilon}{4}\cot\left(\frac{k}{2}\right)\sin\left(\frac{\Delta k}{2}n\right)e^{i\frac{\Delta k}{2}n}e^{i2\theta_n} + \text{c.c.} \tag{29}$$

The amplitudes of the harmonic components at first and second order must be $B_{2\Omega} \simeq O(1)$ and $C_\Omega \simeq O(1)$, respectively, to keep the expansion (18) valid. According to Eq. (26), $(1/4)\cot(k/2) \simeq 1$, which imposes a minimum value of the wave number and frequency, $k_{\min} \simeq 1/2$ and $\Omega_{\min} \simeq 1/4$. Consequently, the solution is not valid in the low frequency limit (where $B_{2\Omega}$ and C_Ω diverge to infinity), and dispersion becomes small.

In the derivation procedure, the time-independent term was assumed to be an order $O(\varepsilon)$ term, Eqs. (18) and (22). However, due to the absence of asynchronism with the linear term, it is nonlinearly accumulated with distance, such that it can, in principle, become of the order 1. This effect, as stated before, corresponds to a shift in the equilibrium (static) position of the beads and a constant strain. However, it is easy to show that only the static strain contributes to Eqs. (19b) and (19c), and it remains small compared to the oscillating linear one (itself small compared to $\delta_0/2R$).

In Fig. 2 the spatial growth rate of the stationary mode for a given frequency and nonlinearity parameter is plotted, as obtained from the numerical integration of Eq. (3) (symbols) and from the theoretical prediction given by Eq. (25) (solid line). The one-dimensional FPU lattice has been numerically simulated by implementing an explicit fourth-order Runge-Kutta scheme [25] with a harmonic boundary condition imposed to the first bead. The numerical results were performed for a chain with $n = 512$ beads and a time step $\Delta t = 0.001$. Nevertheless, numerical tests with higher number of beads

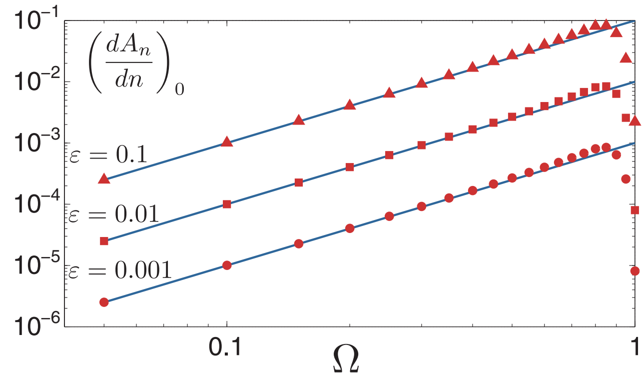

FIG. 2. (Color online) Spatial slope of the static mode at the excitation boundary as a function of the excitation frequency for different nonlinear parameters ε. The numerical simulations are shown in symbols and the theoretical results from Eq. (25) are in continuous line.

and smaller time steps confirmed the convergence to the same results. In Fig. 2 the slope amplitude of the static mode spatial growth is evaluated at the boundary $n = 0$, as a function of the frequency Ω and for different values of the nonlinear parameter ε. The numerical results (in symbols) exhibit a continuous increase as a function of Ω up to the cutoff frequency $\Omega = 1$. The plotted theoretical results shown in continuous lines correspond to the derived Eq. (25), in the case of a propagating second harmonic, i.e., $\Omega < 0.5$. A good agreement is observed between numerical and theoretical results in the validity region of Eq. (25), $\Omega < 0.5$, for all the presented values of the nonlinear parameter ε. For $\Omega > 0.5$ a good agreement is observed for sufficiently small values of ε [as predicted also by Eq. (30) for evanescent second harmonic]. However, a growing mismatch is observed for $\Omega > 0.5$ and values of $\varepsilon > 0.05$. In this case of evanescent second harmonic and sufficiently large nonlinearity, higher order terms are needed to describe the initial growth of the static mode. The observed discrepancy between the analytical results and the numerical simulations for Ω close to 1 could be attributed to the fact that more and more time is required for the numerical solution evaluation. This diverging time could be related to the wave energy travel time (over a given distance), which scales as the inverse of the group velocity $c_g^{-1}(\Omega) \propto (1 - \Omega^2)^{-1/2}$ [10]. Considering that Ω approaches 1, i.e., $\Omega = 1 - \mu$ with $\mu \ll 1$, the wave energy travel time diverges as $\propto\mu^{-1/2}$. Moreover, close to the band gap frequency, modulational wave instability could occur. These phenomena are not accounted for in the developed model and go beyond the present study.

The prediction for the other propagating modes is also investigated. In Fig. 3 the theoretical (solid lines) and numerical (dots) results are compared, for a chain driven with frequency $\Omega = 0.4$ and nonlinearity parameter $\varepsilon = 0.1$. Note that, as predicted by Eq. (29), the effect of finite dispersion causes a beating in the amplitudes of the different harmonics. The displacement in the fundamental frequency wave is maximum when the contribution of the second harmonic vanishes, which occurs when $(\Delta k/2)n = m\pi$, or alternatively at $n = 2m\pi/\Delta k$. For the case shown in Fig. 3 it follows that $n \simeq 30, 60, 90$, which is in good agreement with the numerical simulation.

