Criterion for the emergence of explosive synchronization transitions in networks of phase oscillators

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The emergence of explosive synchronization transitions in networks of phase oscillators recently has become one of the most interesting topics. It is widely believed that the large frequency mismatch of a pair of oscillators (also known as disassortativity in frequency) is a direct cause of an explosive synchronization. It is found that, besides the disassortativity in frequency, the disassortativity in node degree also shows up in connection with the first-order synchronization transition. In this paper, we simulate the Kuramoto model on top of a family of networks with different degree-degree and frequency-frequency correlation patterns. Results show that only when the degrees and natural frequencies of the network's nodes are both disassortative can an explosive synchronization occur.

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I. INTRODUCTION

Synchronization of coupled dynamical units has been studied for many years [1–6]. These phenomena arise in various fields of science, ranging from natural to social and artificial systems [7–12]. The emergence of explosive synchronization of coupled phase oscillators has recently attracted much attention due to the discovery of an explosive percolation transition in complex networks [13–20]. Here the explosive synchronization refers to the first-order phase transition where all nodes in the network abruptly get to the synchronous state once the coupling strength is larger than a threshold value [21–24]. The phase transition processes towards synchronization have been widely studied by considering topological structures and interaction patterns of networks [25–32].

The first-order synchronization transition has been observed in a scale-free network of phase oscillators and of Rössler circuits when the natural frequencies of units are positively correlated to their connections [21,23]. Moreover, it has been found that the time delay may enhance the explosive transition to synchronization [33]. So far the correlation between the natural frequency and degree of the network's node is regarded as the main factor contributing to an explosive synchronization in the heterogeneous networks [21,34,35].

However, to our knowledge, the physical picture behind this mechanism leading to a first-order phase transition is not quite clear compared with the original Kuramoto model (KM) where the phase transition is continuous. In Ref. [21] the natural frequency of each oscillator is identified with its own degree, which makes the hubs possess very high frequency, while the leaves possess rather low frequency. The large frequency mismatches between the hubs and their neighbors in the heterogeneous networks inhibit the formation of condensation nuclei, which leads to a first-order phase transition. Very recently I. Leyva *et al.* [36] explicitly imposed certain constraints on the frequency differences between each node and its neighbors, which avoids any oscillator behaving as the core of a clustering process and also leads to an explosive synchronization.

The seminal work of Ref. [21] and the frequency gap-conditioned (FGC) model proposed in Ref. [36] clearly show the large frequency mismatch of a pair of oscillators (also known as disassortativity in frequency) is a direct cause of an explosive synchronization. It is found that, besides the disassortativity in frequency, the disassortativity in node degree also shows up in connection with the first-order synchronization transition, which is not noticed in the previous literature. Thus, several questions about the synchronous transition remain open: How does the assortativity in degree influence the process of synchronization transition? What is the criterion for an explosive synchronization transition in terms of the assortativities in node degree and natural frequency?

The remainder of this paper is organized as follows. In Sec. II we give the model of coupled phase oscillators and propose a global constraint condition. Meanwhile, the building steps of a FGC random network are introduced in detail. We depict the main results obtained through numerical simulations and analyze the influence of the frequency gap on the synchronization process in Sec. III. We give a qualitative explanation for the emergence of an explosive synchronization in Sec. IV. Section V is devoted to study of the effects of different topological structures and dynamical properties on the synchronization transitions. Concluding remarks are made in Sec. VI.

II. MODEL

Let us consider an undirected and unweighted network of N coupled phase oscillators with the following equations of motion [37]:

$$\frac{d\phi_i}{dt} = \omega_i + d\sum_{j=1}^{N} a_{ij}\sin(\phi_j - \phi_i) \quad i = 1,2,\ldots N, \tag{1}$$

where ϕ_i and ω_i are the phase and natural frequency of the ith oscillator, respectively, $d > 0$ is a coupling constant, and $\{a_{ij}\}$ is the entry of the adjacency matrix that uniquely defines the interaction between the nodes. Thus, if there is a edge between node i and node j, then $a_{ij} = 1$, and otherwise, $a_{ij} = 0$.

