Solitons and kinks in a general car-following model

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We study a general car-following model of traffic flow on an infinitely long single-lane road, which assumes that a car's acceleration depends on time-delayed values of its own speed, the headway between it and the car ahead, and the rate of change of headway, but makes minimal assumptions about the functional form of that dependence. We present a detailed characterization of the onset of linear instability; in particular we find a specific limit on the delay time below which the marginal wave number at the onset of instability is zero, and another specific limit on the delay time above which steady flow is always unstable. Crucially, the threshold of *absolute* stability generally does not coincide with an inflection point of the steady-state velocity function. When the marginal perturbation at onset has wave number 0, we show that Burgers and Korteweg–de Vries (KdV) equations can be derived under the usual assumptions, and that corrections to the KdV equation "select" a single member of the one-parameter set of its one-soliton solutions by driving a slow evolution of the soliton parameter. While in previous models this selected soliton has always marked the threshold of a finite-amplitude instability of linearly stable steady flow, we find that it can alternatively be a stable, small-amplitude jam that occurs when steady flow is linearly unstable. The model reduces to the usual modified Korteweg–de Vries (mKdV) equation *only* in the special situation that the threshold of absolute stability coincides with an inflection point of the steady-state velocity function; in general, near the threshold of absolute stability the model reduces instead to a KdV equation in the regime of small solitons, while near an inflection point it reduces to a Hayakawa-Nakanishi equation. Like the mKdV equation, the Hayakawa-Nakanishi equation admits a continuous family of kink solutions, and the selection criterion arising from the corrections to this equation can be written down explicitly.

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I. INTRODUCTION

One approach to investigating the collective behavior of vehicular traffic is to start with a "car-following model," which is a set of equations describing the response of each individual car to what neighboring cars are doing. (For reviews of other approaches to the theory of vehicular traffic, see [1–3].) A typical car-following model describes a single line of cars traveling along a long, uniform, straight or circular road, with all drivers taken to behave identically. An important prototype of this class of models is the "optimal velocity model" (OVM) of Bando *et al.* [4], which is embodied in the equations

$$\tau \frac{dv_n}{dt} = V_s(x_{n+1} - x_n) - v_n, \quad v_n = dx_n/dt.$$

Here the cars move in the positive x direction, they are numbered consecutively, with car $n+1$ ahead of car n, and x_n denotes the position of, say, the back of car n. The "optimal velocity" function $V_s(h)$ is the speed at which a driver prefers to drive when the next car is ahead by a distance, or "headway," of h. The model assumes that each driver relaxes his or her car's speed to that preferred velocity with some fixed time constant τ.

A model like the OVM, of course, ignores many features of the behavior of real human drivers, who not only have different reaction times, but also have individual preferred velocity functions which can change because of road conditions, fatigue, distractions, and the like. However, such a model would give a better description of a line of cars controlled by on-board adaptive cruise control systems, which would use a small radar to monitor the headway to the next car. The equation of motion would then be implemented by the programming of the cruise control.

The OVM was quickly generalized [5] to account for the fact that the proper response to make to a car that is a distance h ahead must also depend on that car's speed—the response of car n must be very different if the next car in line is stopped than it would be if that car is moving at some speed near v_n. A particularly simple model that incorporates this effect is the "full velocity difference model" (FVDM) of Jiang *et al.* [6], given by

$$\tau \frac{dv_n}{dt} = V_s(x_{n+1} - x_n) - v_n + \lambda\,(v_{n+1} - v_n),$$

where λ is a constant which specifies the relative importance of the size of the headway and its rate of change in determining the response of car n. Tian, Jia, and Li [7] suggested replacing the $v_{n+1} - v_n$ in the FVDM with a general function of $v_{n+1} - v_n$, naming their model the "comprehensive optimal velocity model" (COVM); they carried out calculations in the specific case where the function was a hyperbolic tangent. A number of specific car-following models have been proposed in the literature, some incorporating finite time delays [8], look-ahead [9,10], look-back [11], and a dependence on the acceleration of the next car [12]. For reviews, see [1,2,13].

All these models have a one-parameter family of solutions that describe uniform traffic flow, with the headway in front of each car being a constant Δ (the free parameter) and all cars traveling at speed $V_s(\Delta)$. Most studies begin by analyzing the linear stability of this uniform steady flow, and many [7–12, 14–22] then reduce the model to a Burgers, Korteweg–de Vries (KdV), or modified Korteweg–de Vries (mKdV) equation in

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various parameter ranges. These analyses yield results which are strikingly similar from model to model:

(1) Steady, uniform flow with spacing Δ is linearly stable if and only if $V_s'(\Delta)$, the derivative of the preferred velocity function, is below some critical value which depends on the parameters of the model. (The preferred velocity function is often taken to have a sigmoidal, hyperbolic-tangent-like shape, in which case this means that light traffic, with a large spacing Δ, is stable, and very heavy traffic, with small Δ, is also stable albeit slow, while instability can happen for intermediate traffic densities.) "Absolute stability"—linear stability for all Δ—is obtained when the slope $V_s'(\Delta)$ never exceeds the critical value. Otherwise, when $V_s'(\Delta)$ just reaches the critical value, the marginal perturbations have vanishingly small wave numbers.

