Global interactions, information flow, and chaos synchronization

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We investigate the relationship between the emergence of chaos synchronization and the information flow in dynamical systems possessing homogeneous or heterogeneous global interactions whose origin can be external (driven systems) or internal (autonomous systems). By employing general models of coupled chaotic maps for such systems, we show that the presence of a homogeneous global field, either external or internal, for all times is not indispensable for achieving complete or generalized synchronization in a system of chaotic elements. Complete synchronization can also appear with heterogeneous global fields; it does not requires the simultaneous sharing of the field by all the elements in a system. We use the normalized mutual information and the information transfer between global and local variables to characterize complete and generalized synchronization. We show that these information measures can characterize both types of synchronized states and also allow us to discern the origin of a global interaction field. A synchronization state emerges when a sufficient amount of information provided by a field is shared by all the elements in the system, on the average over long times. Thus, the maximum value of the top-down information transfer can be used as a predictor of synchronization in a system, as a parameter is varied.

I. INTRODUCTION

Global interactions in a system occur when all its elements are subject to a common influence or field. Global interactions appear naturally in the description of many physical, biological, and social systems, such as coupled oscillators [1,2], Josephson junction arrays [3], charge density waves [4], multimode lasers [5], parallel electric circuits, neural dynamics, ecological systems, evolution models [6], economic exchange [7], social networks [8], mass media models [9], cross-cultural interactions [10], etc. A global interaction field may consist of an external environment acting on the elements, as in a driven dynamical system; or it may originate from the interactions between the elements, in which case, we talk of autonomous dynamical systems. In many cases, global interaction fields coexist with local or short-range interactions.

Although systems with global interactions possess a simple topological connectivity structure—a fully connected network—they can exhibit a variety of collective behaviors, such as chaos synchronization, dynamical clusters, nontrivial collective behavior, chaotic itinerancy [6,11], quorum sensing [12], etc. These behaviors have been studied in models of globally coupled maps [13] and have been experimentally investigated in globally coupled oscillators in chemical, physical, and biological systems [14–17].

In particular, chaos synchronization is a fundamental phenomenon in dynamical systems [18,19]. Its investigation has provided insights into many natural processes and motivation for practical applications such as secure communications and control of nonlinear systems [20–22]. Complete synchronization in a system of dynamical elements subject to a global interaction field, either external or autonomous, occurs when the state variables of all the elements and the global field converge to a single orbit in phase space. Generalized chaos synchronization, originally discovered in driven chaotic systems, arises when all the state variables of the elements

in the system get synchronized into an orbit that is different from that of the drive [23,24]. The concept of generalized synchronization of chaos has also been extended to the context of autonomous systems [25]. This means that the chaotic state variables in a dynamical system can synchronize to each other but not to a coupling function containing information from those variables.

The occurrence of both forms of chaos synchronization in driven and in autonomous systems with global interactions suggests that the nature, either external or endogenous, of the global field acting on the elements in a system is irrelevant. At the local level, each element in the system is subject to a field that eventually induces some form of synchronization between that field and the element. In general, the local dynamics in systems with global interactions can be seen as a single drive-response system [11,26]. In particular, if the time evolution of an external global field acting on a system is identical to that of an autonomous global field acting on a replica system, the corresponding local drive-response dynamics in both systems should be indistinguishable, and therefore the corresponding synchronized states are equivalent; i.e., they occur for the same parameter values in both systems [27].

In many systems it is important not only to detect synchronized or other collective states, but also to understand the relationships between global and local scales that lead to such behaviors. For example, it has recently been argued that top-down causation—where information flows from higher levels to lower levels in complex systems—may be a major contributor to evolutionary transitions and to the emergence of behaviors in living systems [28], and synchronization in neural systems has been described as a top-down information processing driven by a stimulus [29].

The above results suggest that the emergence of collective behaviors, such as a synchronized state, in a system is associated with the reception by its elements of some amount

of information provided by a source, either external or endogenous to the system. In this article we investigate the relationship between information flow between the global and local variables and the emergence of complete and generalized synchronization of chaos in dynamical networks with global interactions of different types. We employ information measures [30,31] that have been widely applied to quantify drive-response causal relationships between subsystems and interdependences between data sets in many fields of science, including linguistics [32], electroencephalographic signals [33], neuroscience [34], communication systems [35], dynamical systems [36], and climate networks [37]. We show that these information measures can characterize complete and generalized synchronized states and also allow us to discern the origin, either external or endogenous, of a global interaction field. A given synchronization state emerges when a sufficient amount of the information transmitted by a field is shared by all the elements in the system, on the average over long times. Thus, the maximum value of the top-down information transfer can be employed as a predictor of synchronization as a parameter in the system, such as the coupling strength to the field, is varied.