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The system size N is set to 400 for all calculations in the present work, since the results described below do not change qualitatively for larger systems. Meanwhile, the average degree $\langle k \rangle$ of each node is set to 50 for any one realization. Generally, an order parameter $R(t)$ is defined to quantify the degree of synchronization among the N coupled phase oscillators [1],

$$R(t)e^{i\psi(t)} = \frac{1}{N}\sum_{j=1}^{N} e^{i\phi_j(t)}, \tag{2}$$

where $R(t) \in [0,1]$ measures the degree of the phase coherence of populations and $\psi(t)$ denotes an average phase of the system. When $R(t) = 1$, the system reaches a complete synchronous state, and when $R(t) \approx 0$, it drifts into an incoherent state.

In order to achieve an explosive synchronization, we have to avoid that any one oscillator behaves as the core of a clustering process in which its neighbors begin to aggregate to the synchronous state smoothly and progressively, as in the classical routes described in previous papers [38–40]. In other words, the frequency gap between the connected oscillators cannot be too small. Otherwise, these units easily form a condensation nucleus. Thus, it is feasible to impose certain constraints on the frequency gaps between the linked nodes. Here a global constraint condition of frequency gaps can be set as follows:

$$\frac{1}{L}\sum_{i=1}^{L} |p_i - q_i| > \gamma, \tag{3}$$

where p_i, q_i are the natural frequencies of the nodes at the ends of the ith edge and L is the total number of edges in the network. This model allows us to construct networks with the same average degree $\langle k \rangle$, interpolating from random graphs to disassortative networks by tuning the single parameter $\gamma \in [0,1)$ [21].

The construction of network consists of the following steps: (1) assign to the N oscillators natural frequencies ω_i randomly spaced spanning the interval $[0,1]$; (2) randomly pick a pair of nodes (i,j), and form a link between them only if their values satisfy the condition (3); and (3) repeat step 2 until the desired number L of links in the network is obtained.

III. EMERGENCE OF AN EXPLOSIVE SYNCHRONIZATION

In this section, we will focus on the influence of different values of γ on the process of synchronization transition. As shown in Fig. 1, we have simulated two synchronization trajectories, labeled as forward (solid circle) and backward (hollow circle) continuations in each panel [21]. The former is done by calculating stationary value of R via varying d from 0 to 0.02 in steps of 0.0004, and using the outcome of the last run as the initial condition of the next one, while the latter is performed by decreasing d from 0.02 to 0 with the same step.

Figure 1 reports the results obtained by setting $p(\omega) = 1$ as a uniform frequency distribution in the interval $[0,1]$. Panels (a) and (b) of Fig. 1 show a typical second-order phase transition with a perfect match between the backward and forward synchronization trajectories for relatively small values of γ.

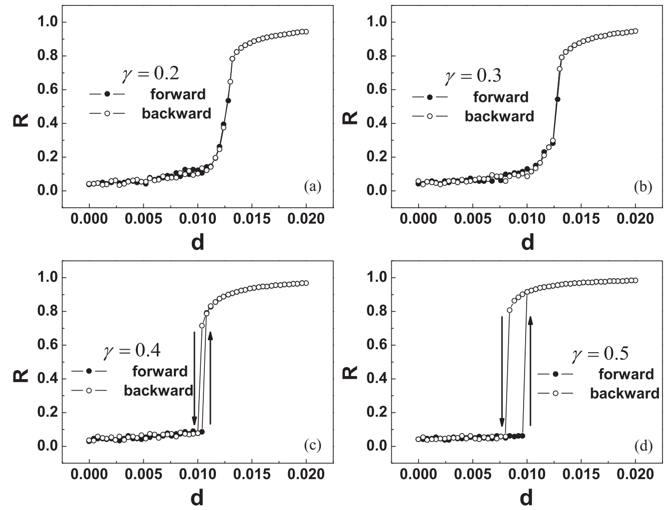

FIG. 1. The order parameter R vs the coupling strength d for different values of γ. (a) $\gamma = 0.2$, (b) $\gamma = 0.3$, (c) $\gamma = 0.4$, (d) $\gamma = 0.5$.