(2) If the headway between cars varies only slightly from the uniform spacing Δ, and only varies slightly from car to car, then one can derive a Burgers equation for headway as a function of time and car number, with the diffusion coefficient proportional to the difference between $V_s'(\Delta)$ and its critical value. When uniform traffic is linearly stable, the diffusion coefficient is positive, and the Burgers equation then describes how slowly varying deviations from uniformity diffuse away (via an interesting intermediate-asymptotic evolution [23]). When uniform flow is linearly *unstable*, the diffusion coefficient is negative, so the Burgers equation becomes a *backwards* diffusion equation. Smooth initial perturbations then grow sharper, rapidly invalidating the assumptions under which the Burgers equation is derived.

(3) When $V_s'(\Delta)$ is close to critical, one can reduce the model to a KdV equation for the headway, plus small corrections. Of the one-parameter family of one-soliton solutions of the KdV equation, only a single "selected" member remains a solution once the correction terms are applied, and this only when uniform flow is linearly stable. Thus the selected soliton seems to mark the threshold of a nonlinear instability of linearly stable traffic.

(4) Near the threshold of *absolute* stability—that is, when the derivative of V_s at its inflection point is close to the critical value—the headway is described by a mKdV equation, plus correction terms. The mKdV equation has kink solutions which represent transition zones between regions of different traffic densities. As for the one-soliton solutions of the KdV equation, there is a continuous family of these one-kink solutions, and the correction terms select a single member of this family.

The above results are often regarded as universal features of near-uniform traffic flow, yet some caution is called for. The reduced equations (Burgers, KdV, and mKdV) are obtained from rather long perturbation series, which may well be sensitive to "higher-order" effects that are typically left out of the models. For instance, the derivations would be sensitive to a nonlinearity in the dependence of the acceleration of car n on its velocity. Would this lead to any important changes to the results, or is the term $(V_s - v)/\tau$ in the OVM an adequate representation of the tendency of a driver to relax to the preferred velocity, as the models implicitly assume? Is it important to allow the time constant τ to depend on headway, as suggested by Gasser *et al.* [24]? The λ parameter in the FVDM surely should depend on headway—the speed of car $n+1$ must be irrelevant if it is, say, several kilometers ahead of car n. Does this have consequences? Are there other effects that are relevant?

To address these questions, we analyze a very general car-following model with time delay, embodied in the equation [25]

$$\frac{dv_n(t+t_d)}{dt} = A(x_{n+1}-x_n, v_{n+1}-v_n, v_n). \tag{1.1}$$

Here t_d is a fixed delay time, and the acceleration function $A(h,\dot{h},v)$ is a general function of the speed $v = v_n$ of the car under consideration, the headway $h = x_{n+1} - x_n$ between it and the next car ahead, and the rate $\dot{h} = v_{n+1} - v_n$ at which the headway is changing. Thus in this model each driver responds, at time t, to the position and velocity of the next car relative to his own, and his own velocity, at some fixed reaction time t_d earlier. To be a realistic model of driver behavior, the acceleration function must satisfy some minimal conditions which we will discuss in Sec. II. Note that the model still assumes a single line of cars, so that the behavior of car n depends only on the single car ahead of it, and it also assumes a uniform road, in that there is no position dependence in the acceleration function. The model does not include look-ahead or look-back effects (viz., a dependence of the behavior of car n on car $n+2$ or car $n-1$).

We find that many, but not all, of the results found in specific car-following models to date are in fact universal at the level of generality of (1.1), provided the delay time t_d is not too large, in a sense that will be made precise in Sec. II. Specifically, the onset of linear stability is correctly given by the results from the FVDM, given appropriate definitions of the steady velocity function $V_s(\Delta)$ and the parameters τ and λ, and the derivations of the Burgers and KdV equations go through as described above. On the other hand, some of the results above are *not* universal; rather, the equation of motion (1.1) can lead to behavior that has not been seen in specific models in the literature that are of that general form. Most of these new possibilities arise from the fact that the linear stability parameters τ and λ, and therefore the critical value of V_s', can be nontrivial functions of the uniform traffic spacing Δ.

While (1.1) still reduces to the KdV equation plus higher-order corrections when parameters are near the onset of linear instability, the increased generality leads to an extra term in the coefficient of one of the correction terms, which can lead to qualitatively different behavior. With this term absent, we show that the corrections "select" a soliton only when uniform traffic flow is linearly stable, and they then lead to slow evolution of the other one-soliton solutions *away* from the selected soliton. Thus the onset of linear instability is a reverse bifurcation, and the selected soliton marks the threshold of a finite-amplitude instability of the linearly stable flow. With the new term, however, it is possible for this to reverse: the corrections can select a soliton when uniform flow is linearly *unstable*, with the slow evolution of other soliton solutions taking them *toward* the selected soliton. The onset of linear instability then becomes a forward bifurcation to a state with small-amplitude jams.