In Sec. II we present a general coupled map model for systems with external or endogenous global interactions. and define the quantities to characterize synchronized states and information flow in such systems. Homogeneous global interaction fields, which may act intermittently, are considered in Sec. III. We extend the concept of a global field to include heterogeneous global interactions in Sec. IV. Section V contains the conclusions of this work.

II. GLOBAL INTERACTION FIELDS

We describe a global interaction in a system as a field that can influence all the elements in the system. As a simple model of a dynamical system subject to a global interaction, we consider a system of N coupled maps of the form

$$
\begin{aligned}
x_{t+1}^i &= w\left(x_t^i, y_t\right), \\
y_{t+1} &= \phi\left(y_t, x_t^j\right),
\end{aligned}
\tag{1}
$$

where x_t^i ($i = 1, 2, \ldots, N$) represents the state variable of the ith map in the system at discrete time t, y_t is a global interaction field that can affect each map at time t, and $j \in Q$ where Q is a subset of elements in the system. Equation (1) describes a system of elements interacting with a common dynamical environment that can receive feedback from the system. For simplicity, we shall focus on the presence of global interactions and will not include local interactions.

An external global field y_t possesses its own dynamics, independent from the dynamics of the elements, given by

$$
\phi\left(y_t, x_t^j\right) = g(y_t).
\tag{2}
$$

On the other hand, an internal global field y_t can be represented by

$$
\phi\left(y_t, x_t^j\right) = h\left(x_t^j \big| j \in Q_t\right),
\tag{3}
$$

where h is a function of the states of a given subset Q_t of elements in the system at time t. The coupling function h may represent a constraint or a conservation law in the system.
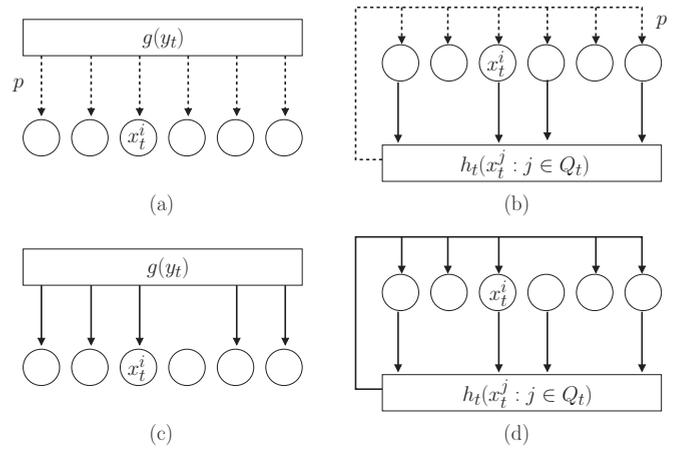

FIG. 1. Top panels: homogeneous global interactions. (a) External field $g(y_t)$ acting with probability p on all elements. (b) Internal field $h(x_t^j | j \in Q_t)$ acting with probability p on all elements. Bottom panels: heterogeneous global interactions. (c) External field $g(y_t)$ acting on a fraction p of elements chosen at random at every time. (d) Internal field $h(x_t^j | j \in Q_t)$ acting on a fraction p of elements chosen at random at every time.

We shall consider the coupling of the maps to the global interaction field in the diffusive form

$$
w\left(x_t^i, y_t\right) = (1 - \epsilon) f\left(x_t^i\right) + \epsilon \phi\left(y_t, x_t^j\right),
\tag{4}
$$

where f describes the local dynamics of the maps, and the parameter ϵ is the strength of the coupling to the global field. Since we are particularly interested in chaos synchronization, we choose for the local dynamics the logistic map $f(x_t^i) = 4x_t^i(1 - x_t^i)$, so that $f(x_t^i)$ is fully chaotic for $x_t^i \in [0,1]$. In this paper, we consider both driven and autonomous systems, subject to global interactions, whose schemes are illustrated in Fig. 1.

A. Synchronization states

Synchronization in the system Eq. (1) at a time t corresponds to a state $x_t^i = x_t^j$, $\forall\, i, j$. Thus, a synchronized state can be described by the condition $x_t^i = \bar{x}_t$, $\forall\, i$, where $\bar{x}_t$ is the instantaneous mean field of the system,

$$
\bar{x}_t = \frac{1}{N} \sum_{i=1}^{N} x_t^i.
\tag{5}
$$

To characterize the occurrence of synchronization, we shall consider the asymptotic time average $\langle \sigma \rangle$ (after discarding a number of transients) of the instantaneous standard deviations σ_t of the distribution of state variables x_t^i, defined as

$$
\sigma_t = \left[\frac{1}{N} \sum_{i=1}^{N} \left(x_t^i - \bar{x}_t\right)^2 \right]^{1/2}.
\tag{6}
$$