Notably, the first important result is observed for sufficiently large values of γ [see panels (c) and (d) of Fig. 1], in which an abrupt first-order synchronization transition appears. Two different paths leading to the synchronization also imply that there is a critical value γ_c, above which the first-order phase transition occurs, otherwise the second-order phase transition happens. Furthermore, for each oscillator i, we denote by $\mathcal{N}(i)$ the set of oscillators linked to it. In Figs. 2 and 3, the symbol $\langle\cdot\rangle$ indicates the average value over the ensemble $\mathcal{N}(i)$, that is, the average neighbor connectivity $\langle k_j \rangle = (1/k_i)\sum_{j\in\mathcal{N}(i)} k_j$ and the average neighbor frequency $\langle \omega_j \rangle = (1/k_i)\sum_{j\in\mathcal{N}(i)} \omega_j$.

The second significant result is the spontaneous emergence of degree-degree and frequency-frequency correlation features with the changeover of the order of phase transitions. By inspecting the average degree of each oscillator's neighbors, comparison between panels (a, b) and panels (c, d) of Fig. 2 manifests that condition (3) leads to the emergence of a

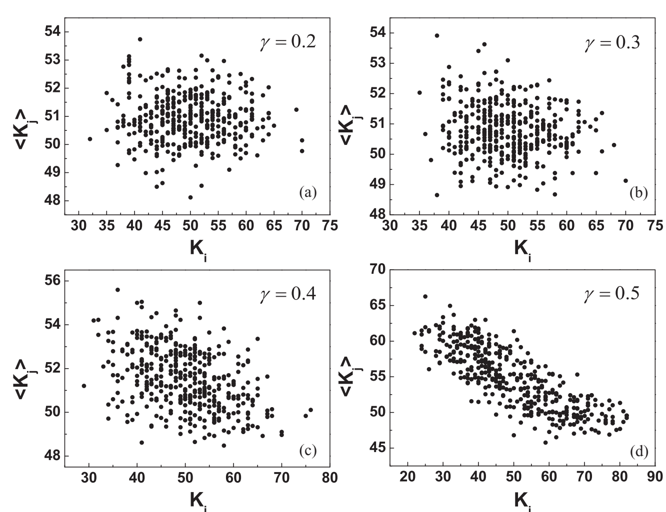

FIG. 2. The average neighbor connectivity $\langle k_j \rangle$ vs the degree k_i of the ith node for different values of γ. (a) $\gamma = 0.2$, (b) $\gamma = 0.3$, (c) $\gamma = 0.4$, (d) $\gamma = 0.5$.

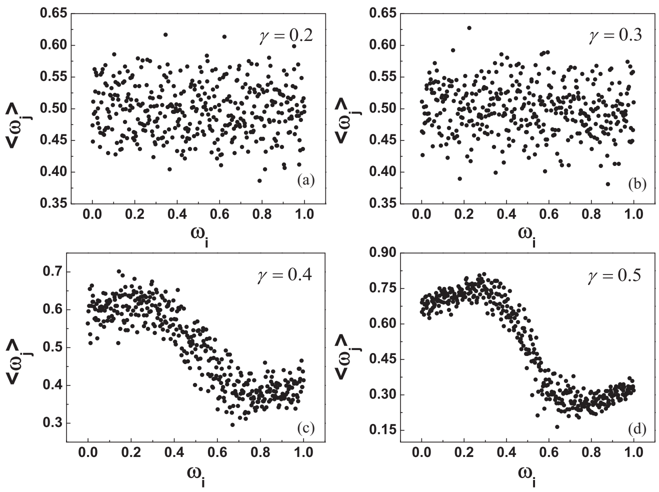

FIG. 3. The average neighbor frequency $\langle \omega_j \rangle$ vs the natural frequency ω_i of the ith node for different values of γ. (a) $\gamma = 0.2$, (b) $\gamma = 0.3$, (c) $\gamma = 0.4$, (d) $\gamma = 0.5$.

disassortative network where low-degree (high-degree) nodes are mainly coupled to high-degree (low-degree) nodes. Panels (a) and (b) of Fig. 3 refer to the cases $\gamma = 0.2$ and $\gamma = 0.3$, in which no frequency-frequency correlation is present. In panels (c) and (d) of Fig. 3, instead, we report the cases $\gamma = 0.4$ and 0.5 (the values for which a first-order phase transition occurs), which give evidence of the emergence of a very pronounced zigzag-shaped relationship between the frequency of the network's node and the average frequency of its neighbors. The results show that the high-frequency (low-frequency) nodes are prone to attaching to other low-frequency (high-frequency) nodes with the increase of γ.