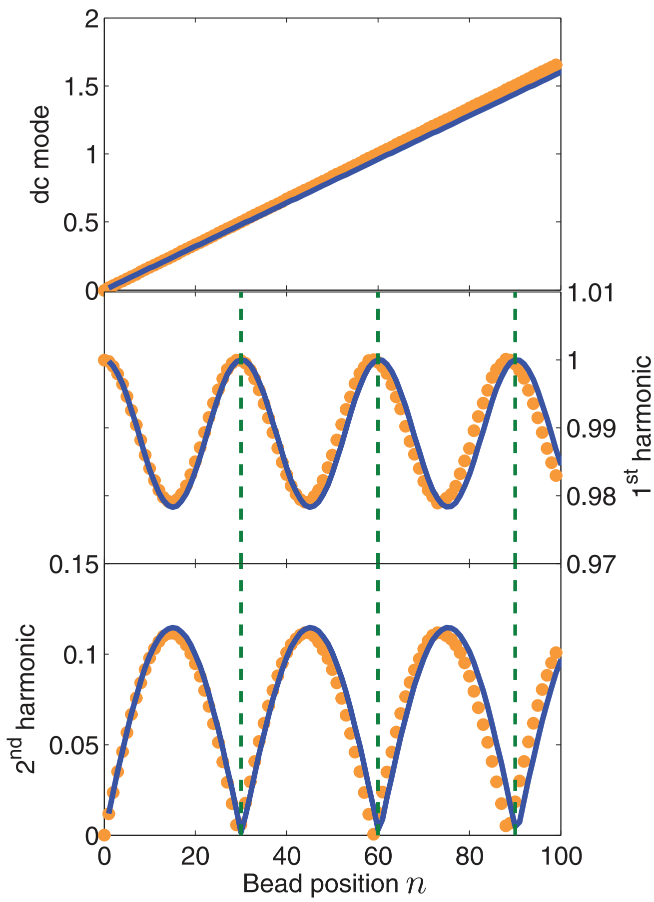

FIG. 3. (Color online) Comparison of the theoretical result from Eq. (20) (lines) and the numerical simulation of Eq. (3) (circles) for $\Omega = 0.4$ and $\varepsilon = 0.1$. Vertical green dashed lines represent beat wavelengths for $m = 1$, 2, and 3.

The study also reveals that the amplitude of the fundamental wave as it propagates along the chain is nontrivially influenced by the presence of the second harmonic, since a part of the energy injected at the boundary flows into higher order modes. The resulting fundamental amplitude is determined by the transfer function Eq. (28), which vanishes as the driving frequency Ω approaches the edge of the propagation band, and becomes complex when Ω is at the center of the propagation band (denoting the transition from propagative to evanescent second harmonic). We note that in [10] similar conclusions were obtained concerning the influence of a biharmonic excitation approaching the cutoff frequency on the nonlinearly generated difference frequency signal.

The time dependence of the displacement at different positions along the chain can be also obtained from Eq. (29), by fixing the value of n. In Fig. 4 the temporal profiles (wave forms) are shown, as obtained from numerical solutions of Eq. (3) (left column) and from analytical solution Eq. (29) (right column) for three different distances, $n = 20$, 50, and 100. The case $\varepsilon = 0.3$, $\Omega = 0.3$ is presented.

Note that, the temporal profiles change along the lattice, but repeat after some distance. This behavior is related to the effect of dispersion modifying the phase with distance between the different harmonics. Note also that the inclusion of only the first two harmonics is accurate enough to describe the wave forms and other wave propagation characteristics.

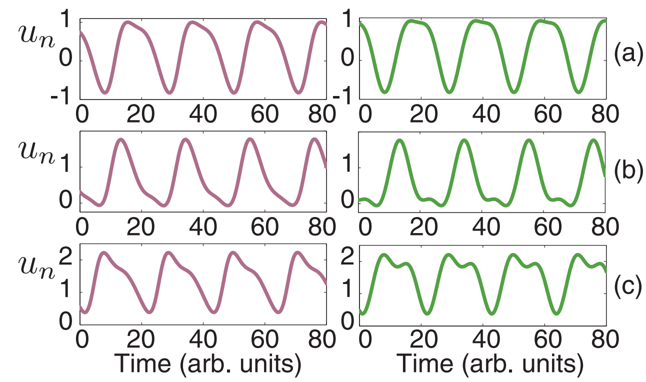

FIG. 4. (Color online) Temporal wave forms from numerical simulation of Eq. (3) (left column) and from Eq. (29) (right column) for $\Omega = 0.3$ and $\varepsilon = 0.3$, at different locations along the chain. (a) $n = 20$, (b) $n = 50$, (c) $n = 100$.