A synchronization state corresponds to $\langle \sigma \rangle = 0$. In addition, we define the asymptotic time average $\langle \delta \rangle$ (after discarding a number of transients) of the quantity

$$
\delta_t = |\bar{x}_t - y_t|.
\tag{7}
$$

Two forms of synchronization can take place in the system Eq. (1) in relation to the global field y_t: (i) *complete*

synchronization, given by the condition $x_t^i = \bar{x}_t = y_t$, i.e., all elements are synchronized to the field, and characterized by $\langle \sigma \rangle = 0$ and $\langle \delta \rangle = 0$; and (ii) *generalized synchronization*, corresponding to the condition $x_t^i = \bar{x}_t \neq y_t$, i.e., all elements are synchronized to each other but not the field, and described by $\langle \sigma \rangle = 0$ and $\langle \delta \rangle \neq 0$. It has been shown that both types of synchronization can occur in systems with global interactions, for either autonomous or driven systems [25]. In this paper we shall use the numerical criteria $\langle \sigma \rangle < 10^{-7}$ and $\langle \delta \rangle < 10^{-7}$ for characterizing the zero values of these quantities.

In order to characterize the information exchange between the global field and the local dynamics in the system, we consider the following quantities:

(1) the normalized mutual information between two variables y_t and x_t, based on Shanon's mutual information [30],

$$M_{y,x} = -\frac{\sum_{x_t,y_t} P(x_t,y_t) \log\left(\frac{P(x_t,y_t)}{P(x_t)P(y_t)}\right)}{\sum_{x_t} P(x_t) \log P(x_t)}; \qquad (8)$$

(2) the information transfer from a variable y_t to a variable x_t, defined as [31]

$$T_{y,x} = \sum_{x_{t+1},x_t,y_t} P(x_{t+1},x_t,y_t) \log\left(\frac{P(x_{t+1},x_t,y_t)P(x_t)}{P(x_t,y_t)P(x_{t+1},x_t)}\right), \qquad (9)$$

where $P(x_t)$ means the probability distribution of the time series of the variable x_t, $P(x_t,y_t)$ is the joint probability distribution of x_t and y_t, and so on. The quantity $M_{y,x}$ measures the overlap of the information content of the variables y_t and x_t; it represents how much the uncertainty about x_t decreases if y_t is known. The quantity $T_{y,x}$ measures the degree of dependence of x_t on the variable y_t; i.e., the information required to represent the value x_{t+1} from the knowledge of y_t. Note that the information transfer is nonsymmetrical, i.e., $T_{y,x} \neq T_{x,y}$. The normalized mutual information $M_{y,x}$ is symmetrical, i.e., $M_{y,x} = M_{x,y}$, and does not indicate the direction of the flow of information between two interacting dynamical variables, as $T_{y,x}$ does. When the two variables are synchronized, $x_t = y_t$. Then we obtain $M_{y,x} = 1$ and $T_{y,x} = 0$.

III. HOMOGENEOUS GLOBAL INTERACTIONS

We describe a homogeneous global interaction as a field shared simultaneously by all the elements in a system. Since, in general, the interaction with the field may not occur for all times, we consider a coupled map system subject to a homogeneous, intermittent, global interaction of the form

$$\forall\, i,\; x_{t+1}^i = \begin{cases} w(x_t^i, y_t) & \text{with probability } p, \\ f(x_t^i) & \text{with probability } (1-p). \end{cases} \qquad (10)$$

Each map in the system Eq. (10) is subject to the presence (or absence) of the same influence at any time. Then, the occurrence of complete or generalized synchronization between a local map and the global field y_t implies the same form of synchronization between the mean field of the system $\bar{x}_t$ and y_t, regardless of the nature, either external or endogenous, of the global field y_t.

A system subject to a homogenous external field [Fig. 1(a)], corresponds to

$$\forall\, i, x_{t+1}^i = \begin{cases} (1-\epsilon)f(x_t^i) + \epsilon g(y_t) & \text{with probability } p, \\ f(x_t^i) & \text{with probability } (1-p), \end{cases}$$
$$y_{t+1} = g(y_t). \qquad (11)$$

The auxiliary system approach introduced in Ref. [24] implies that a driven map can synchronize on identical orbits with another, identically driven map. The system Eq. (11) can be regarded as one of multiple realizations for different initial conditions of a single, intermittently driven map. Thus, by extension, the elements in this system should synchronize with the external field in the same form as a single local map driven by that field does.

A complete synchronized state in the system Eq. (11) is given by $x_t^i = \bar{x}_t = y_t$, $\forall\, i$, and it can occur when the external field is equal to the local dynamics, $g = f$. If $g \neq f$, generalized synchronization, characterized by the condition $x_t^i = \bar{x}_t \neq y_t$, $\forall\, i$, may also arise in this system.