IV. ASSORTATIVITY IN NETWORKS

In view of the analysis in Sec. III, it is natural for us to introduce the assortativity coefficients of degree and frequency to depict the topological structures and dynamical properties of the networks [41,42].

A network is said to show assortative (disassortative) mixing of structure if the nodes in the network that have many connections tend to be connected to other nodes with many (few) connections. In a disassortative network, the explosive transition persists, while in an assortative network the transition becomes a second-order one if the assortative coefficient of degree is large enough [24]. Here, we use the same form as Ref. [43] to characterize the frequency-frequency correlation of networks:

$$r_\omega = \frac{L^{-1}\sum_{i=1}^{L} p_i q_i - \left[L^{-1}\sum_{i=1}^{L}(p_i+q_i)/2\right]^2}{L^{-1}\sum_{i=1}^{L}\left(p_i^2+q_i^2\right)/2 - \left[L^{-1}\sum_{i=1}^{L}(p_i+q_i)/2\right]^2}, \tag{4}$$

where p_i and q_i are the natural frequencies of the nodes at the ends of the ith edge. Here $r_\omega > 0$ implies that the natural frequency of network is assortative, whereas it is disassortative.

Figure 4(a) displays the change trend of two kinds of assortativity coefficients with the increase of γ. One can see that a dramatic drop occurs when γ changes from 0.34 to 0.36, which is consistent with the findings in Figs. 2 and 3.

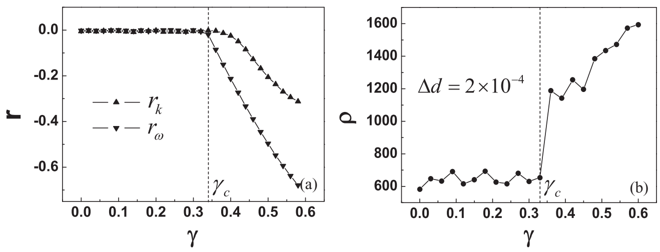

FIG. 4. (a) The assortativity coefficients (r_k and r_ω) of degree and natural frequency vs the continuous frequency gap γ. (b) The maximum slope ρ of order parameter as a function of γ.

Additionally, in order to detect the critical value of γ, we introduce a maximum slope of order parameter at different values of γ [24]. It is defined as follows:

$$\rho = \lim_{\Delta d \to 0} \left.\frac{\Delta R}{\Delta d}\right|_{\max}. \tag{5}$$

One may find that a sudden jump occurs when γ changes from 0.33 to 0.36, as shown in Fig. 4(b). We can conjecture that the critical value of γ is about 0.33.

Comparing with panels (a) and (b) of Fig. 4, we find that the two values of γ are very close. Are both phenomena inevitable or occasional? Does it mean an association between the onset of an explosive synchronization and the transition points of assortativities in degree and frequency? In the following section, we will continue our discussion.

In order to give a qualitative explanation for the emergence of an explosive synchronization in Fig. 1, we plot ω_i as a function of k_i, and a parabola-shaped relationship between the frequencies and the degrees of the network's nodes starts to take shape and becomes clearer with the increase of γ, as shown in Fig. 5. When $\gamma > \gamma_c$, the high-degree (low-degree) nodes locate at the ends (center) of the frequency spectrum. Considering degree-degree correlation, more coupling terms contribute to the high-degree nodes, but they are far away from

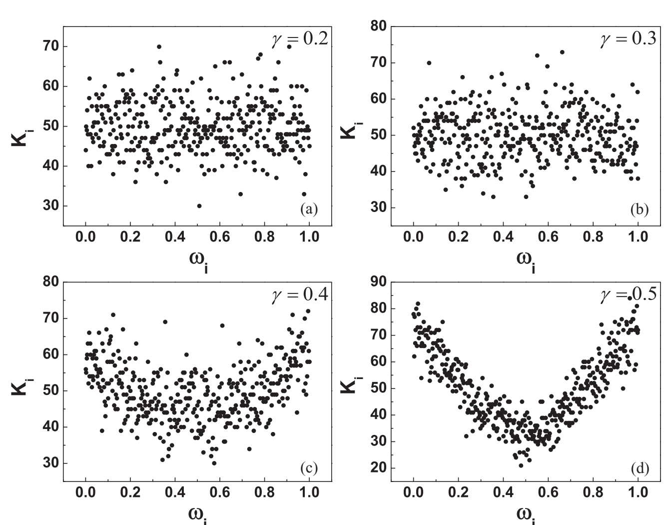

FIG. 5. The relationship between degree k_i and natural frequency ω_i of each node is obtained for different values of γ. (a) $\gamma = 0.2$, (b) $\gamma = 0.3$, (c) $\gamma = 0.4$, (d) $\gamma = 0.5$, after the network construction is completed.

