Restricted solid-on-solid model with a proper restriction parameter N in $4+1$ dimensions

Jin Min Kim and Sang-Woo Kim
Department of Physics and Institute for Integrative Basic Sciences, Soongsil University, Seoul 156-743, Korea
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A restricted solid-on-solid growth model is studied for various restriction parameters N in $d = 4+1$ dimensions. The interface width W grows as t^{β} with $\beta = 0.158 \pm 0.006$ and W follows $W \sim L^{\alpha}$ at saturation with $\alpha = 0.273 \pm 0.009$, where L is the system size. The dynamic exponent $z \approx 1.73$ is obtained from the relation $z = \frac{\alpha}{\beta}$. The estimated exponents satisfy the scaling relation $\alpha + z = 2$ very well. Our results indicate that the upper critical dimension of the Kardar-Parisi-Zhang equation is larger than $d = 4+1$ dimensions. With a proper choice of the restriction parameter N, we can reduce the discrete effect of the height to the width and obtain the values of the exponents accurately.

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I. INTRODUCTION

Over the last several decades, there has been great interest in the surface roughening of nonequilibrium interfaces [1–4]. It is related to a wide variety of phenomena, such as interfaces of a burning paper, domain walls in the two-dimensional random bond Ising model [5], a randomly stirred fluid [6], ballistic aggregation [7], and directed polymers in random potentials [8,9].

One of the interesting quantities under stochastic growth is interface width W, which characterizes the roughness of the interface. W is defined as the standard deviation of the surface height:

$$W(L,t) \equiv \overline{\langle [h(x,t) - \overline{h}(t)]^2 \rangle}^{1/2}, \tag{1}$$

where $h(x,t)$ is the local height variable of the surface. $\overline{A}$ represents the spatial average of A at time t, and $\langle A \rangle$ denotes the average over many samples.

In general, the surface structure of many growth processes is self-affine and the surface configurations have a scaling behavior. Starting from a flat initial condition, the interface width grows in time following a power law $W(L,t) \sim t^{\beta}$ in the early time of growth and it saturates in the late time. The saturation width exhibits a power law $W(L,t) \sim L^{\alpha}$, with L being the linear size of the system. The interface width $W(L,t)$, therefore, follows the Family-Visek scaling formula [10]

$$W^2(L,t) = L^{2\alpha} f(t/L^z) \to t^{2\beta} \quad \text{for } t \ll L^z \\ \to L^{2\alpha} \quad \text{for } t \gg L^z, \tag{2}$$

where β and α are the growth exponent and the roughness exponent, respectively. Since the scaling function behaves as $f(x) \sim x^{2\beta}$ for $x \ll 1$ and constant for $x \gg 1$, the dynamic exponent z has a relation $z = \alpha/\beta$.

Much of the theoretical effort has been focused on the study of roughening surfaces for various growth models which are related to Langevin-type equations. For example, a restricted solid-on-solid (RSOS) model [11] is well described by the Kardar-Parisi-Zhang (KPZ) equation [12]. It takes into account the nonlinearities of the model as

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(\vec{r},t), \tag{3}$$

where η is a random variable that satisfies

$$\langle \eta(\vec{r},t)\eta(\vec{r}',t') \rangle = 2D\delta(\vec{r} - \vec{r}')\delta(t - t'), \tag{4}$$

with D describing local variations of noise.

The invariance of the KPZ equation under an infinitesimal tilt of the interface yields a scaling relation $\alpha + z = 2$ [12]. So, there is only one independent exponent to be determined. Most of the recent effort on this problem has been devoted to verifying these exponents. In $d = 1+1$, which means one substrate dimension and one height dimension, the exponents $\beta = 1/3$ and $z = 3/2$ are known [12]. The nonlinear processes controlling the KPZ equation are not yet completely understood. In particular, the scaling exponents in higher dimensions are still under debate [9,11,13–22]. Based on the computer simulation results of the RSOS model, one of us suggested $\beta = \frac{1}{d+1}$ [11]. There are some variations among the values of the exponents quoted by various authors in higher dimensions [17–27].

Another interesting subject is to look for phase transitions between a strong-coupling fixed point and an Edwards-Wilkinson-type trivial point [28]. In $d = 2+1$ there is no phase transition and only the strong-coupling phase exists. There is a controversy about the existence of the phase transition in $d = 4+1$. Some mode coupling calculations of the KPZ equation suggest that the upper critical dimension is less than or equal to $4+1$ [13,14], where the exponents take the values $z = 2$ and $\alpha = 0$. However, the simulation result of the RSOS model [15,26] and the nonperturbative renormalization group calculation for the KPZ equation [22] show the existence of a strong-coupling regime. The existence of an upper critical dimension is still under debate.