On the other hand, a system subject to an autonomous homogeneous global field [Fig. 1(b)] can be described as

$$\forall\, i,\; x_{t+1}^i$$
$$= \begin{cases} (1-\epsilon)f(x_t^i) + \epsilon h(x_t^j \mid j \in Q_t) & \text{with probability } p, \\ f(x_t^i) & \text{with probability } (1-p), \end{cases} \qquad (12)$$

where Q_t is a subset consisting of $q \leqslant N$ elements of the system that may be chosen at random at each time t. Each map receives the same input from the endogenous global field $y_t = h$ at any t with probability p. Complete synchronization in the system Eq. (12) occurs when $f(x_t^i) = f(\bar{x}_t) = h$, while generalized synchronization appears if $f(x_t^i) = f(\bar{x}_t) \neq h$, $\forall\, i$.

A. Complete synchronization

As examples of complete chaotic synchronization in systems having homogeneous global interactions, we consider the driven system Eq. (11) with $g = f$, and the autonomous system Eq. (12) subject to a partial mean field coupling function defined as

$$h(x_t^j \mid j \in Q_t) = \frac{1}{q} \sum_{j=1}^{q} f(x_t^j), \qquad (13)$$

where $q \leqslant N$ maps are randomly chosen at each time t. For these systems, the condition $\langle \delta \rangle = 0$ implies $\langle \sigma \rangle = 0$ and, therefore, complete synchronization.

Figure 2(a) shows the quantity $\langle \delta \rangle$ as a function of the coupling parameter ϵ, for both the homogeneous driven system and the homogeneous autonomous system, with fixed values of p and q/N. Complete synchronization for both systems takes place at a critical value $\epsilon_c = 0.579$, for which $\langle \delta \rangle < 10^{-7}$.

Figures 2(b) and 2(c) show, respectively, the normalized mutual information M_{y_t,x_t^i} and the information transfer T_{y_t,x_t^i} between the homogeneous global field and one map, averaged over 50 randomly chosen maps, for both systems as functions of ϵ. These averaged quantities give practically the same
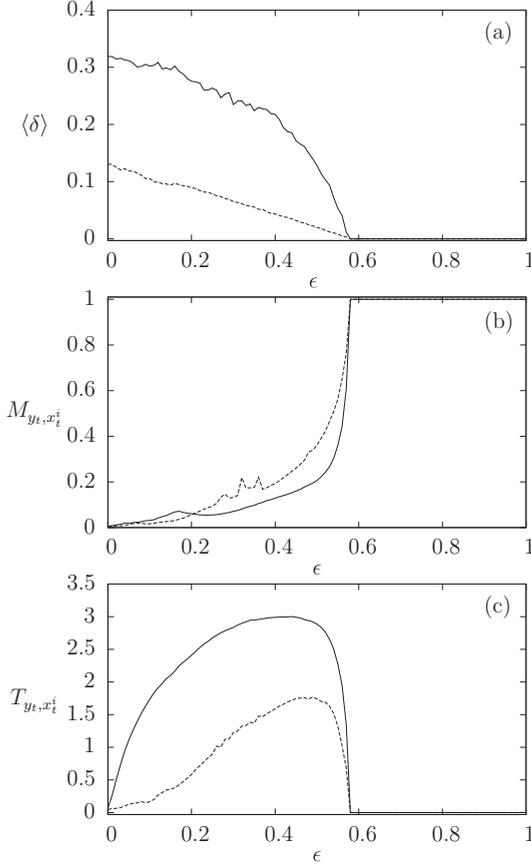
FIG. 2. Complete chaos synchronization in systems with homogeneous global fields. (a) $\langle \delta \rangle$ vs ϵ, (b) mutual information M_{y_t,x_t^i} vs ϵ, and (c) information transfer T_{y_t,x_t^i} vs ϵ. On each panel, the continuous line corresponds to the homogeneous driven system Eq. (11) with $g = f$ and the dashed line corresponds to the homogeneous autonomous system Eqs. (12) and (13). Both information measures are calculated with 2×10^5 points in the time series, after discarding transients, and averaged over 50 randomly chosen maps. The number of states used to calculate the corresponding probability distributions is 100. The same conditions are used in Figs. 3 and 5. Fixed parameters: $p = 0.8$, $N = 10^4$, $q/N = 0.4$.

result as for just one randomly chosen map. The results shown are also independent of q for large enough system system size N. We observe that, as the coupling strength ϵ increases, the global field and the local variables become more correlated, and the normalized mutual information for both systems increases until $M_{y_t,x_t^i} = 1$ at the value ϵ_c. In the complete synchronization region $\epsilon \geqslant \epsilon_c$, we find the constant values $M_{y_t,x_t^i} = 1$ and $T_{y_t,x_t^i} = 0$ for both systems, signaling complete synchronization in each case. Once complete chaos synchronization is established, the evolution of the global field, regardless of its source, is identical to that of the maps. Thus, the mutual or the transfer information cannot distinguish between the driven and the autonomous systems in a regime of complete synchronization.