the center of the frequency spectrum, while the low-degree nodes are in the opposite situation. The pattern of interaction inhibits a condensation center from generating and makes the oscillators either all drift or all phase-locked [44,45]. From the standpoint of frequency-frequency correlation, the emergence of an explosive synchronization is due to the high probability of connections with large frequency mismatches.

V. EFFECTS OF DIFFERENT MIXING PATTERNS

In order to reveal the relationship between the two phenomena in Fig. 4, we construct four types of networks with different degree-degree and frequency-frequency correlation patterns. (a) The model of a FGC random network is constructed according to the condition $\gamma = 0$. It falls back to the random graph of Erdös and Rényi (ER) [46]. (b) First, the model of a FGC random network is constructed according to the condition $\gamma = 0$. Then we intentionally swap natural frequencies of nodes, until $r_\omega \approx -0.3$ is obtained. (c) First, the model of a FGC random network is constructed according to the condition $\gamma = 0.5$. Then we use a randomly distributed natural frequency to replace the original frequency distribution. The operation keeps the degree sequence, and thus the heterogeneity of the network's degree remains unchanged. (d) The model of a FGC random network is constructed according to the condition $\gamma = 0.54$.

It is shown in Fig. 6 that the degree-degree and the frequency-frequency correlations of the networks have a significant impact on an explosive transition. The first-order phase transition arises due to the disassortativities in degree and natural frequency in the FGC networks. Return to Fig. 4(a), the dramatic drop can be regarded as the onset of an explosive synchronization, which is in good agreement with the simulation of Fig. 4(b).

From the above analysis, we can conclude that the disassortativities in degree and natural frequency are the underlying factors to excite an explosive synchronization. In order to verify our assertion, we adopt two kinds of control strategies.

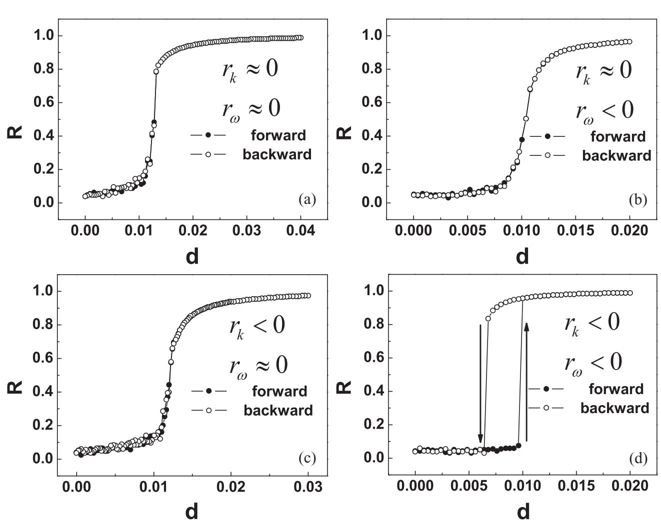

FIG. 6. Four types of mixing patterns (a) $r_k = -0.005258$, $r_\omega = -0.000536$; (b) $r_k = -0.001057$, $r_\omega = -0.299152$; (c) $r_k = -0.213452$, $r_\omega = 0.000797$; (d) $r_k = -0.269381$, $r_\omega = -0.593702$.

