Collisionless energy-independent kinetic equilibria in axisymmetric magnetized plasmas

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The proof of existence of Vlasov-Maxwell equilibria which do not exhibit a functional dependence in terms of the single-particle energy is established. The theory deals with the kinetic treatment of multispecies axisymmetric magnetized plasmas, with particular reference to plasma systems which are slowly time varying. Aside from collisionless laboratory plasmas, the theory concerns important aspects of astrophysical scenarios, such as accretion-disk and coronal plasmas arising in the gravitational field of compact objects. Qualitative properties of the solution are investigated by making use of a perturbative kinetic theory. These concern the realization of the equilibrium kinetic distribution functions in terms of generalized Gaussian distributions and the constraints imposed by the Maxwell equations. These equilibria are shown to be generally non-neutral and characterized by the absence of the Debye screening effect. As a further application, the stability properties of these equilibria with respect to axisymmetric electromagnetic perturbations are addressed. This permits us to establish absolute stability criteria holding in such a case.

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I. INTRODUCTION

Recently, significant theoretical progress has been achieved concerning the kinetic theory of equilibria occurring in collisionless magnetized plasmas. This type of description, based on the Vlasov-Maxwell kinetic treatment, concerns the investigation of quasistationary plasmas subject to the action of both gravitational and electromagnetic (EM) fields [1–7]. It is important to stress that kinetic equilibria of this type necessarily correspond also to fluid magnetohydrodynamic (MHD) equilibria (kinetic-MHD equilibria). However, as a notable feature, in the framework of the Vlasov-Maxwell theory, such a MHD description is not arbitrary or subject to the "ad hoc" prescription of appropriate closure conditions for the fluid equations, but rather the latter are identified with moments of the Vlasov kinetic equation which follow uniquely in a consistent way in terms of the kinetic equilibrium. In this work, a new class of collisionless kinetic equilibria is pointed out which concerns to nonrelativistic axisymmetric plasmas, which is applicable in principle both to astrophysical and laboratory systems. In particular, the conditions of existence of kinetic equilibria which do not exhibit functional dependencies in terms of single-particle energy are investigated, showing that these equilibria can be characterized by generalized Gaussian kinetic distribution functions (KDFs). As a remarkable feature, it is found that, independent of the particular realization of the energy-independent KDF, these equilibria are absolutely stable with respect to axisymmetric EM perturbations. Such a result extends the conclusions pointed out in Ref. [8].

In detail, we consider here multispecies collisionless plasmas for which binary Coulomb interactions and EM radiation effects are negligible as far as the microscopic single-particle dynamics is concerned (see Refs. [9,10] and the references reported therein). For collisionless plasmas, the appropriate statistical description is provided by the kinetic treatment, which allows for both phase-space single-particle dynamics as well as phase-space plasma collective phenomena to be properly taken into account. Applications of the theory are wide ranging. In astrophysical context, they include the kinetic treatment of plasmas arising in the solar environment or in accretion-disk systems around compact objects, like the so-called radiatively inefficient accretion flows (RIAFs) [11] and active galactic nuclei [12], low-density hot magnetized coronas [13–15], funnel-flow plasmas [16], and current-carrying string-loop plasmas [6,17,18]. In the case of laboratory context, they pertain in principle both to axisymmetric devices, such as tokamaks [3,19], and the nonaxisymmetric systems, such as Stellarators and quasisymmetric confinement machines [20,21].

The background to this study is provided by the kinetic theory recently established in Refs. [1–7] for the treatment of collisionless plasmas in quasistationary configurations. In these works, the development of a solution method based on the use of particle invariants ("method of invariants") has been obtained, which allows one to determine equilibrium solutions consistent with microscopic conservation laws and to uniquely prescribe the functional dependencies of the species distribution functions and the related fluid fields. Several physical configurations have been identified which allow us to represent these solutions in terms of generalized Gaussian kinetic distribution functions (KDFs) and to cast them, by means of suitable perturbative expansions, in terms of analytical tractable forms. In the case of axisymmetric systems, solutions have been obtained which apply to accretion disks and tokamak devices [1–4,6,7]. As a main outcome, it has been shown that according to the kind of invariants used when representing the species distributions, different plasma phenomenologies can be treated and retained consistently in the description. More precisely, inclusion of energy introduces an isotropic dependence on particle velocity and permits us to have KDFs which can be of Maxwellian type [1–4]. The conservation of the canonical momentum following from the axisymmetry assumption allows for the existence of equilibrium azimuthal flow velocities [1,7], while the use of its guiding-center representation is associated with the existence of equilibrium poloidal flows [2,3]. Remarkably, the adiabatic conservation of the particle magnetic moment as predicted by gyrokinetic theory contributes as a source of temperature anisotropy and determines the bi-Maxwellian character of the equilibrium solutions [1–3]. Finally, action variables make possible the description of non-Maxwellian features that affect the statistical dynamics of plasmas in epicyclic motion [6].

Other notable effects have been identified which are expected to characterize the phenomenology of collisionless magnetized plasmas. The first one is the kinetic dynamo, a mechanism responsible for the self-generation of quasistationary poloidal and toroidal magnetic fields by the charge currents arising in quasineutral plasmas [1–3,6]. In axisymmetric configurations, the azimuthal currents are associated with the conservation of the canonical momentum. Instead, in the case of toroidal field components, the corresponding currents are driven by temperature anisotropy. Intrinsically kinetic effects have also been pointed out which contribute to the occurrence of such a mechanism and which have been identified with energy-correction, diamagnetic, and finite Larmor radius (FLR) effects (see, in particular, Refs. [2,3,6]).

The second relevant feature is the occurrence of temperature anisotropy in quasistationary collisionless plasmas [1–3,6,7]. It has been shown that the latter phenomenon is associated with phase-space or velocity-space anisotropies carried by the equilibrium KDFs. These arise in generalized bi-Maxwellian distributions [1–3] and in the case of axisymmetric plasmas characterized either by epicyclic motion [6] or by strong rotation phenomena such as supersonic flows and/or strong velocity shear [7].

As a relevant development, the extension of these results to the treatment of spatially nonsymmetric systems in astrophysical and laboratory contexts has been achieved [5]. The corresponding kinetic-MHD equilibria have been shown to recover the relevant phenomenology pointed out for axisymmetric systems, and in particular to exhibit nonvanishing species flow velocities, temperature and pressure anisotropies, as well as a kinetic dynamo mechanism.

Among the variety of physical conditions considered so far, a common feature of all the treatments mentioned above is the ubiquitous dependence of the species equilibrium KDFs on the single-particle energy. However, the question arises as to whether possible kinetic equilibria may exist also in configurations for which the KDF does not contain any kind of functional dependencies (both explicit or implicit via possible other invariants or the imposition of suitable kinetic constraints on the same KDFs) in terms of the single-particle energy. In fact, physical situations might occur in which plasmas undergo transient, energy nonconserving processes (such as radiation phenomena) giving rise to the relaxation of the phase-space probability distributions to energy-independent states. In such a case, the KDF should depend exclusively on the set of remaining particle phase-space invariants to be suitably identified. In magnetized plasmas, these should include at least the particle magnetic moment [5]. In the following, such a type of equilibrium KDFs that is energy independent will be referred to in short as "no-energy kinetic equilibria." Solutions of this type, if they exist at all, are expected to require suitable physical conditions and to exhibit peculiar non-Maxwellian features. The problem posed is novel and its treatment in principle is nontrivial. In fact, one must assure that even in the absence of single-particle energy dependencies, the equilibrium KDF remains defined, summable, and smooth in the whole phase space when it is expressed in terms of the remaining invariants. In addition, a basic requirement concerns the simultaneous possibility of recovering the phenomenology found for the previous kinetic solutions and ascribed to the conservation of adiabatic invariants other than the energy. The issue is also closely connected with the investigation of corresponding fluid equilibria. Notice again that here the approach relies, rather than on a self-standing fluid theory, on the analysis of the fluid fields derived directly from the equilibrium KDFs which are obtained in the framework of the Vlasov-Maxwell description. This technique is expected to lead to a better understanding of the physical systems to which they actually apply, e.g., astrophysical and laboratory plasmas, sonic and supersonic flows, neutral and non-neutral environments, etc.

In connection with the discussion introduced above, a related basic issue is represented by the kinetic stability properties of the no-energy equilibria. A reference work for this problem is given by Ref. [8], where a kinetic description of low-frequency and long-wavelength (with respect to the Larmor time and length scales) axisymmetric EM perturbations was addressed. Such a theory applies to nonrelativistic, collisionless, and axisymmetric accretion-disk plasmas belonging to the strongly magnetized and gravitationally bound regime and occurring in quasineutral subsonic kinetic regimes (for a discussion on the issue see also Refs. [2,4]). The kinetic stability analysis carried out there takes into account consistently the existence of phase-space single-particle invariants on which the equilibrium KDF necessarily must depend. As a major result, it was proved that for the regimes considered, these kinetic equilibria are actually stable against axisymmetric kinetic perturbations. In this paper, the problem is posed of extending these conclusions to arbitrary kinetic regimes possibly characterized by the simultaneous presence of sonic or supersonic flow velocities and local violation of quasineutrality. The goal is reached here by investigating the axisymmetric linear stability properties of the no-energy equilibria and by extending to such a case the theory developed in Ref. [8]. In particular, this concerns the search of stability criteria for no-energy KDFs which are absolute, i.e., they apply for arbitrary equilibrium KDFs in a suitable functional class, and hold for arbitrary axisymmetric EM perturbations with prescribed range of frequencies and wavelengths.

Aside from its conceptual importance for theoretical plasma physics and mathematical physics, the answer to these questions can be relevant for a better understanding of collisionless plasma dynamics in both astrophysical and laboratory scenarios. The possible explicit realization of no-energy kinetic equilibria and the establishment of their stability with respect to axisymmetric perturbations could in fact provide a result of reference for the interpretation of the complex phenomenology that characterize these systems.