Here, we present a more detailed analysis of the numerical data on the RSOS model with a proper choice of restriction parameter N specifically in $d = 4+1$. By reducing the discrete height effect to the surface width, we can obtain accurate values of the exponents β, α, and z. Also, our result insists that the upper critical dimension of the KPZ equation should be larger than $d = 4+1$ dimensions.

II. RESTRICTED SOLID-ON-SOLID MODEL WITH VARIOUS RESTRICTION PARAMETER N

The dynamic rule of the discrete RSOS model [11] is to randomly select a site r on a substrate and then to add a

particle $h(r) \to h(r) + \delta h$ within the solid-on-solid condition, provided that the restriction on the local height difference

$$|\nabla h| = |h(r) - h(r')| \leqslant N \tag{5}$$

is obeyed at all sites on the substrate where r' is the nearest neighbor site of r. If this RSOS condition is not satisfied, the corresponding deposition event is forbidden. No relaxation or hopping of the deposited atom is allowed [11]. There exist two independent parameters: δh is a discrete unit of the height and N is a restriction parameter which confines the height difference between the nearest neighbors. Without loss of any generality, we choose the discrete unit of the height as $\delta h = 1$ and carry out Monte Carlo simulations of the RSOS model for various N. The height restriction parameter N, which directly controls the interface width, has a strong influence on the exponents obtained from simulations in higher dimensions specifically for $N = 1$. With a proper choice of N, here we could obtain the exponents accurately. Also the effect of N to the width is discussed.

III. SURFACE WIDTH

Starting from a flat initial condition on four-dimensional hypercubic substrates with a periodic boundary condition, we performed simulations for various values of the height difference restriction parameter N. The simulation time is defined in units of the number of trial Monte Carlo steps. To determine the growth exponent β, we measured $W^2(t)$ as a function of time. Plotted in Fig. 1 is the mean-square surface width, $W^2(t)$ for $L = 128$, averaged over 500 independent runs for various N. Notice that, except for the case of $N = 1$, the curves in log-log plot have quite similar slopes which are the values of 2β. One can estimate β from the relation $W^2(t) \sim t^{2\beta}$.

The RSOS model with $N = 1$ seems to yield accurate values of the exponents in $1 + 1$ dimensions [11]. However, in higher dimensions such as $d = 4 + 1$ dimensions, the width for $N = 1$ shows clear oscillatory behavior at the early time due to the discrete unit of the height where $W^2(t)$ is not larger than one up to very long time. The oscillatory behavior becomes stronger when the width is relatively small compared to the unit height $\delta h = 1$. If we consider a perfect layer by layer growth, then the surface width becomes $1/2$ for half-filled layers and zero for filled layers. The remnant of this effect in the RSOS model produces the oscillation behavior which prevents us from measuring the growth exponent accurately. We obtain $\beta = 0.137 \pm 0.009$ with simple fitting function $W^2(t) \sim t^{2\beta}$ for $N = 1$. This value is consistent with the previous results $\beta(N = 1) \approx 0.145 \sim 0.150$ for the RSOS model on larger system sizes [25,26]. Also, the data of $W^2(t)$ in log-log plot are slightly curved upwards for $N = 1$. Due to the discrete height effect, we think the estimated β using the interface width for $N = 1$ tends to underestimate the true value of β.

Since the height h can take only integer value, we should look at the regime $W^2 > 1$ to reduce the artifact of the discrete height. We study the model for various N to eliminate the oscillation behavior. For $N \geqslant 3$, $W^2(t)$ becomes larger than 1 as shown in Fig. 1. There is an initial region dominated by a random deposition exponent $\beta = 1/2$, followed by a quick crossover towards a good power law behavior without any oscillation. We calculate the effective running exponent $\beta_{\text{eff}}(t)$, defined by $\frac{\partial \ln W(t)}{\partial \ln t}$ against t, which should approach the true exponent β for $1 \ll t \ll L^z$ as shown in Fig. 2. For both $N = 1$ and $N = 3$, the $\beta_{\text{eff}}(t)$ increases with time and then decreases due to the finite size effect. However, it decreases with time for $N = 5$ and 7. The effective β remains constant for quite a long time for $N = 4$. Furthermore, Fig. 2 suggests that there exists a value of N for which the leading correction to the scaling $[W^2(t) \sim t^{2\beta}]$ is quite small. For this value of $N = 4$ the asymptotic behavior sets in for relatively short times. The width has a negligible discrete effect and shows a good straight line in log-log plot. Fitting to the relation $W^2(t) \sim t^{2\beta}$ without any correction terms for the data in the early time, provides

$$\beta(N = 4, d = 4 + 1) = 0.158 \pm 0.006. \tag{6}$$

Figure 2 indicates that there is an approaching value of $\beta \approx 0.158$ for $1 \ll t \ll L^z$. One can believe that the value of N only changes the short wavelength fluctuations which should be irrelevant as far as the scaling exponents are concerned. We also try to fit the data using a scaling form with a correction