On the other hand, just before vanishing at the critical value ϵ_c, the information transfer for both systems becomes maximum. This indicates that, as the critical values of the parameters for the onset of complete chaos synchronization are approached, the flow of information from the global field

to the local maps must be large. Figure 2(c) shows that the maximum value of the information transfer for the driven system is greater than the corresponding maximum value for the autonomous system. Thus, in the vicinity of parameter values for the emergence of complete synchronization, an autonomous global field needs to convey less information to the local maps than an external driving field. This suggests that the information transfer T_{y_t,x_t^i} can serve as a predictor of a state of complete synchronization in the parameter space of driven and autonomous systems with homogenous global interactions. Moreover, this quantity can distinguish between these two types of systems near the onset of complete synchronization.

B. Generalized synchronization

If the functional form of the global field is different from that of the local dynamics, generalized synchronization may occur in a system subject to a homogeneous global interaction. For example, consider an external field in a driven system Eq. (11) such as

$$g(y_t) = \frac{\mu}{2}(1 - |2y_t - 1|), \tag{14}$$

with $\mu = 1.98$ and $y_t \in [0,1]$. Then at synchronization in the driven system Eqs. (11) and (14), we have $x_t^i = \bar{x}_t \neq y_t$. Similarly, in an autonomous system Eqs. (12), consider a homogeneous global interaction different from a mean field, such as the coupling function

$$h\big(x_t^j \big| j \in Q_t\big) = \frac{\mu}{2}\left[1 - \left| 2 \left(\frac{1}{q} \sum_{j=1}^{q} x_t^j \right) - 1 \right| \right], \tag{15}$$

with $\mu = 1.98$, where $q \leqslant N$ elements are chosen at random at each time t. Then, a synchronized state in the autonomous system Eqs. (12) and (15) corresponds to $f(x_t^i) = f(\bar{x}_t) \neq h$. For the fields chosen above, the functional form of the autonomous field in a synchronized state is similar to that of the drive, $h = g(\bar{x}_t)$. However, the time evolution of h at synchronization is not necessarily identical to that of $g(y_t)$.

Figure 3(a) shows the quantity $\langle \sigma \rangle$ as a function of the coupling parameter ϵ for both systems with homogeneous global interactions, the driven system with g given by Eq. (14) and the autonomous system with h given by Eq. (15). These systems get synchronized at different values of ϵ for which $\langle \sigma \rangle < 10^{-7}$. The inset in Fig. 3(a) shows that the quantity $\langle \delta \rangle$ for both systems does not vanish when ϵ is varied, indicating that the synchronized state in both cases corresponds to generalized synchronization.

Figure 3(b) shows M_{y_t,x_t^i} for both systems, as a function of ϵ. In contrast to the constant value $M_{y_t,x_t^i} = 1$ exhibited by the normalized mutual information for both systems in a state of complete synchronization [Fig. 2(b)], the behavior of M_{y_t,x_t^i} in the regime of generalized synchronization is different for each system. The normalized mutual information for the autonomous system in a generalized synchronized state reaches an almost constant value $M_{y_t,x_t^i} = 0.695 < 1$, since the time series of the local maps and the coupling function h are not identical. For the driven system, M_{y_t,x_t^i} increases monotonically with increasing ϵ, but the values of M_{y_t,x_t^i} are below the value of this quantity for the autonomous system
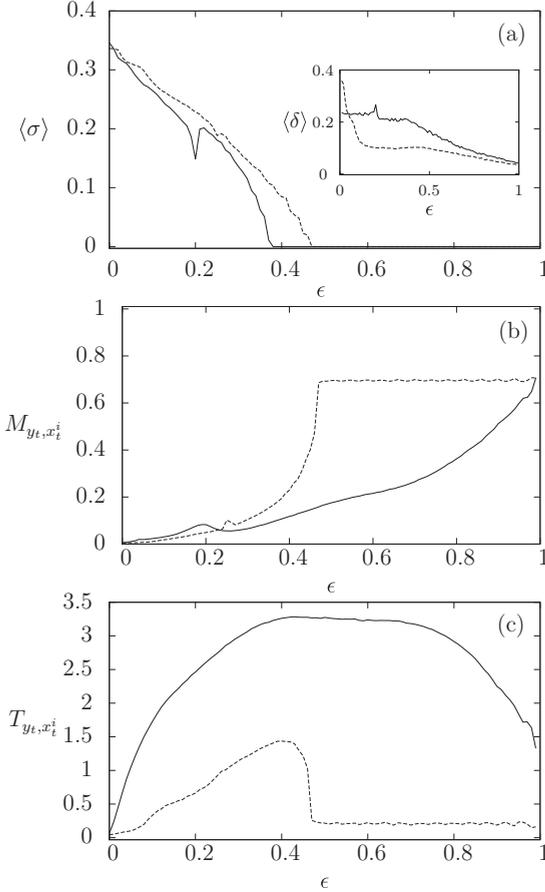
FIG. 3. Generalized chaos synchronization in systems with homogeneous global fields. (a) $\langle\sigma\rangle$ vs ϵ (inset: $\langle\delta\rangle$ vs ϵ); (b) M_{y_t,x_t^i} vs ϵ; and (c) T_{y_t,x_t^i} vs ϵ. On each panel, the continuous line corresponds to a homogeneously driven system Eq. (11) with g given in Eq. (14) and the dashed line corresponds to the homogeneous autonomous system Eqs. (12) and (15). Fixed parameters are $p = 0.8$, $N = 10^4$, $q/N = 0.4$.