II. GOALS AND SCHEME OF THE PAPER

Putting all these issues in perspective, the goals of this study are as follows: (1) To identify the set of integrals of motion and adiabatic invariants that characterize quasistationary axisymmetric collisionless plasmas and are appropriate for the study of no-energy equilibria. (2) By implementing the method of invariants [1–7], to prove the existence of no-energy kinetic equilibria and to determine the functional form of the corresponding KDFs for a multispecies plasma. (3) To construct an explicit representation for these solutions and to show that no-energy equilibria admit a representation in terms of generalized Gaussian distributions, while allowing for the treatment of nonuniform fluid fields and temperature anisotropies. (4) To point out that a necessary condition for the validity of such KDFs is the presence of a nonvanishing azimuthal component of the equilibrium magnetic field. (5) To develop a suitable perturbative theory for the representation of the equilibrium KDFs in terms of Chapman-Enskog series that can allow for the analytical treatment of the implicit phase-space dependencies contained in the same distributions. (6) To estimate the leading-order constitutive equations of the fluid fields corresponding to the kinetic equilibrium determined here and to prove that such a configuration admits generally nonuniform differential flow velocity as well as temperature anisotropy. (7) To investigate the constraints imposed on the kinetic equilibria by the Maxwell equations. In particular, this concerns first the investigation of the conditions of validity of the quasineutrality condition as implied by the Poisson equation. This allows one to show that no-energy kinetic equilibria are generally non-neutral and do not exhibit any Debye screening effect. Second, this requires analysis of the Ampere equation and its implications for the possible occurrence of dynamo mechanisms. (8) To address the issue of the stability for the no-energy equilibria with respect to axisymmetric EM perturbations. In this reference, the main achievement is the proof of stability criteria which hold independently of the plasma regime being realized and of the relative strength of azimuthal and poloidal magnetic fields. This outcome makes also possible a comparison with the results obtained in Ref. [8] as far as the physical relevance of the conclusions is concerned for theoretical plasma physics and astrophysics.

In detail, the scheme of the paper is as follows. In Sec. III, the basic assumptions and definitions underlying the construction of no-energy equilibria are introduced. Section IV deals with the specification of the plasma orderings and the definition of the plasma collisionless regime considered in this study. In Sec. V, the fundamentals of the solution method adopted here are recalled and the relevant adiabatic invariants for the problem of interest are determined. In Sec. VI, the proof of existence of no-energy equilibria is provided together with the general functional dependencies carried by the corresponding KDFs. An explicit realization of the equilibrium distributions in terms of generalized Gaussian functions is also determined. Section VII presents the development of the perturbative theory appropriate for the Chapman-Enskog representation of the no-energy KDFs and their analytical treatment. As an application, in Sec. VIII, the fluid constitutive equations for the species number density, flow velocity, and pressure tensor are evaluated to the leading order and the characteristic physical properties of the corresponding fluid solution are pointed out. In Sec. IX, the implications on the kinetic solution determined by the Maxwell equations are investigated, while in Sec. X the issue of the linear stability properties of the no-energy kinetic equilibria with respect to axisymmetric EM perturbations is addressed and a stability property is pointed out. Finally, Sec. XI contains a summary of the results and the concluding remarks.

III. ASSUMPTIONS AND DEFINITIONS

Ignoring possible weakly dissipative effects (Coulomb collisions and turbulence) and EM radiation effects [9,10], it is assumed that the KDF and the EM fields associated with the plasma obey the system of Vlasov-Maxwell equations. For definiteness, we shall consider here a plasma consisting of s species of charged particles which are characterized by proper mass M_s and total charge $Z_s e$.

We consider here a collisionless plasma, which means that the mean free path of the plasma particles is much longer than the largest characteristic scale length associated with the plasma fluid fields or with the gravitational and EM fields. Within the kinetic description of collisionless plasmas, the fundamental dynamical variable is represented by the KDF $f_s = f_s(\mathbf{r},\mathbf{v},t)$, which is defined in the phase space $\Gamma = \Gamma_\mathbf{r} \times \Gamma_\mathbf{v}$, with $\Gamma_\mathbf{r} \subset \mathbb{R}^3$ and $\Gamma_\mathbf{v} \equiv \mathbb{R}^3$ denoting the configuration and velocity space, respectively. The phase-space evolution of f_s is then determined by the Vlasov equation

$$\frac{d}{dt} f_s(\mathbf{r},\mathbf{v},t) = 0. \tag{1}$$

In addition, in this treatment the plasma is taken to be as follows: (a) subject to both gravitational and EM fields; (b) nonrelativistic, in the sense that it has nonrelativistic particles and species flow velocities, that the gravitational field can be treated within the classical Newtonian theory, and that the nonrelativistic Vlasov kinetic equation is used as the dynamical equation for the KDF; (c) axisymmetric, so that the relevant dynamical variables characterizing the plasma (e.g., the fluid fields) are independent of the azimuthal angle φ, when referred for example to a set of cylindrical coordinates (R,φ,z). From this assumption, as a shortcut in the following we shall denote with $\mathbf{x} = (R,z)$ the configuration state vector.

We are concerned here with quasistationary configurations, namely, solutions which are slowly varying in time. This condition is also referred to as equilibrium configuration. It must be stressed that this kind of slow-time dependence is allowed in the present treatment of kinetic equilibria and is consistent with the assumptions introduced above. From the physical point of view, the slow dynamics can be a consequence of the assigned externally produced EM fields acting on the plasma, which may exhibit such a feature, or can be an intrinsic property of the same equilibrium KDF, when the latter is expressed in terms of single-particle adiabatic invariants. A detailed discussion of this issue and the related mathematical treatment will be addressed in Sec. V [see in particular Eqs. (10)–(12)]. For a generic physical quantity G which depends on spatial coordinates $\mathbf{x}$ and time t, the quasistationarity is expressed by letting in the following $G = G(\mathbf{x},\xi^k t)$, with $\xi \ll 1$ being a small dimensionless parameter to be suitably defined (see below) and $k \geqslant 1$ being an integer. Similar considerations apply for the equilibrium KDF f_s, which is denoted in the following as $f_s = f_s(\mathbf{x},\mathbf{v},\xi^k t)$.

We focus on solutions for the equilibrium magnetic field $\mathbf{B}$ which admit, at least locally, a family of nested axisymmetric toroidal magnetic surfaces $\{\psi(\mathbf{x},\xi^k t)\} \equiv \{\psi(\mathbf{x},\xi^k t) = \text{const}\}$, where ψ denotes the poloidal magnetic flux of $\mathbf{B}$. The magnetic surfaces can be either locally closed [1] or locally open [2] in the configuration domain occupied by the plasma. In both cases, a set of magnetic coordinates (ψ,φ,ϑ) can be defined locally, where ϑ is a curvilinear anglelike coordinate on the magnetic surfaces $\psi(\mathbf{x},\xi^k t) = \text{const}$. Each relevant physical quantity $G(\mathbf{x},\xi^k t)$ can then be conveniently expressed either in terms of the cylindrical coordinates or as a function of the magnetic coordinates, i.e., $G(\mathbf{x},\xi^k t) = \overline{G}(\psi,\vartheta,\xi^k t)$.

Consistent with these assumptions, we assume the magnetic field to be represented as

$$\mathbf{B} \equiv \nabla \times \mathbf{A} = \mathbf{B}^{\text{self}}(\mathbf{x},\xi^k t) + \mathbf{B}^{\text{ext}}(\mathbf{x},\xi^k t), \tag{2}$$

where $\mathbf{B}^{\text{self}}$ and $\mathbf{B}^{\text{ext}}$ denote the self-generated magnetic field produced by the plasma and a finite external axisymmetric magnetic field (vacuum field). For definiteness, both contributions are assumed to exhibit generally nonvanishing azimuthal and poloidal components. Hence, they are represented as

$$\mathbf{B}^{\text{ext}} = I_{\text{ext}}(\mathbf{x},\xi^k t)\nabla\varphi + \nabla\psi_{\text{ext}}(\mathbf{x},\xi^k t) \times \nabla\varphi, \tag{3}$$

$$\mathbf{B}^{\text{self}} = I_{\text{self}}(\mathbf{x},\xi^k t)\nabla\varphi + \nabla\psi_{\text{self}}(\mathbf{x},\xi^k t) \times \nabla\varphi, \tag{4}$$

so that the total magnetic field takes the form

$$\mathbf{B} = I(\mathbf{x},\xi^k t)\nabla\varphi + \nabla\psi(\mathbf{x},\xi^k t) \times \nabla\varphi, \tag{5}$$

where $\mathbf{B}_T \equiv I(\mathbf{x},\xi^k t)\nabla\varphi$ and $\mathbf{B}_P \equiv \nabla\psi(\mathbf{x},\xi^k t) \times \nabla\varphi$ are the corresponding toroidal (i.e., azimuthal) and poloidal components, respectively, with $I \equiv I_{\text{ext}} + I_{\text{self}}$ and $\psi \equiv \psi_{\text{ext}} + \psi_{\text{self}}$. For greater generality, at this point no ordering assumptions are introduced between $\mathbf{B}^{\text{ext}}$ and $\mathbf{B}^{\text{self}}$, nor between $\mathbf{B}_T$ and $\mathbf{B}_P$.

Charged particles are also generally subject to both electrostatic (ES) and gravitational fields. In the present theory, the latter can be dealt with in terms of the effective potential $\Phi_s^{\text{eff}}(\mathbf{x},\xi^k t)$ defined as

$$\Phi_s^{\text{eff}}(\mathbf{x},\xi^k t) \equiv \Phi(\mathbf{x},\xi^k t) + \frac{M_s}{Z_s e}\Phi_G(\mathbf{x},\xi^k t), \tag{6}$$

with $\Phi(\mathbf{x},\xi^k t)$ and $\Phi_G(\mathbf{x},\xi^k t)$ denoting the ES and gravitational potentials, which are in principle generated both by the plasma charge and mass density and by external sources. In particular, in the external domain to the configuration space $\Gamma_{\mathbf{r}}$ occupied by the plasma, all the external sources of the external EM and gravitational fields, in particular the external charge and current densities $\rho^{\text{ext}}(\mathbf{x},\xi^k t)$ and $\mathbf{J}^{\text{ext}}(\mathbf{x},\xi^k t)$, are assumed to be prescribed deterministically and independent of Φ, and to be slowly time dependent in the sense indicated above. The origin of external charge and current densities can be explained, for example, as being due to possible additional equilibrium plasma in the configuration domain which can belong to a different plasma regime and exists independently of the no-energy equilibria. For the simplicity of the treatment, but without loss of generality, in the following the contribution of the plasma to Φ_G will be neglected. For an axisymmetric accretion-disk plasma, the gravitational potential can be conveniently assumed to coincide with the potential associated with the central compact object. In this regard, and consistent with the present assumptions, relevant relativistic effects due to the curvature of space-time can be retained consistently by use of pseudo-Newtonian or effective potentials, such as the Paczýnski-Wiita potential [22,23]. We also notice that, in the case of laboratory plasmas in axisymmetric devices, the contribution of the gravitational potential becomes negligible, so that one has from Eq. (6) that in this context $\Phi_s^{\text{eff}}(\mathbf{x},\xi^k t) \equiv \Phi(\mathbf{x},\xi^k t)$, namely, the effective potential can be taken to coincide identically with the ES potential. Consistent with the Vlasov-Maxwell description, the ES potential generated by the nonvanishing plasma charge density is uniquely determined by the Poisson equation, with the corresponding source being consistently prescribed as a velocity moment of the equilibrium KDF solution of the Vlasov equation. This issue will be investigated in greater detail in Sec. IX.