B. Evanescent second harmonic

When the driving frequency belongs to the upper half of the propagation band, its second harmonic lies above the cutoff frequency. Therefore, the second harmonic is strongly damped and one may expect that it plays no role in the chain behavior at long distances ($n \gg 1$). Under this assumption, in the analysis of similar problems (see, e.g., Ref. [6]) the presence of the second harmonic was ignored (i.e., the amplitude was assumed to vanish). Next we show that, even being an evanescent wave, the second harmonic must be taken into account in the expansion since it has a non-negligible contribution to the amplitude of the fundamental mode, even at long distances. This contribution comes from the forced solution at the second harmonic, which reaches a stationary but finite value along the chain.

Consider now a driving frequency in the range $1/2 < \Omega < 1$. In this case $2\Omega > 1$, the dispersion relation Eq. (4), does not support purely real solutions, so the corresponding wave number should be complex. Then, from Eq. (4) and as it is proposed in Ref. [10], it can be written that $k(2\Omega) = 1 - ik''$, with $k'' = 2\,\text{acosh}(2\Omega)$. Now the mismatch term takes the form $\Delta k = 2k(\Omega) - k(2\Omega) = 2k(\Omega) - 1 + ik'' = k' + ik''$, where $k' = 2k(\Omega) - 1$. Since we keep the same order $O(\varepsilon^2)$ solution as defined in Eq. (27), the expressions found in the propagative case remain valid, but now the solution given by Eq. (29) reads

$$u_n = \varepsilon\Omega^2 n + \frac{1}{2}[1 - C_\Omega\varepsilon^2(1 - e^{-k''n}e^{ik'n})]e^{i\theta_n} - \frac{\varepsilon}{8}\cot\left(\frac{k}{2}\right)(1 - e^{-k''n}e^{ik'n})e^{i2\theta_n} + \text{c.c.}, \quad (30)$$

and the transfer function C_Ω, given by Eq. (28), becomes complex.

In Fig. 5 the analytical results (lines) are shown together with the results of the numerical simulation (dots) for $\Omega = 0.6$.

Note that, in the evanescent regime shown in Fig. 5 there is good agreement with the theoretical results except for the amplitude of the stationary mode. The theoretical solution represents here a rather good approximation close to the

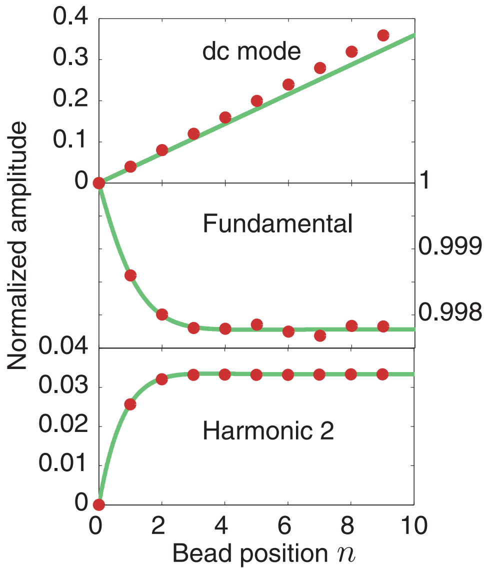

FIG. 5. (Color online) Displacement amplitudes of the harmonics versus distance. Comparison of the theoretical result from Eq. (30) (lines) and the numerical simulation of Eq. (3) (discs) for $\Omega = 0.6$ and $\varepsilon = 0.1$.

boundary, but higher order terms may be needed for a more accurate solution.

V. CONCLUSIONS

In this work we have considered the propagation of nonlinear compressional waves in a one-dimensional granular chain, consisting of an array of spherical beads in contact, driven at one end by a harmonic excitation. The nonlinearity of the chain, located at the contacts and modeled by a Hertz type law, leads to the appearance of higher harmonics of the driving (fundamental) frequency as the wave propagates along the chain. Also, the discrete nature of the chain results in a highly dispersive regime for the propagating waves, which obey a particular dispersion relation. An analytical study of the chain behavior is performed in the weakly nonlinear limit (corresponding to small amplitude injected signal, or also to a strongly compressed chain), where the chain dynamics can be described by a (discrete) FPU equation with quadratic nonlinearity. By means of perturbative techniques we obtain analytical expressions for the amplitudes of the fundamental and second-harmonic modes, and also for the stationary, dc mode whose amplitude grows linearly with distance. The second harmonic is shown to have a finite amplitude even when its frequency is above the band edge, i.e., when it is an nonpropagating or evanescent mode. These theoretical predictions are validated by the numerical simulation of the FPU equation. These results on the analytical treatment of the harmonic generation in a one-dimensional granular chain constitute the first step towards the in-depth study of nonlinear phononic processes for dispersive acoustic waves. It is expected that spectacular nonlinear effects (such as harmonic total suppression or oppositely, total energy transfer from one frequency component to the other, interaction between waves with opposite group velocities) could be conceived. These results could be extended to the case of two-dimensional granular membranes [26] (and even three-dimensional granular crystals) and can be useful for the design of granular protectors, waves' sensors, and nonlinear acoustic lenses, as well as acoustic switching and rectification devices [20,21].

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