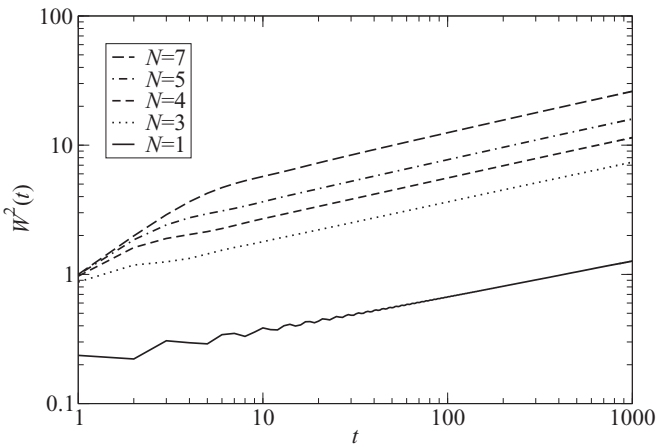

FIG. 1. The data for the interface width $W^2(t)$ as a function of time t plotted on a logarithmic scale for $N = 1$, 3, 4, 5, and 7 in $4 + 1$ dimensions.

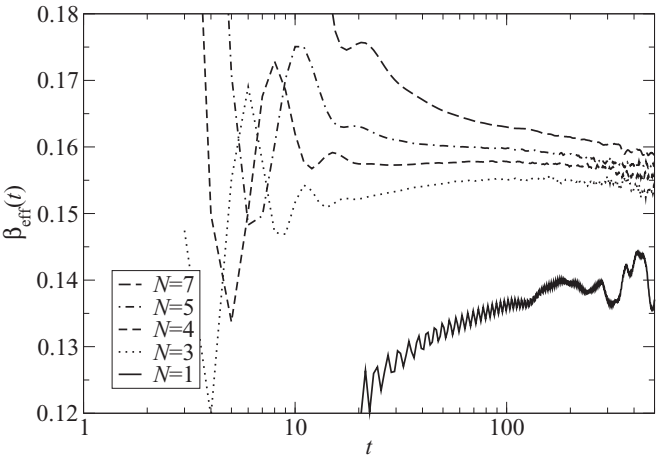

FIG. 2. The effective exponent $\beta_{\text{eff}}(t)$ as a function of t for various values of N in $d = 4 + 1$ (from bottom to top: $N = 1$, 3, 4, 5, and 7).

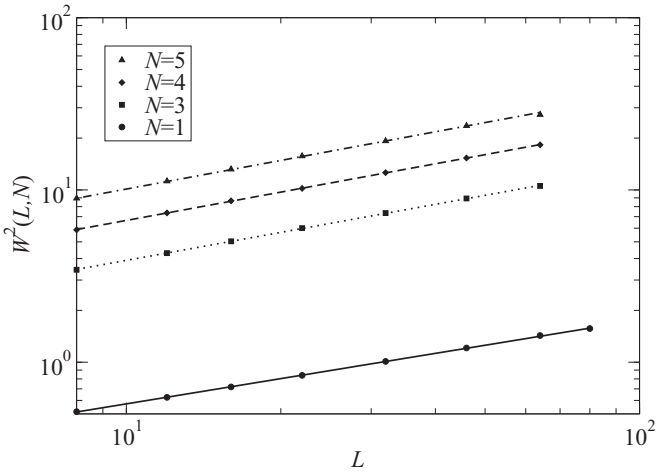

FIG. 3. The data for the saturated interface width W^2 as a function of the system size of $L = 8$, 12, 16, 22, 32, 46, 64, and 80 plotted on a logarithmic scale for $N = 1$, 3, 4, and 5 in $4 + 1$ dimension.