IV. PLASMA ORDERINGS AND REGIMES

In this section, we introduce the fundamental orderings which are needed for the construction of no-energy equilibria. These follow from the treatment of single-particle dynamics in magnetized plasmas and are expressed through the definition of the dimensionless species parameters $\varepsilon_{M,s}$ and ε_s. Following the treatment in Refs. [3,4,6,7], these parameters are prescribed in such a way to be all independent of single-particle velocity and at the same time to be related to the characteristic species thermal velocities. It is important to remark that the latter requirement is meaningful when the equilibrium KDF is of Gaussian type, as it is the case here (see Sec. VI). In such a configuration, both perpendicular and parallel thermal velocities (defined with respect to the local magnetic field direction) can be consistently introduced. They are defined, respectively, by $v_{\perp\text{ths}} = (T_{\perp s}/M_s)^{1/2}$ and $v_{\parallel\text{ths}} = (T_{\parallel s}/M_s)^{1/2}$, with $T_{\perp s}$ and $T_{\parallel s}$ denoting here the species perpendicular and parallel temperatures. For definiteness, in the following we shall regard $T_{\perp s}$ and $T_{\parallel s}$ to be of the same order of magnitude.

In detail, the parameter $\varepsilon_{M,s}$ is defined as $\varepsilon_{M,s} \equiv \frac{r_{Ls}}{L}$, where $r_{Ls} \equiv v_{\perp\text{ths}}/\Omega_{cs}$ is the species average Larmor radius, with L being the minimum scale length characterizing the spatial variations of the fluid fields associated with the KDF and of the EM fields and $\Omega_{cs} \equiv \frac{Z_s eB}{M_s c}$ being the Larmor frequency for the species s. The parameter ε_s is related to the particle canonical momentum $p_{\varphi s}$ conjugate to the azimuthal angle φ:

$$p_{\varphi s} = M_s R v_\varphi + \frac{Z_s e}{c}\psi(\mathbf{x},\xi^k t) \equiv \frac{Z_s e}{c}\psi_{*s}, \tag{7}$$

where $v_\varphi = \mathbf{v} \cdot \mathbf{e}_\varphi$. Denoting by $v_{\text{ths}} \equiv \sup\{v_{\parallel\text{ths}}, v_{\perp\text{ths}}\}$, ε_s is identified with $\varepsilon_s \equiv |\frac{M_s R v_{\text{ths}}}{\frac{Z_s e}{c}\psi}|$. Hence, ε_s effectively measures the ratio between the toroidal angular momentum $L_{\varphi s} \equiv M_s R v_\varphi$ and the magnetic contribution to the toroidal canonical momentum, for all particles in which v_φ is such that $v_\varphi \sim v_{\text{ths}}$ while ψ is assumed as being nonvanishing.

Depending on the magnitude of these parameters, several different plasma regimes can be identified, as described in Ref. [4]. Following the classification scheme proposed therein, in this work the equilibrium plasma is assumed to belong to the *strongly magnetized regime* for which the asymptotic orderings

$$0 < \varepsilon_{M,s} \sim \varepsilon_s \ll 1 \tag{8}$$

apply. At this point, we notice that one can consistently identify the small parameter ξ introduced above with $\varepsilon_{M,s}$. From the validity of Eq. (8), the following asymptotic expansion for ψ_{*s} is implied:

$$\psi_{*s} = \psi[1 + O(\varepsilon_s)]. \tag{9}$$

Before concluding, it is necessary to comment on the physical realization of the strongly magnetized regime in real systems, and in particular in astrophysical ones. We notice that the magnetic field enters the two parameters in a different way. In fact, ε_s contains the poloidal flux ψ which contributes to the toroidal canonical momentum $p_{\varphi s}$, while $\varepsilon_{M,s}$ depends on the magnitude of the total magnetic field. Indeed, the parameter ε_s determines the particle spatial excursion from a magnetic flux surface $\psi(\mathbf{x},\xi^k t) = \text{const}$, while $\varepsilon_{M,s}$ measures the amplitude of the Larmor radius with respect to the inhomogeneities of the background fluid and EM fields. These two effects correspond to different physical magnetic-related processes, due respectively to the Larmor-radius and magnetic-flux surface confinement mechanisms. As pointed out in Ref. [4], the ordering conditions (8) are expected to be easily verified in accretion-disk systems for a wide range of magnetic-field magnitudes that can be present in these scenarios. This supports the choice of the strongly magnetized regime also in the present treatment.

V. SOLUTION METHOD

In this section, we recall the fundamentals of the solution method implemented in this work for the construction of no-energy equilibria, which is referred to here as the "method of invariants." More detailed discussions can be found in Refs. [1–7]. The technique consists in the search of equilibrium KDFs for collisionless plasma species such that, in a suitable subset of phase space, each of them can be realized in terms of appropriate generalized Gaussian distributions. Specifically, the method of invariants permits the determination of particular solutions of the Vlasov equation for the KDF of the form $f_s = f_{*s}$, where f_{*s} is prescribed as a function of particle invariant phase functions $\{K_j, j = 1,n\}$. This target is reached by making explicit use of the characteristic system invariants in combination with the introduction of suitable *kinetic constraints* (see definition in Sec. VI). Here, we follow the definitions given in Ref. [5]. Thus, K_j is regarded as a first integral if it does not depend explicitly on time and satisfies the equation

$$\frac{d}{dt}K_j(\mathbf{z}) = 0 \tag{10}$$

for a properly defined state $\mathbf{z}$. Instead, K_j represents an adiabatic invariant of order $k \geqslant 1$ when it depends at most slowly on time, in the sense $K_j = K_j(\mathbf{z},\xi^k t)$, and satisfies the asymptotic equation

$$\frac{d}{dt}K_j(\mathbf{z},\xi^k t) = 0 + O(\xi^k), \tag{11}$$

where the dynamical variable K_j is considered of $O(\xi^0)$ and $k \geqslant 1$. Hence, f_{*s} is a first integral, and therefore a stationary KDF, when it is expressed as a function of first integrals only. Instead, more generally f_{*s} can be regarded itself as an adiabatic invariant which depends explicitly on a given set of adiabatic invariants $\{K_j, j = 1,n\}$ and moreover is allowed to depend slowly on time (through the same invariants), or in other words it is quasistationary, namely, it is of the form

$$f_{*s} = f_{*s}(\{K_j, j = 1,n\},\xi^k t). \tag{12}$$

In the following, we shall denote f_{*s} as equilibrium KDF.

It is important to stress that a basic requirement of the method of invariants is the possibility of determining "*a posteriori*" a perturbative representation of the KDF in terms of a Chapman-Enskog series expansion. In reference with this, it is worth recalling that the use of phase-space perturbative techniques in plasma physics is well known. Typical examples of this type are represented by nonrelativistic and relativistic gyrokinetic theories [5,24,25]. In the present case, the use of a perturbative method of the Chapman-Enskog type is motivated by the existence of Gaussian-type equilibrium KDFs and the possibility of introducing appropriate expansion parameters. It will be proved below that this method can be realized also in the present case, determining an asymptotic expression for the equilibrium species KDF which permits also an analytical estimation of the corresponding fluid fields.

Several advantages characterize the technique based on the method of invariants. In particular, note the following: (1) The solution can be written in closed form and can be dealt with analytically, by performing "*a posteriori*" a series expansion on f_{*s} according to the perturbative theory. (2) It permits us to obtain an equilibrium solution which includes consistently the constraints arising from single-particle phase-space conservation laws. (3) It determines the functional form of the equilibrium fluid fields carried by the species KDFs. (4) It also uniquely prescribes the leading-order distribution in the corresponding Chapman-Enskog representation. Remarkably, for collisionless plasmas, the latter was proved to be represented by a bi-Maxwellian KDF [1–3], while it remains generally non-Maxwellian in the absence of spatial symmetries [5], in the presence of epicyclic motion [6], or in the presence of strong rotation phenomena [7]. This feature represents an aspect of critical importance for the physical properties of equilibrium collisionless plasmas and a point of major difference with the customary Chapman-Enskog solution method, where the leading-order KDF is usually identified with an isotropic Maxwellian distribution.

We conclude this section by identifying the relevant invariant phase functions which characterize single-particle dynamics and which are required here to determine the no-energy equilibria. Because of axisymmetry, the toroidal canonical momentum $p_{\varphi s}$ defined by Eq. (7) is a first integral of motion. Then, from the quasistationarity condition one has that the single-particle energy E_s expressed as

$$E_s = \frac{M_s}{2}v^2 + Z_s e\Phi_s^{\text{eff}}(\mathbf{x},\xi^k t) \tag{13}$$

defines an adiabatic invariant of prescribed order. However, in the context of no-energy equilibria, by definition the energy E_s will not contribute to the functional dependence of the KDFs.

Then, given validity of the condition (8), additional adiabatic invariants can be determined for magnetized plasmas by using gyrokinetic (GK) theory. A variational nonperturbative formulation of GK theory can be found in Ref. [5] for nonrelativistic charged particles in the presence of both EM and gravitational fields. It is proved that, when EM radiation-reaction effects are neglected [9,10], the particle magnetic moment m'_s associated with the Larmor rotation of charges around magnetic field lines is an adiabatic invariant. According to standard notation, here and in the rest of the paper, quantities labeled with a prime refer to dynamical variables which are evaluated at the particle guiding-center position. We notice that, in the framework of an asymptotic formulation of GK theory carried out by means of a Larmor-radius expansion in terms of the parameter $\varepsilon_{M,s}$ (see details in Ref. [5]), the magnetic moment can be in principle determined with arbitrary accuracy. In particular, to the leading order m'_s can be represented as

$$m'_s = \mu'_s[1 + O(\varepsilon_{M,s})], \tag{14}$$

where $\mu'_s \equiv \frac{M_s w'^2}{2B'}$, with w' denoting the magnitude of the component of the guiding-center particle velocity orthogonal to the magnetic field direction.