in the region of generalized synchronization. Therefore, in a generalized synchronization state, the amount of information shared between the field h and the local maps in the autonomous system is greater than that between the external field g and the maps in the driven system.

Figure 3(c) shows the information transfer T_{y_t,x_t^i} versus ϵ for both systems. Similarly to the behavior observed for complete synchronization, as the coupling strength approaches the critical value ϵ_c for the emergence of generalized synchronization, the information transfer in the autonomous system becomes maximum. Also, the values of T_{y_t,x_t^i} for the driven system are greater than the values of this quantity for the autonomous system. However, in the generalized chaos synchronization regime, for $\epsilon > \epsilon_c$, the information transfer in both systems does not vanish; and the values of T_{y_t,x_t^i} for the driven system are greater than the values of this quantity for the autonomous system. This means that the autonomous field must provide less information to the local maps than an external drive for sustaining generalized synchronization. This behavior should be expected since the autonomous field h already contains information about the dynamics of the elements in the system. At the onset of generalized synchronization, both T_{y_t,x_t^i} and

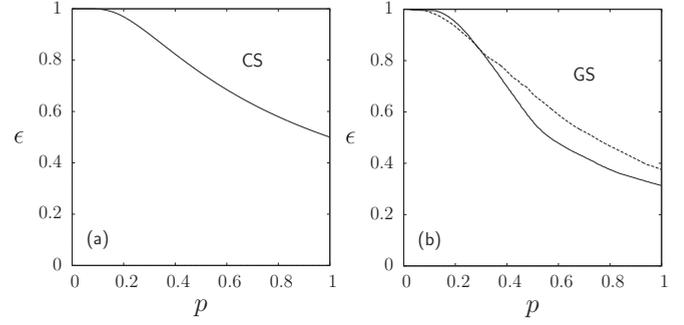FIG. 4. Regions for chaos synchronization on the plane (p,ϵ) for systems with homogeneous global interactions. (a) Complete synchronization (CS) for both the homogeneous driven system Eq. (11) with $g = f = 4x(1 - x)$, and the homogeneous autonomous system Eqs. (12) and (13). The boundary of the region where complete synchronization takes place is given by $\epsilon = 1 - e^{-\lambda_f/p}$, with $\lambda_f = \ln 2$ for the map f. (b) Generalized synchronization (GS) for both the homogeneously driven system Eqs. (11) and (14) (continuous line) and for the homogeneous autonomous system Eqs. (12) and (15) (dashed line), with $N = 10^4$, $q/N = 0.4$.

M_{y_t,x_t^i} for the driven system are continuous while they are discontinuous for the autonomous system. Thus, the quantities M_{y_t,x_t^i} and T_{y_t,x_t^i} can distinguish between the driven and the autonomous systems in a state of generalized synchronization, in contrast to the case of complete synchronization.

C. Dynamics at the local level

At the local level in a system with a homogeneous global field, each element is subject to a field that eventually induces some form of synchronization between that element and the field, similarly to a single master-slave system. Thus, the local dynamics can be seen as a single drive-response map system where a drive g acts with probability p on a map f. In particular, the linear stability analysis of the complete synchronized state for the single driven map yields the condition [25]

$$p \ln |1 - \epsilon| + \lambda_f < 0, \qquad (16)$$

where λ_f is the Lyapunov exponent of the map f. A stable completely synchronized state occurs when this condition is fulfilled. On the other hand, a stable generalized synchronized state in both kinds of homogeneous system can be numerically determined with the criterion $\langle\sigma\rangle < 10 < -7$ on the space of parameters (p,ϵ).

Figure 4 shows the regions where complete and generalized synchronization can be found on the plane (p,ϵ) for the systems with homogeneous global interactions considered here. The region of parameters for complete synchronization is the same for both the autonomous and the driven systems. The regions corresponding to generalized synchronization are not identical for these systems with the chosen functional forms of their global fields.