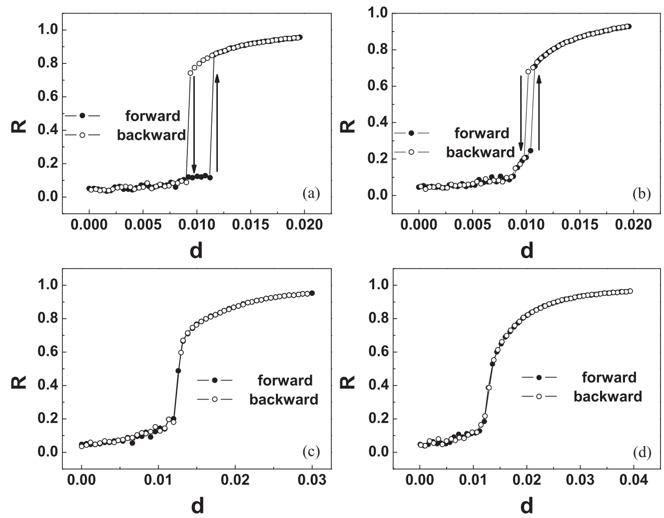

FIG. 7. Synchronization diagrams $R(d)$ for different assortativities in natural frequency. The values of r_ω in each panel are (a) $r_\omega = -0.286715$, (b) $r_\omega = -0.105545$, (c) $r_\omega = 0.100128$, (d) $r_\omega = 0.250056$. The four realizations share the same disassortativity in degree $r_k = -0.321414$.

On the one hand, a network with the disassortative degree is constructed according to the condition $\gamma = 0.58$. Then we intentionally swap natural frequencies of nodes, until the predetermined value of r_ω is obtained. The swap operation keeps the disassortative degree unchanged. As a result, the size of the hysteresis loop gradually shrinks until completely disappears with the increase of the assortativity in frequency (Fig. 7). On the other hand, a network with the disassortative frequency is constructed according to the initial condition $\gamma = 0.44$. Then we reshuffle the random network and keep its disassortative frequency unchanged [47]. As expected, a similar phenomenon happens with the increase of the assortativity in degree (Fig. 8). It is clear that the transition

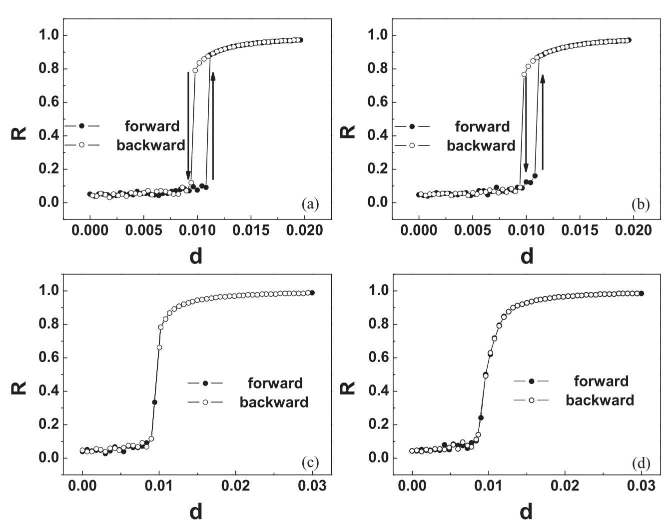

FIG. 8. Synchronization diagrams $R(d)$ for different assortativities in degrees. The values of r_k in each panel are (a) $r_k = -0.250372$, (b) $r_k = -0.149809$, (c) $r_k = 0.100203$, (d) $r_k = 0.200252$. The four realizations share the almost identical disassortativity in natural frequency $r_\omega \approx -0.305276$.

points of assortativities in degree and natural frequency are also the demarcation points of the order of phase transitions in the FGC random networks.

VI. CONCLUSIONS

In summary, we proposed a new scheme to induce an explosive synchronization and demonstrated a prerequisite for the occurrence of this phenomenon in networks of phase oscillators. It should be pointed out that the single disassortativity is not uniquely sufficient to determine the emergence of an explosive synchronization transition. The disassortativities in degree and natural frequency are indispensable to excite an explosive synchronization in the FGC random networks. It is worth mentioning that the two transition points of assortativities in degree and natural frequency almost overlap in the FGC random networks. In addition, the more disassortative a network is, the more easily an explosive synchronization may occur.

Our study generalizes some previous results and extends the possibility of encountering first-order phase transitions to a general standard of topological structures and dynamical properties. This suggests a feasible scheme for engineering networks able to display critical phenomena and the emergence of explosive transition in their macroscopic states. Finally, the evidence for the spontaneous emergence of degree-degree and frequency-frequency correlations in connection with these abrupt transitions may shed light on the microscopic roots behind these phenomena. Our findings will pave the way for the study with similar dynamical context in real-world networks.

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