VI. NO-ENERGY EQUILIBRIA

In this section, we proceed with the proof of existence of no-energy equilibria and the explicit determination of the corresponding KDFs, to be represented in terms of generalized Gaussian distributions. Consistent with the assumptions underlying the method of invariants outlined in Sec. V, the general functional form of the species equilibrium KDF is assumed of the type

$$f_{*s} = f_{*s}\left[\left(p_{\varphi s} - \frac{Z_s e}{c}\psi_0\right)^2, p_{\varphi s}, m'_s, \Lambda_{*s}, \xi^k t\right], \tag{15}$$

which by construction depends on the guiding-center magnetic moment m'_s and where, for greater generality, explicit functional dependencies on both $(p_{\varphi s} - \frac{Z_s e}{c}\psi_0)^2$ and $p_{\varphi s}$ are considered, with ψ_0 denoting a suitable constant having the dimension of a poloidal magnetic flux. The physical motivation behind the inclusion of such a dependence on ψ_0 will be discussed in Sec. VIII when the fluid constitutive equations are calculated. In Eq. (15), Λ_{*s} denotes the so-called structure functions [1–3,5–7], i.e., functions which depend implicitly on the particle state $(\mathbf{x},\mathbf{v})$ and which are suitably related to a finite set of observable variables, namely, fluid fields carried by f_{*s}. For definiteness, both f_{*s} and Λ_{*s} are assumed to be analytic functions. In order for f_{*s} to be an adiabatic invariant, Λ_{*s} must also be a function of the adiabatic invariants. This restriction is referred to as a kinetic constraint, and its precise choice is imposed according to the specific form of the solution and its conditions of existence, including the possibility of an analytical treatment of f_{*s} and the calculation of the corresponding fluid fields. In particular, depending on the physical properties of the system to be studied, such a constraint must be consistent with the implementation of a perturbative theory for the equilibrium KDF and the determination of its Chapman-Enskog representation. Following the treatment in Refs. [1–3,5–7] and in agreement with the strongly magnetized regime considered here, the kinetic constraints are expressed as follows:

$$\Lambda_{*s} = \Lambda_{*s}(\psi_{*s}). \tag{16}$$

It must be noticed though that Eq. (16) does not represent the most general form of kinetic constraint. As shown in Refs. [5,7], additional functional dependencies on the remaining invariants may also be included, provided they are suitably ordered, e.g., in terms of the parameter ξ, so that their contribution can be treated perturbatively.

By construction, f_{*s} given in Eq. (15) and subject to the constraint (16) does not contain any implicit or explicit dependence on the particle energy. As a result, the following constraint equation remains always identically satisfied in this case:

$$\frac{\partial f_{*s}}{\partial E_s} = 0. \tag{17}$$

On the other hand, manifestly the same f_{*s} is an acceptable equilibrium kinetic solution of the Vlasov equation for collisionless plasmas in axisymmetric configurations since it depends uniquely on particle invariants. In addition, it will be shown below that the velocity dependencies contained in Eq. (15) are sufficient to warrant that f_{*s} remains defined and summable on the whole velocity space $\Gamma_{\mathbf{v}}$.

Let us now proceed with the construction of explicit representations for f_{*s} according to Eq. (15). To this aim, we introduce the following additional requirements: (1) The species KDF f_{*s} must be characterized by nonuniform fluid fields, including in particular nonuniform number density, flow velocity, and nonisotropic temperature. (2) The equilibrium KDF f_{*s} must be expressed in terms of generalized Gaussian distributions, which can be generally different from isotropic Maxwellian functions. (3) The solution is required to admit an asymptotic representation of the Chapman-Enskog type for the treatment of implicit phase-space dependencies contained in the structure functions due to Eq. (16). (4) The KDF f_{*s} must be a strictly positive real function and it must be summable, in the sense that the velocity moments of the form

$$\Xi_s(\mathbf{x},\xi^k t) = \int_{\Gamma_{\mathbf{v}}} d\mathbf{v}\, K_s(\mathbf{x},\mathbf{v},\xi^k t) f_{*s} \tag{18}$$

must exist for a suitable ensemble of weight functions $\{K_s(\mathbf{x},\mathbf{v},\xi^k t)\}$, to be prescribed in terms of polynomials of arbitrary degree defined with respect to components of the velocity vector field $\mathbf{v}$.

In line with the previous requirements and following the approach of Refs. [1–3,5–7], an explicit solution for the species KDF f_{*s} is realized by the species distribution

$$f_{*s} = \beta_{*s} e^{-(p_{\varphi s} - \frac{Z_s e}{c}\psi_0)^2\gamma_{*s} - \delta_{*s} p_{\varphi s} - m'_s\alpha_{*s}}, \tag{19}$$

which we refer to as the *no-energy generalized Gaussian KDF*. In Eq. (19), the structure functions are identified with the set of dimensional functions

$$\{\Lambda_{*s}\} \equiv \{\beta_{*s}, \gamma_{*s}, \delta_{*s}, \alpha_{*s}\}, \tag{20}$$

which are subject to the constraint (16). Here, $\alpha_{*s} \equiv \frac{B}{T_{\perp *s}}$, with $T_{\perp *s}$ being referred to as the generalized perpendicular temperature, while β_{*s}, γ_{*s}, and δ_{*s} are related, respectively, to the definition of species number density, parallel temperature, and flow velocity. The connection between the structure functions $\{\Lambda_{*s}\}$ and the corresponding fluid fields will be established through the perturbative theory developed in Sec. VII. Here, we remark that because of the functional form of the structure functions in Eq. (16), which are defined in phase space, at this stage the set $\{\Lambda_{*s}\}$ can not be directly identified with particular fluid fields. The latter in fact must be necessarily computed in a consistent way as velocity moments of the KDF from Eq. (18).

The representation (19) allows one to point out the following features: (i) Apart from the behavior of the structure functions, the species KDFs (19) contain both linear and quadratic explicit dependencies in terms of the canonical momentum $p_{\varphi s}$. (ii) The missing energy contribution together with the quadratic dependence on $p_{\varphi s}$ are the major departures from the solutions earlier considered in Refs. [1–7]. (iii) The absence of energy dependence in the functional form (15) implies that generally the same KDF is nonisotropic in velocity space. (iv) The condition of existence of an equilibrium KDF of the type (15), which can be normalized in the three-dimensional Euclidean velocity space, requires that there must exist a strictly nonvanishing component of the azimuthal magnetic field. This condition applies independent of the specific form of the equilibrium KDF (15).

VII. PERTURBATIVE THEORY

In this section, a perturbative kinetic theory is developed, which is appropriate for the treatment of implicit phase-space dependencies contained in the equilibrium KDF by means of the structure functions $\{\Lambda_{*s}\}$. We stress that such a perturbative theory is permitted specifically thanks to the assumption of generalized Gaussian distribution introduced above. In fact, as a consequence, it follows that the behavior of the KDF can be evaluated asymptotically in suitable subsets of the velocity space. The perturbative theory follows by invoking Eq. (8) and the related expansion (9) for the canonical momentum.

In detail, thanks to the choice (16) of the kinetic constraints, the structure functions $\{\Lambda_{*s}\}$ can be Taylor expanded in terms of the dimensionless parameter ε_s to give

$$\Lambda_{*s} = \Lambda_s(\psi)[1 + O(\varepsilon_s)]. \tag{21}$$

The perturbative theory is therefore obtained by performing on f_{*s} a Taylor expansion for $\{\Lambda_{*s}\}$, which correct to first order in the expansion parameter is of the form

$$\Lambda_{*s} = \Lambda_s(\psi) + (\psi_{*s} - \psi)\left[\frac{\partial \Lambda_{*s}}{\partial \psi_{*s}}\right]_{\psi_{*s}=\psi}. \tag{22}$$

Hence, neglecting corrections of $O(\varepsilon_s^k)$, with $k \geqslant 2$, one obtains the Chapman-Enskog representation of f_{*s} in the form

$$f_{*s} = f'_{0s}[1 + \varepsilon_s h_{Ds}], \tag{23}$$

where f'_{0s} denotes the leading-order KDF

$$f'_{0s} = f'_{0s}\left[\left(p_{\varphi s} - \frac{Z_s e}{c}\psi_0\right)^2, p_{\varphi s}, m'_s, \Lambda_s, \xi^k t\right], \tag{24}$$

which generally depends on the guiding-center invariant m'_s. This is found to be

$$f'_{0s} = \beta_s e^{-\gamma_s (p_{\varphi s} - \frac{Z_s e}{c}\psi_0)^2 - \delta_s p_{\varphi s} - m'_s \alpha_s}, \tag{25}$$

which is referred to as the *no-energy Gaussian KDF*. In the previous equation, the leading-order structure functions are identified with the set $\{\Lambda_s\} \equiv \{\beta_s, \gamma_s, \delta_s, \alpha_s\}$ which are by construction flux functions of ψ. Here, $\alpha_s \equiv \frac{B}{T_{\perp s}}$, with $T_{\perp s}$ representing now the leading-order species perpendicular temperature, while β_s, γ_s, and δ_s are related to the definition of the leading-order species number density, parallel temperature, and flow velocity, respectively.

In addition, h_{Ds} results from the perturbative treatment of the structure functions in terms of ψ_{*s} and is referred to as the diamagnetic correction. This is given by

$$\begin{aligned} h_{Ds} = {} & \frac{cM_s R}{Z_s e}\left[A_{1s} - \left(p_{\varphi s} - \frac{Z_s e}{c}\psi_0\right)^2 \gamma_s A_{2s}\right] v_\varphi \\ & - \frac{cM_s R}{Z_s e}[p_{\varphi s}\delta_s A_{3s} + m'_s \alpha_s A_{4s}] v_\varphi, \end{aligned} \tag{26}$$

where the following definitions have been introduced:

$$\begin{aligned} A_{1s} &\equiv \frac{\partial \ln \beta_s}{\partial \psi}, \quad A_{2s} \equiv \frac{\partial \ln \gamma_s}{\partial \psi}, \\ A_{3s} &\equiv \frac{\partial \ln \delta_s}{\partial \psi}, \quad A_{4s} \equiv \frac{\partial \ln \alpha_s}{\partial \psi}. \end{aligned} \tag{27}$$

The quantities A_{is}, $i = 1$–4, represent the gradients of the structure functions Λ_s across ψ surfaces and are referred to as generalized thermodynamic forces (see also Refs. [2,4]). Since Λ_s are uniquely associated with physically observable fluid fields (see Sec. VIII), the inclusion of the first-order term h_{Ds} in the equilibrium solution permits the consistent treatment of collisionless plasmas characterized by nonuniform fluid fields.

VIII. CONSTITUTIVE EQUATIONS FOR THE FLUID FIELDS

In this section, we estimate the constitutive equations for the species fluid fields associated with the equilibrium KDFs determined above and which are generally expressed by velocity integrals of the form given by Eq. (18). In particular, we focus here on the physical observables identified with the plasma species number density, flow velocity, and pressure tensor. While exact calculations of the fluid fields can be carried out (e.g., numerically) once the KDFs f_{*s} and their structure functions are assigned, here we proceed with an asymptotic analytical estimation by adopting the perturbative theory for f_{*s} outlined in the previous section. As a consequence, the conclusions of this section apply in the subset of phase space where the perturbative theory holds.