IV. HETEROGENEOUS GLOBAL INTERACTIONS

The concept of a global field can be extended beyond the concept of spatial homogeneity. In this respect, we consider a

system with heterogeneous global interactions, as follows:

$$x_{t+1}^i = \begin{cases} w(x_t^i, y_t) & \text{if } i \in R_t, \\ f(x_t^i) & \text{if } i \notin R_t. \end{cases} \quad (17)$$

where R_t is a subset containing pN elements of the system, with $p \leqslant 1$, which may be chosen at random at each time t. Thus, the average fraction of elements coupled to the field in Eq. (17) at any given time is p, so that not all the maps in the system receive the same influence at all times. In comparison, the coupling of the elements to the field in systems with homogeneous global interactions, Eq. (10), is simultaneous and uniform; each map receives the same influence from the field y_t at any t with probability p. At the local level, each map in the system with heterogeneous global interactions, Eq. (17), is subject, on the average, to the global field y_t with probability p over long times. For $p = 1$, the homogeneous system Eq. (10) and the heterogeneous system Eq. (17) are identical.

In the case of an external field [Fig. 1(c)], Eq. (17) takes the form

$$x_{t+1}^i = \begin{cases} (1 - \epsilon)f(x_t^i) + \epsilon g(y_t) & \text{if } i \in R_t, \\ f(x_t^i) & \text{if } i \notin R_t, \end{cases} \quad (18)$$
$$y_{t+1} = g(y_t).$$

For an autonomous field [Fig. 1(d)], the coupled map system Eq. (17) becomes

$$x_{t+1}^i = \begin{cases} (1 - \epsilon)f(x_t^i) + \epsilon h(x_t^j : j \in Q_t) & \text{if } i \in R_t, \\ f(x_t^i) & \text{if } i \notin R_t, \end{cases} \quad (19)$$

where, again, Q_t is a subset consisting of $q \leqslant N$ elements of the system that may be chosen at random at each time t. Each map in Eq. (19) is subject, on the average, to the same coupling function h with probability p over long times. The same condition holds for each map with respect to the drive g in the heterogeneously driven system Eq. (18). Then, if g exhibits the same temporal evolution as h, the synchronization behavior of the autonomous system Eq. (19) should be similar to the behavior of the driven system Eq. (18) over long times.

A. Complete synchronization

When the heterogeneously driven system Eq. (18) gets synchronized, we have $x_t^i = \bar{x}_t$. However, the synchronized solution exists only if $g = f$. Therefore, only complete synchronization $x_t^i = \bar{x}_t = y_t$ can take place in this system. On the other hand, a synchronized state in the heterogeneous autonomous system Eq. (19) occurs when $f(x_t^i) = f(\bar{x}_t)$. However, this synchronized solution exists only if $h = f(\bar{x}_t)$. Therefore, as in the case of the heterogeneous driven system, only complete synchronization, where $f(x_t^i) = f(\bar{x}_t) = h$, can emerge in the heterogeneous autonomous system Eq. (19). As an example of a coupling function $h(x_t^j : j \in Q_t)$ leading to complete synchronization in the heterogeneous autonomous system Eq. (19), we choose the partial mean field Eq. (13).

Figures 5(a)–5(c) show the quantities $\langle \delta \rangle$, M_{y_t, x_t^i}, and T_{y_t, x_t^i}, respectively, as functions of ϵ for both heterogeneous systems, driven and autonomous, with global interactions. Both systems
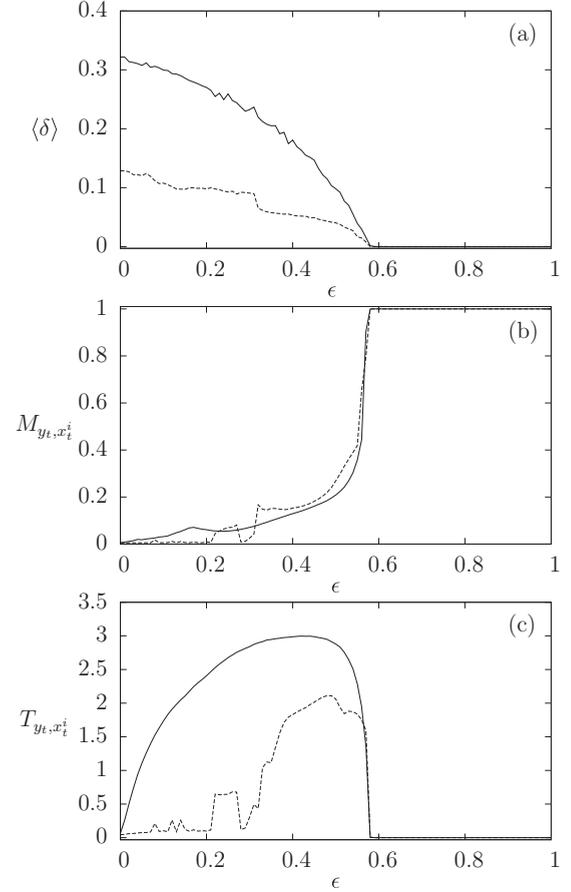