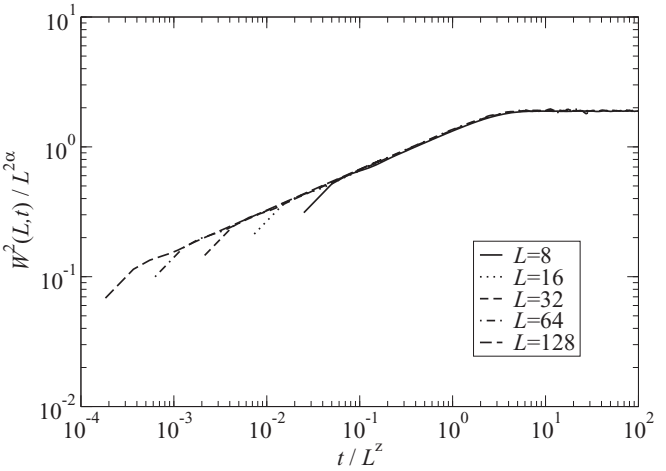

FIG. 4. The scaled interface width with $\alpha = 0.273$ and $z = 1.73$ for system sizes of $L = 8$, 16, 32, 64, and 128 in the RSOS model for $N = 4$ in $4 + 1$ dimension.

$W^2(t) = ct^{2\beta} + m$ where m is a term like an intrinsic width. We find $\beta = 0.158(7)$ which is quite robust for a various range of $3 \leqslant N \leqslant 5$. The m becomes very small for $N = 4$ implying that the correction to scaling is quite negligible. Our estimated value β is larger than that of $N = 1$, but a little bit smaller than $1/6$. However, it is almost consistent with the conjecture $\beta = \frac{1}{6}$ within the error bar.

To measure the roughness exponent α describing the saturation of the interfacial fluctuations in the late time region, we monitor the interface width for various system sizes in the saturated regime as shown in Fig. 3. We calculate α by fitting the data to the relation $W^2(L) \sim L^{2\alpha}$ without any correction term, and obtain

$$\alpha = 0.273 \pm 0.009 \tag{7}$$

for $N = 4$. Almost the same exponent is obtained even for $N = 3$ and 5. As given in Table I, the measured α is quite robust against various N ($3 \leqslant N \leqslant 5$) except $N = 1$. It seems that the measured $\alpha \approx 0.243$ for $N = 1$ is quite a bit smaller than that of $N = 4$ due to the discrete height effect. Since our model for $N = 4$ is closer to the continuum version of the KPZ equation than that of $N = 1$, we could get better estimates of the exponent.

With the estimates of β and α for $N = 4$, the exponent z is estimated as $z = \alpha/\beta \approx 1.73$. Our results for critical exponents are summarized in Table I. They satisfy the scaling relation $\alpha + z = 2$ very well. As Fig. 4 shows, the data scaled with $\alpha = 0.273$ and $z = 1.73$ collapse onto a single curve, supporting the scaling behavior in Eq. (2). It seems that the width for $N = 4$ gives better scaling behavior even for smaller system sizes.

TABLE I. Summary of critical exponents α, β, z, and $\alpha + z$ obtained through numerical simulations for $N = 1$, 3, 4, and 5 in $4 + 1$ dimension.

N	α	β	z	$\alpha + z$
1	0.243(9)	0.137(9)	1.77	2.01
3	0.270(9)	0.155(6)	1.74	2.01
4	0.273(9)	0.158(6)	1.73	2.00
5	0.276(9)	0.159(6)	1.74	2.01

IV. CONCLUSIONS

It is generally known that the RSOS model [11] is well described by the KPZ equation [12] due to the restriction on the height difference. In this Brief Report, the RSOS model with various restriction parameter N is studied to reduce the artifact of the discrete height, and the critical exponents are measured as $\beta \approx 0.158$, $\alpha \approx 0.273$, and $z \approx 1.73$ for $N = 4$. The growth exponent is slightly smaller than, but is almost consistent with our conjecture $\beta = \frac{1}{6}$ [11] within the error bars. Actually the value of N only changes the short wavelength fluctuations which should be irrelevant as far as the scaling exponents are concerned. $\beta \approx 0.137$ for $N = 1$ is much smaller than that of the present work, indicating that the asymptotic results in the infinite system limit were not achieved due to the small $W(t)$ compared to the unit of the height. We choose $N = 4$ to reduce this discrete height effect and obtain a good scaling collapse of the surface width. Also, the numerical results of the exponents satisfy the scaling relation $z + \alpha = 2$ very well, which is known to be valid for the KPZ equation. Therefore, we believe our results with proper choice of N are more reliable. Our results indicate that $d = 4 + 1$ is not the upper critical dimension of the KPZ equation. Since we obtain $z \approx 1.73$ which is much less than 2, there exists a strong-coupling fixed point in $d = 4 + 1$. So, the upper critical dimension should be larger than $4 + 1$ dimension. Further analytical work and additional numerical studies for larger simulations are necessary for a greater understanding.

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