Since the species equilibrium KDF f_{*s} depends on the guiding-center magnetic moment m'_s, before calculating the fluid fields it is necessary to express the same invariant at the actual particle position. This can be done by implementing an inverse guiding-center transformation, according to the theory developed in Ref. [5]. Accordingly, one can adopt a general representation for the particle velocity as

$$\mathbf{v} = u\mathbf{b} + \mathbf{w} + \mathbf{V}_{Ds}, \tag{28}$$

where $\mathbf{V}_{\mathrm{Ds}} \equiv \frac{c}{B}\mathbf{E}_{\mathrm{s}}^{\mathrm{eff}} \times \mathbf{b}$ is the effective drift velocity, with $\mathbf{E}_{\mathrm{s}}^{\mathrm{eff}} \equiv -\nabla \Phi_{\mathrm{s}}^{\mathrm{eff}}(\mathbf{x},\xi^k t)$. In addition, u is the magnitude of the component of the particle velocity along $\mathbf{b} \equiv \mathbf{B}/B$ in the frame moving with velocity $\mathbf{V}_{\mathrm{Ds}}$. Finally, $\mathbf{w} = w\cos\phi\mathbf{e}_1 + w\sin\phi\mathbf{e}_2$, with ϕ denoting the gyrophase angle, w being the magnitude of the velocity perpendicular to the magnetic field as measured in the same frame, and ($\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3 \equiv \mathbf{b}$) forming a set of orthogonal axis. The procedure introduces corrections of $O(\varepsilon_{\mathrm{M,s}})$ in the corresponding KDFs that identify the so-called finite Larmor-radius (FLR) contributions. In the following, we assume the inverse transformation to be applied on $f_{*\mathrm{s}}$ when the latter is treated perturbatively according to Eq. (23). As a consequence, given validity of the ordering assumption (8), one obtains FLR corrections which are comparable with the diamagnetic terms. Hence, in this case Eq. (23) yields the perturbative representation

$$f_{*\mathrm{s}} = f_{0\mathrm{s}}[1 + \varepsilon_{\mathrm{s}} h_{\mathrm{Ds}} + \varepsilon_{\mathrm{M,s}} h_{\mathrm{FLRs}}], \tag{29}$$

where h_{FLRs} denotes the FLR contribution of $O(\varepsilon_{\mathrm{M,s}})$, while all quantities on the right-hand side in the previous equation are now expressed at the actual particle position. In detail, the leading-order term $f_{0\mathrm{s}}$ is given by

$$f_{0\mathrm{s}} = \beta_{\mathrm{s}} e^{-\gamma_{\mathrm{s}}(p_{\varphi\,\mathrm{s}} - \frac{Z_{\mathrm{s}}e}{c}\psi_0)^2 - \delta_{\mathrm{s}} p_{\varphi\mathrm{s}} - m_{\mathrm{s}}\alpha_{\mathrm{s}}}, \tag{30}$$

which does not depend on guiding-center variables since the magnetic moment is calculated at the effective particle position. Here, we omit to report the expression of h_{FLRs} as it is beyond the scope of this section. In fact, for the purpose of this work, we restrict the calculation of the fluid fields only to the leading order with respect to both the expansion parameters ε_{s} and $\varepsilon_{\mathrm{M,s}}$. As discussed below, this is sufficient to reveal the relevant physical properties of the fluid system corresponding to the no-energy kinetic equilibria determined here. In particular, this allows one to understand which features characterize the plasma flow velocity and pressure tensor, in connection with the possible occurrence of sonic and supersonic flows as well as of temperature and pressure anisotropies. Finally, the calculation is also a basic prerequisite for the subsequent study of the consistency of the kinetic solution with the constraints posed by the Maxwell equations and the existence of a dynamo mechanism for the self-generation of equilibrium EM fields.

In view of these considerations, in the subset of phase space where the perturbative theory holds, for no-energy equilibria the leading-order fluid fields can be evaluated in terms of $f_{0\mathrm{s}}$, which can be conveniently expressed as

$$f_{0\mathrm{s}} = \overline{\beta}_{\mathrm{s}} e^{-\frac{(u-u_{0\mathrm{s}})^2}{u_{\parallel\mathrm{ths}}^2} - \frac{(\mathbf{w}-w_{0\mathrm{s}}\mathbf{e}_\varphi)^2}{u_{\perp\mathrm{ths}}^2}}. \tag{31}$$

Equation (31) follows from Eq. (25) invoking the representation (28) and after substituting the explicit expression of $p_{\varphi\mathrm{s}}$ given by Eq. (7) and the definition of the structure function α_{s}, while the magnetic moment has been approximated with $\mu_{\mathrm{s}} \equiv \frac{M_{\mathrm{s}} w^2}{2B}$. In addition, the following definitions have been introduced:

$$\overline{\beta}_{\mathrm{s}} \equiv \beta_{\mathrm{s}} e^{\frac{\delta_{\mathrm{s}}^2}{4\gamma_{\mathrm{s}}} + \delta_{\mathrm{s}}(M_{\mathrm{s}} R \mathbf{V}_{\mathrm{Ds}}\cdot\mathbf{e}_\varphi - \frac{Z_{\mathrm{s}}e}{c}\psi_0)}, \tag{32}$$

$$u_{0\mathrm{s}} \equiv -\frac{RB}{I}(\mathbf{w} + \mathbf{V}_{\mathrm{Ds}})\cdot\mathbf{e}_\varphi - \frac{\Omega_{\mathrm{cs}}}{I}(\psi - \psi_0) - \frac{\delta_{\mathrm{s}} B}{2\gamma_{\mathrm{s}} M_{\mathrm{s}} I}, \tag{33}$$

$$u_{\parallel\mathrm{ths}}^2 \equiv \frac{1}{\gamma_{\mathrm{s}}\left(M_{\mathrm{s}}\frac{I}{B}\right)^2}, \tag{34}$$

$$u_{\perp\mathrm{ths}}^2 \equiv \frac{2T_{\perp\mathrm{s}}}{M_{\mathrm{s}}}, \tag{35}$$

$$w_{0\mathrm{s}} \equiv \delta_{\mathrm{s}} R T_{\perp\mathrm{s}}, \tag{36}$$

where Ω_{cs} is the cyclotron frequency. We then start with the calculation of the leading-order species number density $n_{0\mathrm{s}}$, which is defined as the velocity integral

$$n_{0\mathrm{s}}(\mathbf{x},\xi^k t) = \int_{\Gamma_{\mathrm{v}}} d\mathbf{v}\, f_{0\mathrm{s}}. \tag{37}$$

An explicit calculation gives

$$n_{0\mathrm{s}}(\mathbf{x},\xi^k t) = \overline{\beta}_{\mathrm{s}} \pi^{3/2} u_{\parallel\mathrm{ths}} u_{\perp\mathrm{ths}}^2 = 2\pi^{3/2}\frac{T_{\perp\mathrm{s}}}{M_{\mathrm{s}}^2}\frac{B}{I}\frac{\overline{\beta}_{\mathrm{s}}}{\sqrt{\gamma_{\mathrm{s}}}}. \tag{38}$$

We then consider the leading-order flow velocity $\mathbf{V}_{0\mathrm{s}}$ defined as

$$\mathbf{V}_{0\mathrm{s}}(\mathbf{x},\xi^k t) = \frac{1}{n_{0\mathrm{s}}}\int_{\Gamma_{\mathrm{v}}} d\mathbf{v}\,\mathbf{v} f_{0\mathrm{s}}. \tag{39}$$

Adopting the representation (28), a straightforward calculation yields

$$\mathbf{V}_{0\mathrm{s}}(\mathbf{x},\xi^k t) = V_{\parallel\mathrm{s}}\mathbf{b} + w_{0\mathrm{s}}\mathbf{e}_\varphi + \mathbf{V}_{\mathrm{Ds}}, \tag{40}$$

where $V_{\parallel\mathrm{s}}$ is defined as

$$V_{\parallel\mathrm{s}} \equiv -\frac{RB}{I}\mathbf{V}_{\mathrm{Ds}}\cdot\mathbf{e}_\varphi - \frac{\Omega_{\mathrm{cs}}}{I}(\psi - \psi_0) - \frac{\delta_{\mathrm{s}} B}{2\gamma_{\mathrm{s}} M_{\mathrm{s}} I}. \tag{41}$$

Equation (40) displays the dependence of the flow velocity in terms of the structure function $\delta_{\mathrm{s}} = \delta_{\mathrm{s}}(\psi)$, which in turn is associated with the linearly dependent term (occurring in the KDF) with respect to $p_{\varphi\mathrm{s}}$. Since the value of δ_{s} at this stage is arbitrary, it follows that generally $V_{\parallel\mathrm{s}}$ and $w_{0\mathrm{s}}$ (and therefore $\mathbf{V}_{0\mathrm{s}}$) are species dependent. In addition, we notice here that Eq. (40) determines the contribution of the gravitational field in the no-energy equilibria and its interplay with the magnetic and ES fields. This is uniquely associated with the species-dependent drift velocity $\mathbf{V}_{\mathrm{Ds}}$ entering the flow velocity $\mathbf{V}_{0\mathrm{s}}$.

It can be instructive to estimate the order of magnitude of the contribution $\frac{\Omega_{\mathrm{cs}}}{I}\psi$ which appears in $\mathbf{V}_{0\mathrm{s}}$ with respect to the corresponding species thermal velocity v_{ths}. To this aim, we introduce the following estimations:

$$\nabla\psi \sim \frac{\psi}{L}, \tag{42}$$

$$\psi \sim B_{\mathrm{P}} R L, \tag{43}$$

$$I \sim B_{\mathrm{T}} R, \tag{44}$$

so that here L is a measure of the characteristic length of variation of the poloidal flux ψ, while B_{P} and B_{T} are the magnitudes of the poloidal and toroidal magnetic fields, respectively. Concerning the estimation of the thermal velocity, consistent with the assumption introduced in Sec. IV, we take $v_{\mathrm{ths}} \sim \frac{Z_{\mathrm{s}}e}{M_{\mathrm{s}}Rc}\psi\varepsilon_{\mathrm{s}}$, where v_{ths} is defined above after Eq. (7). Then, it is possible to show that the following ordering applies:

$$\frac{\left|\frac{\Omega_{\mathrm{cs}}}{I}\psi\right|}{v_{\mathrm{ths}}} \sim \frac{1}{O(\varepsilon_{\mathrm{s}})}\frac{\Omega_{\mathrm{cs}}}{\Omega_{\mathrm{cs}}^{\mathrm{T}}}, \tag{45}$$

where $\Omega_{cs}^{T} \equiv \frac{Z_s e B_T}{M_s c}$ is the contribution to the cyclotron frequency determined by the toroidal magnetic field only. From this analysis one can conclude that in the particular case in which $\psi_0 = 0$ and the contributions proportional to δ_s are negligible in Eq. (40), the equilibrium KDF is characterized by a supersonic species-dependent flow velocity. More generally, it follows that for suitable choices of the initial constant ψ_0 and the structure functions δ_s, the species flow velocities remain sonic or subsonic with respect to the thermal velocity v_{ths}. It follows that the equilibrium KDF (15) describes in principle kinetic equilibria with flow velocities of arbitrary magnitudes.