FIG. 5. Complete chaos synchronization in systems with heterogeneous global fields. (a) $\langle \delta \rangle$ vs ϵ; (b) M_{y_t, x_t^i} vs ϵ; and (c) T_{y_t, x_t^i} vs ϵ. On each panel, the continuous line corresponds to the heterogeneously driven system Eq. (18) with $g = f$, and the dashed line corresponds to the heterogeneous autonomous system Eqs. (19) and (13). Fixed parameters: $p = 0.8$, $N = 10^4$, $q/N = 0.4$.

reach complete chaos synchronization at the critical value $\epsilon_c = 0.579$.

The information transfer in Fig. 5(c) becomes maximal previous to the synchronization threshold, similarly to the behavior observed in homogeneous systems. Thus, a maximum in the information transfer T_{y_t, x_t^i} in the space of parameters can be regarded as a precursor to a state of synchronization, either complete or generalized. Figures 2(c), 3(c), and 5(c) reveal that a lesser amount of information flow from the global field to the local maps is necessary for the emergence of synchronization in autonomous systems, in comparison to that required for synchronization in driven systems possessing similar functional forms of their global fields and identical parameter values.

In either homogeneous or heterogeneous autonomous systems, complete synchronization occurs independently of the number q of elements randomly chosen in the function h, or if the q chosen elements are always the same. Thus, the reinjection of an autonomous coupling function h, although containing partial information about the system, to a fraction of randomly selected elements suffices to achieve complete synchronization. If the elements in subset R_t receiving the coupling function h or the drive g are always the same, then

only elements in this subset reach complete synchronization, since only those elements share the same information, on the average.

V. CONCLUSIONS

We have investigated the relationship between the emergence of synchronization and the information flow in dynamical systems possessing global interactions. We have used the normalized mutual information M_{y_t,x_t^i} and the information transfer T_{y_t,x_t^i} between global and local variables to characterize complete and generalized synchronization in models of coupled chaotic maps for such systems.

We have found that the presence of a homogeneous global field, either external or internal, for all times is not indispensable for achieving complete or generalized synchronization in a system of chaotic elements. Complete synchronization can also appear with heterogeneous global fields; it does not requires the simultaneous sharing of a global field by all the elements in the system. Furthermore, the global coupling function in autonomous systems does not need to depend on all the internal variables for reaching synchronization and, in particular, its functional form is not determinantal for generalized synchronization.

In both systems with homogeneous or heterogeneous global fields, at the local level each element is subject, on the average, to a field that eventually induces some form of synchronization between that element and the field, similarly to a single drive-response system. Then, a set of elements identical to the response and subject to a global field that behaves as the drive also synchronizes in a similar manner.

What becomes essential for the emergence of a given synchronization state is that all the elements in the system share a sufficient amount of information provided by a field, on the average, over time. This amount is characterized by the maximum value of the information transfer T_{y_t,x_t^i} previous to the critical values of parameters for either complete or generalized synchronization. Therefore, the quantity T_{y_t,x_t^i} could be employed to anticipate the occurrence of a state of synchronization in the space of parameters of a system possessing a global interaction field. Furthermore, the form in which information flows from macroscopic to microscopic scales for the emergence of synchronization, as measured by the quantities M_{y_t,x_t^i} and T_{y_t,x_t^i}, differs between a driven and an autonomous system with global interactions, even if they have similar functional forms for their local dynamics or for their global fields. In summary, we have found that (i) near the onset of complete synchronization when a parameter is varied, the maximum of the information transfer T_{y_t,x_t^i} for a driven system is greater than that for an autonomous system; (ii) near the onset of generalized synchronization, the normalized mutual information M_{y_t,x_t^i} and T_{y_t,x_t^i} exhibit sharp changes for an autonomous system, while these quantities exhibit a smooth behavior for a driven system; and (iii) in a state of generalized synchronization, T_{y_t,x_t^i} is greater for a driven system than for an autonomous system and M_{y_t,x_t^i} is smaller for a driven system than for an autonomous system.

Our results suggest that these information measures could be used to characterize, and possibly also to predict, other forms of collective behaviors observed in dynamical systems having global interactions. Further extensions of this work include the investigation of the relationship between top-down information flow between global and local scales, and the emergence of collective behaviors and structures in more complex dynamical networks.

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