Finally, we consider the leading-order species pressure tensor $\underline{\underline{\Pi}}_{0s}$ defined as

$$\underline{\underline{\Pi}}_{0s}(\mathbf{x},\xi^k t) = M_s \int_{\Gamma_v} d\mathbf{v}(\mathbf{v} - \mathbf{V}_{0s})(\mathbf{v} - \mathbf{V}_{0s}) f_{0s}. \quad (46)$$

In this approximation, one finds that $\underline{\underline{\Pi}}_{0s}$ is diagonal but nonisotropic, so that it can be written as

$$\underline{\underline{\Pi}}_{0s}(\mathbf{x},\xi^k t) = P_{\parallel s}\mathbf{bb} + P_{\perp s}(\underline{\underline{\mathbf{I}}} - \mathbf{bb}), \quad (47)$$

where $P_{\parallel s}$ and $P_{\perp s}$ denote, respectively, the parallel and perpendicular pressures. Explicit calculation gives

$$P_{\parallel s} = n_{0s}\frac{1}{\gamma_s}\frac{B^2}{2M_s I^2}, \quad (48)$$

$$P_{\perp s} = n_{0s} T_{\perp s}. \quad (49)$$

From this result, we can conclude the following notable features: (1) The no-energy equilibria are characterized by a nonisotropic pressure tensor. The sources of such anisotropy are the simultaneous conservations of the particle magnetic moment and the canonical momentum, in agreement with the results pointed out, respectively, in Refs. [1–3,5,6] and in Ref. [7] for plasmas characterized by strong rotation phenomena, such as supersonic flows. (2) Equation (49) shows that $T_{\perp s}$ represents the leading-order species perpendicular temperature, which is included in the equilibrium KDF through the structure function α_s. (3) From Eq. (48), one can similarly infer the expression of the corresponding leading-order species parallel temperature. This is given by

$$T_{\parallel s} = \frac{1}{\gamma_s}\frac{B^2}{2M_s I^2}. \quad (50)$$

This result uniquely relates the observable parallel temperature $T_{\parallel s}$ with the leading-order species structure function $\gamma_s = \gamma_s(\psi)$ carried by the no-energy equilibrium KDF, thus assigning a consistent physical meaning to γ_s. (4) The definition of the parallel temperature in Eq. (50) in turn allows one to write the leading-order species number density in Eq. (38) as

$$n_{0s}(\mathbf{x},\xi^k t) = \left(\frac{2\pi}{M_s}\right)^{3/2} T_{\perp s} T_{\parallel s}^{1/2} \overline{\beta}_s, \quad (51)$$

which represents the physical relationship between the structure function $\beta_s = \beta_s(\psi)$ and the observable n_{0s}.

IX. MAXWELL'S EQUATIONS

In this section, we consider the relationship between the no-energy kinetic equilibria and the validity of the Maxwell equations. The latter uniquely solve for the EM fields once the charge and current densities are consistently assigned in terms of velocity moments of the equilibrium KDF. It is important to remark that the Vlasov-Maxwell equations represent a closed set of equations, so that within such a description, the Maxwell equations represent the only possible constraints to be imposed on the kinetic solution, while no other equations can eventually determine the EM fields. Hence, no additional solubility conditions can possibly arise, in particular from the fluid moment equations, since the latter are identically satisfied once the kinetic equilibrium solution f_{*s} is prescribed. We refer to Refs. [2,3] for a proof of this statement.

We start by analyzing the Poisson equation, which uniquely prescribes the ES potential (and, consequently, the ES field) generated by the charge distribution. In the configuration domain $\Gamma_{\mathbf{r}}$ occupied by the collisionless plasma, this is written as

$$\nabla^2\Phi(\mathbf{x},\xi^k t) = -4\pi\rho^{\text{self}}(\mathbf{x},\xi^k t), \quad (52)$$

where $\rho^{\text{self}}(\mathbf{x},\xi^k t) \equiv \sum_s Z_s e n_s(\mathbf{x},\xi^k t)$ denotes the plasma charge density, while $n_s(\mathbf{x},\xi^k t)$ is the species number density that is defined in terms of f_{*s} by Eq. (18) for $K_s(\mathbf{x},\xi^k t) = 1$, namely,

$$n_s(\mathbf{x},\xi^k t) = \int_{\Gamma_v} d\mathbf{v}\, f_{*s}. \quad (53)$$

The general solution for the ES potential $\Phi(\mathbf{x},\xi^k t)$, holding in all the axisymmetric configuration domain S which contains both the plasma described by the no-energy KDF and external sources, is then given by

$$\Phi(\mathbf{x},\xi^k t) = \int_S \rho^{\text{tot}}(\mathbf{x}',\xi^k t) G(\mathbf{x},\mathbf{x}') d\mathbf{x}', \quad (54)$$

where here $G(\mathbf{x},\mathbf{x}') \equiv \frac{1}{|\mathbf{x}-\mathbf{x}'|}$ is the Green's function of the boundary-value problem associated with Eq. (52), with Dirichlet boundary conditions set at infinity. In addition, $\rho^{\text{tot}}(\mathbf{x},\xi^k t) = \rho^{\text{ext}}(\mathbf{x},\xi^k t) + \rho^{\text{self}}(\mathbf{x},\xi^k t)$ is the total charge density, with ρ^{ext} possibly arising from external sources. According to the assumptions introduced in Sec. III, when ρ^{ext} is nonvanishing it is assumed to be known and prescribed pointwise. It must be stressed that Eq. (54) formally represents an exact and unique solution of the ES potential Φ.

In the general case, from Eq. (54) it is not possible to exclude a dependence of the source term ρ^{tot} on the potential Φ itself. In fact, this enters through the charge density ρ^{self} and is carried by the drift velocity $\mathbf{V}_{Ds}$ when the inverse guiding-center transformation is performed on the KDF f_{*s} for the calculation of the species number density [see Eq. (28)]. In addition, the same contribution on $\mathbf{V}_{Ds}$ also determines the way in which the ES potential Φ is affected by the gravitational potential Φ_G, proving the consistency behind the occurrence of a self-generated ES field by the collisionless plasma in the presence of a nonvanishing background gravitational field. On the other hand, when the perturbative representation (29) for the KDF applies to the leading order with respect to all the expansion parameters, the potential Φ can be calculated explicitly in terms of the leading-order charge density $\rho_0^{\rm self}(\mathbf{x},\xi^k t) \equiv \sum_s Z_s e n_{0s}(\mathbf{x},\xi^k t)$ which is independent of $\Phi_s^{\rm eff}$, with n_{0s} being given by Eq. (38). In fact, in this limit, by construction the leading-order no-energy equilibrium KDF f_{0s} does not contain any explicit or implicit functional dependencies on the single-particle energy, and therefore on both Φ and Φ_G [see Eq. (31) in Sec. VIII]. This feature is characteristic of the no-energy equilibria. For comparison, we notice that this condition is never realized (even in an asymptotic way) for the kinetic equilibria treated previously in Refs. [1–7], where the equilibrium KDF always depends on the conserved particle energy, as in the case of generalized bi-Maxwellian distributions.

As a further development, we then consider the problem associated with the possible realization of the quasineutrality condition for the equilibria treated here. To this aim, the asymptotic form of the constitutive equations determined in Sec. VIII is adopted, thus restricting the treatment to the leading-order solution according to Eq. (29). In this approximation, the quasineutrality condition requires

$$\rho_0^{\rm self}(\mathbf{x},\xi^k t) = 0, \quad (55)$$

which yields

$$\sum_s \frac{Z_s e}{M_s^2} \frac{\overline{\beta}_s}{\alpha_s \sqrt{\gamma_s}} = 0, \quad (56)$$

where $\overline{\beta}_s$ is given by Eq. (32). Since $\overline{\beta}_s$ is a function of the type $\overline{\beta}_s = \overline{\beta}_s(\psi,\vartheta)$, it follows that Eq. (56) can not be generally satisfied pointwise. In other words, no-energy kinetic equilibria are in such conditions necessarily non-neutral. We remark that in the literature Eq. (55) is assumed to hold on a suitable spatial scale length, typically identified with the Debye length [26], and usually displays a constraint solution for the potential Φ. However, as detailed above, in the present case, Eq. (56) does not exhibit any dependence on the ES potential Φ. This implies in turn that kinetic equilibria of this type actually can not exhibit the customary Debye screening effect. As a consequence, the equilibrium ES potential must be necessarily determined from Eq. (54), where for non-neutral systems the plasma itself contributes by means of a nonvanishing equilibrium charge density. This conclusion also necessarily applies at all orders of accuracy in the expression of the total charge density $\rho^{\rm self}(\mathbf{x},\xi^k t)$.

It is nevertheless important to investigate whether, under suitable particular conditions, the quasineutrality condition can still be recovered. By inspection of Eq. (32), two different possibilities can be considered in this regard. The first one corresponds to the case of weak drift velocity, in the sense of the ordering $\delta_s M_s R \mathbf{V}_{Ds} \cdot \mathbf{e}_\varphi \sim O(\varepsilon_s)$. In such a case, the leading-order number density becomes a function of ψ-flux functions only, a feature which allows charge neutrality to be fulfilled pointwise. The second case occurs when δ_s vanishes identically at this order for all species. In fact, in this limit the constraint (56) reduces to

$$\sum_s \frac{Z_s e}{M_s^2} \frac{\beta_s}{\alpha_s \sqrt{\gamma_s}} = 0, \quad (57)$$

in which the left-hand side is a ψ-flux function only. Therefore, Eq. (57) can be satisfied identically pointwise by appropriate choice of the structure functions. Under this condition, the general solution of the ES potential is again provided by Eq. (54), where now the source $\rho^{\rm self}(\mathbf{x},\xi^k t)$ takes into account deviations from the quasineutrality (57) due to the higher-order nonvanishing plasma charge density arising from diamagnetic and FLR contributions. As a result, invoking Eqs. (29) and (55), correct to first order in the expansion parameters, one has

$$\rho^{\rm self}(\mathbf{x},\xi^k t) = \sum_s Z_s e \int_{\Gamma_v} d\mathbf{v}\, f_{0s}[\varepsilon_s h_{Ds} + \varepsilon_{M,s} h_{FLRs}]. \quad (58)$$

However, it must be remarked that, for arbitrary initial conditions on $(\alpha_s,\beta_s,\gamma_s)$, the quasineutrality condition is generally not fulfilled, so that the equilibrium remains non-neutral. This result follows consistently with the microscopic particle dynamics and conservation laws and with the validity of Vlasov-Maxwell equations, while the contribution of the gravitational field appears as already built in consistently through the equilibrium KDF. Notably, these conclusions regarding violation of local quasineutrality depart significantly from those obtained in Refs. [1–3,7].

We next consider the Ampere equation for the quasistationary self-generated magnetic field:

$$\nabla \times \mathbf{B}^{\rm self}(\mathbf{x},\xi^k t) = \frac{4\pi}{c} \mathbf{J}^{\rm self}(\mathbf{x},\xi^k t), \quad (59)$$

where the source from the collisionless plasma is given by the current density $\mathbf{J}^{\rm self}(\mathbf{x},\xi^k t)$ which is computed self-consistently in terms of the equilibrium KDF. Precisely, this is defined as

$$\mathbf{J}^{\rm self}(\mathbf{x},\xi^k t) \equiv \sum_s Z_s e n_s(\mathbf{x},\xi^k t) \mathbf{V}_s(\mathbf{x},\xi^k t), \quad (60)$$

where here

$$n_s(\mathbf{x},\xi^k t)\mathbf{V}_s(\mathbf{x},\xi^k t) = \int_{\Gamma_v} d\mathbf{v}\mathbf{v}\, f_{*s}. \quad (61)$$

Following the calculation of the previous section, we can write $\mathbf{J}^{\rm self} = \mathbf{J}_0^{\rm self} + \Delta \mathbf{J}^{\rm self}$, where $\mathbf{J}_0$ represents the leading-order contribution, which is given by

$$\mathbf{J}_0(\mathbf{x},\xi^k t) \equiv \sum_s Z_s e n_{0s}(\mathbf{x},\xi^k t) \mathbf{V}_{0s}(\mathbf{x},\xi^k t), \quad (62)$$

with $\mathbf{V}_{0s}$ being expressed by Eq. (40). Instead, correct to first order, $\Delta \mathbf{J}^{\rm self}$ is defined as

$$\Delta \mathbf{J}^{\rm self}(\mathbf{x},\xi^k t) = \sum_s Z_s e \int_{\Gamma_v} d\mathbf{v}\mathbf{v}\, f_{0s}[\varepsilon_s h_{Ds} + \varepsilon_{M,s} h_{FLRs}]. \quad (63)$$

The validity of Eq. (59) necessarily requires the constraint of solenoid current to be imposed on $\mathbf{J}^{\rm self}$, namely,

$$\nabla \cdot \mathbf{J}^{\rm self}(\mathbf{x},\xi^k t) = 0. \quad (64)$$

This represents the only solubility condition which is implied by the Ampere equation, together with possible additional periodicity conditions to be imposed in the case of locally closed and nested magnetic surfaces (see for example Ref. [5]). However, we notice that in the present description, Eq. (64) is necessarily identically satisfied, neglecting corrections of $O(\xi^k)$, with $k \geqslant 1$. In fact, recalling the discussion given at the beginning of this section and invoking Eq. (11), from the validity of the Vlasov equation resolved in terms of the KDF f_{*s}, it follows that for quasistationary systems for each species the continuity equation

$$\nabla \cdot [n_s(\mathbf{x},\xi^k t)\mathbf{V}_s(\mathbf{x},\xi^k t)] = 0 + O(\xi^k) \tag{65}$$

holds identically. After multiplication by the charge $Z_s e$ and summation over species, Eq. (65) recovers Eq. (64), which proves the statement. Hence, in such a case the Ampere equation does not provide any additional constraint on the solution, so that the charge current is only subject to the kinetic constraints (16) introduced in Sec. VI.

Inspection of Eq. (60) shows that the current $\mathbf{J}^{\text{self}}$ is generally nonvanishing, independent of the validity or the violation of the quasineutrality condition. This feature arises as a characteristic property of collisionless plasmas. In fact, as discussed above, the flow velocity $\mathbf{V}_s$ is species dependent and, in the absence of collisions, it is generally different for each species. In particular, this implies that even at leading order, the current density $\mathbf{J}_0$ is generally nonvanishing since the flow velocity $\mathbf{V}_{0s}$ given by Eq. (40) is mass dependent through the drift velocity $\mathbf{V}_{Ds}$, while the structure functions in the same expression are still arbitrary according to the constraint (21). In addition, it follows that in general the current $\mathbf{J}^{\text{self}}$ has nonvanishing components along all spatial directions. This feature arises also in combination with the inclusion of diamagnetic and FLR correction terms (see detailed discussions in Refs. [2,3,5,6] concerning this point). These conclusions allow us to state that no-energy equilibria are able to sustain a kinetic dynamo mechanism for the self-generation of quasistationary EM fields of the type similarly pointed out in Refs. [1–7], to which specifically kinetic effects contribute.

We conclude by stressing once again that the two main results of this section, concerning the charge non-neutral feature of no-energy equilibria and the kinetic dynamo, have been obtained consistently with the requirement of validity of the Vlasov-Maxwell equations. These conclusions apply to collisionless plasmas even in the presence of a background gravitational field, and can therefore be relevant also for astrophysical problems. In particular, we stress that the so-called condition of strict ambipolarity requiring the simultaneous validity of the neutrality condition $\rho^{\text{self}} = 0$ and the vanishing of the plasma current $\mathbf{J}^{\text{self}} = 0$ does not hold in the present case. In fact, in this formulation the no-energy collisionless plasma is generally non-neutral, i.e., $\rho^{\text{self}} \neq 0$. Nevertheless, the divergence-free condition (64) (weak ambipolarity condition) is found to be satisfied, which still admits $\mathbf{J}^{\text{self}} \neq 0$. We stress that this property should be viewed, in a sense, as a characteristic signature of collisionless plasmas. The same feature, in fact, appears in a variety of energy-dependent collisionless kinetic equilibria investigated previously in Refs. [1–7]. A further characteristic property of the no-energy kinetic equilibrium is that the KDF is characterized by nonuniform species fluid fields together with nonisotropic species pressure tensors. These remarks are characteristic of collisionless plasmas, and more specifically of no-energy equilibria described by phase-space nonisotropic KDFs of the type (15). As a consequence, the present theory departs significantly from treatments based on the adoption of local isotropic Maxwellian or energy-dependent distributions for the plasma species, as for the customary Pannekoek-Rosseland model of ambipolar electric field holding for ideal, isothermal, and electroneutral plasmas in stellar systems and based on fluid equations [27,28].

X. ABSOLUTE STABILITY OF NO-ENERGY EQUILIBRIA

In this section, we address the issue of the linear stability of the no-energy kinetic equilibria with respect to axisymmetric EM perturbations. In particular, the target here is to prove that absolute stability criteria apply for these configurations. As discussed in the following, this conclusion generalizes to the new family of no-energy kinetic solutions the analogous result established for the kinetic equilibria considered in Ref. [8].

The stability of equilibrium collisionless plasmas is a basic subject of theoretical research, which is particularly relevant for the astrophysics of accretion-disk plasmas. The transport phenomena responsible for the accretion flows in these systems are usually ascribed to the occurrence of fluid and/or kinetic instabilities. In particular, candidates for fluid instabilities driving the angular momentum transport include the magnetorotational instability and the thermal instability, caused by unfavorable gradients of rotation and shear and temperature, respectively (see related discussion in Ref. [8]). However, in the case of collisionless plasmas, the specific form of the equilibrium KDF must be taken into account in order to develop a consistent (kinetic) stability analysis.

The starting point concerns the prescription of the specific form of the perturbations and the appropriate ordering assumptions. We follow here the notation adopted in Ref. [8]. Then, given the validity of Eq. (11), for greater generality f_{*s} is allowed to exhibit slow-time variation on the equilibrium slow-time scale $(\Delta t)^{\text{eq}}$, i.e.,

$$\frac{d}{dt} \ln f_{*s} \sim \frac{1}{(\Delta t)^{\text{eq}}}, \tag{66}$$

with f_{*s} to be identified with the quasistationary no-energy KDF expressed by Eq. (15). Consistent with the assumption of having a collisionless plasma, this implies the validity of the inequality $\frac{(\Delta t)^{\text{eq}}}{\tau_{\text{col,s}}} \ll 1$, where $\tau_{\text{col,s}}$ denotes the Spitzer collision time for the species s. The problem of the linear stability of Vlasov-Maxwell equilibria of this type is addressed by considering perturbations of both the EM field and the equilibrium KDF which exhibit appropriate variation time and space scales $\{(\Delta t)^{\text{osc}},(\Delta L)^{\text{osc}}\}$. These perturbations are prescribed to have fast time and fast space dependencies with respect to those of the equilibrium quantities, in the sense that

$$\frac{(\Delta t)^{\text{osc}}}{(\Delta t)^{\text{eq}}} \sim \frac{(\Delta L)^{\text{osc}}}{(\Delta L)^{\text{eq}}} \sim O(\lambda), \tag{67}$$

with λ being a suitable infinitesimal parameter. It is also assumed that these perturbations are *nongyrokinetic*, i.e., they are characterized by typical wave frequencies and wavelengths which are much larger than the Larmor gyration frequency Ω_{cs} and radius r_{Ls}. This implies that the following inequalities must hold:

$$\frac{\tau_{Ls}}{(\Delta t)^{\text{osc}}} \sim \frac{r_{Ls}}{(\Delta L)^{\text{osc}}} \ll 1, \tag{68}$$

with $\tau_{Ls} = 1/\Omega_{cs}$, while λ must be such that $\lambda \gg \varepsilon_s, \varepsilon_{M,s}, \varepsilon$, where $\varepsilon \equiv \max\{\varepsilon_{M,s}\}$. These are referred to as *low-frequency* and *long-wavelength perturbations* with respect to the corresponding Larmor scales. Notice that Eqs. (67) and (68) are independent and complementary, establishing the upper and lower limits for the range of magnitudes of both $(\Delta t)^{osc}$ and $(\Delta L)^{osc}$.

Given the validity of the previous assumptions, it is required that the EM field is subject to axisymmetric EM perturbations of the form

$$\delta\mathbf{B} = \nabla \times \delta\mathbf{A}, \tag{69}$$

$$\delta\mathbf{E} = -\nabla\delta\phi - \frac{1}{c}\frac{\partial\delta\mathbf{A}}{\partial t}, \tag{70}$$

with $\{\delta\phi(\frac{\mathbf{x}}{\lambda},\frac{t}{\lambda}),\delta\mathbf{A}(\frac{\mathbf{x}}{\lambda},\frac{t}{\lambda})\}$ being assumed to be *analytic* (with respect to $\mathbf{x}$ and t), *infinitesimal*, i.e., such that $\frac{\delta\mathbf{E}}{|\mathbf{E}|},\frac{\delta\mathbf{B}}{|\mathbf{B}|} \sim O(\varepsilon)$, and energy eigenfunctions, namely of the form

$$\delta\phi\left(\frac{\mathbf{x}}{\lambda},\frac{t}{\lambda}\right) = \delta\widehat{\phi}\left(\frac{\mathbf{x}}{\lambda}\right)e^{i\omega t}, \tag{71}$$

$$\delta\mathbf{A}\left(\frac{\mathbf{x}}{\lambda},\frac{t}{\lambda}\right) = \delta\widehat{\mathbf{A}}\left(\frac{\mathbf{x}}{\lambda}\right)e^{i\omega t}, \tag{72}$$

where ω is the complex time frequency which, according to Eq. (67), is ordered as $\omega(\Delta t)^{eq} \sim 1/O(\lambda)$. This implies that the corresponding perturbations for the EM potentials must scale as $\frac{\delta\phi}{|\Phi|},\frac{\delta\mathbf{A}}{|\mathbf{A}|} \sim O(\varepsilon)O(\lambda)$, with $\mathbf{A}$ denoting the equilibrium vector potential [see Eq. (2)]. As a consequence, one obtains that, subject to such EM perturbations, the particle energy E_s defined by Eq. (13) varies as

$$\frac{d}{dt}E_s = Z_s e\left[\frac{\partial\delta\phi}{\partial t} - \frac{1}{c}\mathbf{v}\cdot\frac{\partial\delta\mathbf{A}}{\partial t}\right] = i\omega Z_s e\left[\delta\widehat{\phi} - \frac{1}{c}\mathbf{v}\cdot\delta\widehat{\mathbf{A}}\right]. \tag{73}$$

Instead, the canonical momentum $p_{\varphi s}$ and the magnetic moment m'_s remain invariants by construction, although their definition must be modified in accordance with the EM perturbations (for the appropriate definition of m'_s in such a case, see for example Refs. [29,30]). We next consider a perturbed KDF of the form

$$f_s = f_{*s} + \delta f_s. \tag{74}$$

Here, f_{*s} is identified with the no-energy KDF defined by Eq. (15), to be expressed in terms of the modified invariants. Instead, δf_s is the infinitesimal perturbation of the equilibrium KDF. For definiteness, $\delta f_s \equiv \delta f_s(t)$ is taken of the general form

$$\delta f_s \equiv \delta f_s\left(X_{*s},\frac{\mathbf{x}}{\lambda},\frac{t}{\lambda}\right) = \delta\widehat{f_s}\left(X_{*s},\frac{\mathbf{x}}{\lambda}\right)e^{i\omega t}, \tag{75}$$

with $\frac{\delta f_s}{f_{*s}} \sim O(\varepsilon)O(\lambda)$ and where $X_{*s} \equiv (E_s, p_{\varphi s}, m'_s)$ identifies now the set of the dynamical variables which are modified by the EM perturbations. In particular, in Eq. (75) no fast dependence is allowed with respect to the dynamical variables X_{*s}. Indeed, such an assumption is required for the consistency of the same representation (75).

As a consequence, the perturbation δf_s satisfies the equation

$$\frac{d}{dt}\delta f_s = -\frac{d}{dt}f_{*s}, \tag{76}$$

where the right-hand side is simply

$$\frac{d}{dt}f_{*s} \equiv \frac{d}{dt}E_s\frac{\partial f_{*s}}{\partial E_s}, \tag{77}$$

with the time derivative of the energy being provided by Eq. (73). For no-energy equilibria KDFs, the constraint (17) is identically satisfied by construction, so that Eq. (76) reduces to

$$\frac{d}{dt}\delta f_s = 0. \tag{78}$$

Let us now consider two cases. The first one corresponds to assume that the perturbation is produced externally at the initial time $t = t_0$, which requires that $\delta f_s(t_0) = 0$, while at the same time t_0 the EM perturbations $\{\delta\phi,\delta\mathbf{A}\}$ are nonvanishing and of the types (71) and (72). In this case, Eq. (78) requires manifestly that $\delta f_s(t) = 0$ identically, so that stability of the kinetic equilibrium is warranted. The second case is obtained by assuming that $\delta f_s(t_0) \neq 0$ and of the form given by Eq. (75). Then, due to the ordering assumptions in terms of the variables $(\mathbf{x},t)$, it follows that to the leading order in λ Eq. (78) implies that

$$i\omega + \mathbf{v}\cdot\nabla\ln\delta\widehat{f_s} = 0. \tag{79}$$

Equations (78) and (79) therefore yield

$$\omega = 0, \tag{80}$$

$$\delta\widehat{f_s} = \delta\widehat{f_s}(p_{\varphi s}, m'_s), \tag{81}$$

where possible additional slow-time dependencies contained in $\delta\widehat{f_s}$ must be ignored in view of Eq. (75). This means that δf_s is an equilibrium KDF of the same type as f_{*s} and, therefore, it can be absorbed in the very definition of the equilibrium KDF f_{*s} given by Eq. (15). These conclusions provide absolute stability criteria for no-energy kinetic equilibria, which hold in the two cases indicated above. As a consequence, *no analytic unstable perturbations can exist in axisymmetric collisionless plasmas characterized by no-energy kinetic equilibria.*

To conclude this section, let us briefly comment on these results and present a comparison with Ref. [8]. The following remarks are in order:

(i) The present stability analysis and the one presented in Ref. [8] share a number of important features, namely, (a) the specific form of the equilibrium KDF remains in both cases largely arbitrary since only generic forms of the functional dependencies need to be imposed. In particular, in the present case it is sufficient to require that f_{*s} is of the generic form given by Eq. (15). (b) The specific analytic form of the equilibrium EM fields is not needed. Nevertheless, for the existence of the no-energy equilibria, the existence of nonvanishing azimuthal and poloidal magnetic fields must be assured.

(ii) In Ref. [8], kinetic equilibria were considered in which the corresponding KDFs exhibit a functional dependence on the particle energy, both of explicit and implicit type. As a notable departure, in the present case such a dependence is instead excluded by construction as an *a priori* condition.

(iii) The stability criterion established in Ref. [8] applies in configurations characterized by weak toroidal magnetic field (with respect to the poloidal component), the presence of a small fluid accretion velocity compared with the azimuthal one, and sonic or subsonic flows. These constraints are not invoked or do not apply for no-energy equilibria. In fact, in the present case no ordering assumptions on the magnitude of the magnetic fields and/or the components of the flow velocity are needed to reach the stability criteria.

(iv) The analysis in Ref. [8] applies to collisionless plasmas belonging to the strongly magnetized and gravitationally bound regime. In difference with this, in this study the gravitationally bound condition is never invoked while the assumption of strongly magnetized ordering is not needed for the stability analysis. Therefore, in this sense the present analysis extends significantly the results of Ref. [8] since it avoids imposing the asymptotic conditions earlier invoked for stability.

XI. CONCLUDING REMARKS

In this paper, a kinetic treatment of collisionless multispecies axisymmetric magnetized plasmas has been presented, in the framework of the nonrelativistic Vlasov-Maxwell treatment. It has been proved that systems of this type admit the existence of kinetic equilibria which do not exhibit any kind of implicit or explicit functional dependence on the single-particle energy. These solutions have been referred to here as no-energy equilibria. Applications have been pointed out which concern both laboratory devices as well as collisionless plasmas in astrophysical scenarios, such as those arising in the environment of accretion-disk systems around compact objects.

The general functional form of the equilibrium kinetic distribution functions (KDFs) has been determined and shown to depend on a finite set of appropriate particle integrals of motion and adiabatic invariants. The condition of existence of this type of solution is represented by the simultaneous occurrence of nonvanishing azimuthal and poloidal magnetic fields. It has been proved that the same KDFs admit an explicit representation in terms of generalized Gaussian distributions which are subject to prescribed kinetic constraints and have a phase-space nonisotropic character. By implementing a perturbative theory holding in a proper subset of phase space, a Chapman-Enskog series representation has been obtained for the no-energy KDFs. This provides the basis for the analytical calculation of the constitutive equations for the fluid fields and the treatment of nonuniform collisionless plasmas of this type.

A number of significant features have been discussed, which show that no-energy kinetic equilibria are indeed of different character with respect to the quasistationary kinetic solutions considered in previous works which were expressed in terms of generalized bi-Maxwellian distributions. In particular, it has been proved that no-energy kinetic equilibria allow for the description of fluid systems possibly characterized by either supersonic or sonic and subsonic species-dependent flow velocities together with the occurrence of temperature and pressure anisotropies. Both these features have been shown to arise as a consequence of the simultaneous adiabatic conservation of the particle canonical momentum and guiding-center magnetic moment.

As shown here, analysis of the Maxwell equations, and in particular of the quasineutrality condition, proves that the no-energy kinetic equilibria are generally non-neutral and are characterized by the absence of the Debye screening effect. Nevertheless, these equilibria can still sustain self-generated electric currents, thus giving rise to an equilibrium kinetic dynamo mechanism.

As a final result, the absolute stability of the no-energy equilibria with respect to axisymmetric electromagnetic perturbations has been established. The proof concerns analytic axisymmetric perturbations characterized by low frequency and long wavelength with respect to the Larmor scales. It has been concluded that for these systems unstable perturbations of this type remain necessarily excluded. This follows uniquely from the specific functional form of the equilibrium KDFs which do not contain any kind of dependence on single-particle energy. Hence, the stationary configurations considered here have been shown to be absolutely stable against axisymmetric kinetic instabilities of this type. Remarkably, the stability criteria hold independently both of the strength of the azimuthal and poloidal magnetic fields to which the plasma is subject and of the magnetic regime to which the plasma belongs, provided the existence of guiding-center adiabatic invariants remains warranted. As a fundamental consequence, since fluid descriptions of these plasmas can only be based on the present Vlasov-Maxwell statistical theory, also axisymmetric MHD instabilities, such as the axisymmetric magnetorotational instability or thermal instabilities remain forbidden for collisionless plasmas described by no-energy equilibria.

The theoretical study presented in this work extends the kinetic treatment of collisionless plasmas recently developed in Refs. [1–8] to a new class of equilibrium solutions with distinctive novel features and characterized by absolute stability properties. The outcomes of this research can provide the framework for further theoretical and experimental investigations of the dynamics of these systems in both laboratory and astrophysical scenarios.

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