The generic situation near the threshold of *absolute* stability is quite different than that which has been observed previously in specific models of the general form (1.1). Since the critical value of V_s' generally depends on the traffic spacing Δ, the threshold of absolute instability, where the maximum *difference* between $V_s'(\Delta)$ and its critical value is zero, does not generally occur at an inflection point of V_s. The model only reduces to a mKdV equation when these two things *do* coincide. When they differ, different reduced equations are found at the two points. Near the threshold of absolute stability, the model reduces to a KdV equation, in the regime that leads to small solitons. Near an inflection point of V_s, the leading-order reduced equation is a mKdV equation plus an extra term. The possibility of having such an extra term in this context was anticipated by Komatsu and Sasa [18], and the new equation was derived from a model including look-back by Hayakawa and Nakanishi [11]. Like the mKdV equation, the Hayakawa-Nakanishi equation admits a one-parameter family of one-kink solutions, and the correction terms select a single member of this family.

In Sec. II we discuss the conditions that the acceleration function $A(h,\dot{h},v)$ must satisfy in order for (1.1) to be a reasonable model of traffic flow, and we carry out the linear stability analysis of steady traffic flow with a uniform spacing Δ with and without a time delay. We identify the linear stability parameters, and give general results for the onset of instability of steady, uniform traffic flow, including the wave number of the marginal perturbation at onset. We show that the onset condition is identical to that with zero time delay, and the marginal wave number is zero, when the delay time t_d is below a specific limit. In Sec. III we expand (1.1) about the uniform steady state, assuming that the headway varies only slowly from car to car. From the leading order of this expansion we find that the Burgers equation appears exactly as described above. In Sec. IV we modify the expansion from Sec. III to apply when conditions are close to the onset of instability, deriving the KdV equation and its leading-order corrections. We then carry out a multiple-time-scales analysis of the one-soliton solutions of the leading-order KdV equation. In Sec. V we again modify the expansion from Sec. III, this time to apply to a uniform steady state that is close to an inflection point of the steady-state velocity function. In previous models, this has corresponded to the threshold of absolute stability, but at the level of generality of (1.1) this correspondence breaks down. We derive the Hayakawa-Nakanishi equation and its corrections, and carry out the solvability analysis of the one-kink solutions of this equation. We discuss our results in Sec. VI.

II. STEADY STATES AND LINEAR STABILITY

Before deriving and analyzing the steady states of the model embodied in (1.1), we first set forth some conditions which a realistic acceleration function $A(h,\dot{h},v)$ must satisfy [25]. Here and throughout this paper we will assume that $A(h,\dot{h},v)$ is differentiable as many times as necessary. For a given headway h and a given rate of change of headway $\dot{h}$, a driver will be more prone to decelerate or less prone to accelerate the faster his or her car is traveling. Similarly, other things being equal a driver is more prone to accelerate the larger the headway is, or the more rapidly it is increasing. Thus we expect to have

$$\frac{\partial A}{\partial v} \leqslant 0, \quad \frac{\partial A}{\partial h} \geqslant 0, \quad \frac{\partial A}{\partial \dot{h}} \geqslant 0. \tag{2.1}$$

We will in fact assume slightly more, namely that $\partial A/\partial v$ is strictly negative, or at least that there are no finite ranges of v over which $A(h,0,v)$ is constant.

To find simple steady-state solutions of the model, we assume that the cars are equally spaced, and that they all cruise at the same speed, so they remain equally spaced. If the headway in front of each car is Δ, then this uniform flow is a steady state provided the common speed of the cars is $V_s(\Delta)$, which is defined implicitly by

$$A(\Delta,0,V_s(\Delta)) = 0. \tag{2.2}$$

From the first inequality in (2.1) we see that $A(\Delta,0,v)$ is a nonincreasing function of v; given our assumption above that $A(h,0,v)$ does not remain constant over any finite range of v, the steady flow speed for a given Δ will then be unique. Thus we have a one-parameter family of steady states of the model, labeled by the spacing Δ; of course, these steady-flow solutions may be stable or unstable.

The linear stability analysis of steady-flow solutions with no delay ($t_d = 0$) has been carried out at this level of generality by Wilson [25] and by Orosz *et al.* [26], while the analysis with nonzero delay has been considered by Orosz *et al.* [27]. Here we will review the zero-delay analysis and characterize the onset of instability with finite delay. As usual, we begin by taking the initial positions of the cars to differ infinitesimally from the exact steady-flow solution. Since the underlying model is translationally invariant, we take the deviation of x_n to be a linear combination of Fourier modes $\exp(ikn)$, and since it is time-translation invariant we take each mode to vary exponentially in time, as $\exp(\sigma t)$. Substituting into the equation of motion (1.1) and linearizing yields an equation for the (complex) linear growth rate $\sigma(k)$ of the mode of wave number k,

$$\sigma^2\tau(\Delta)e^{\sigma t_d} + \sigma = [V_s'(\Delta) + \lambda(\Delta)\sigma](e^{ik} - 1). \tag{2.3}$$

The new parameters τ and λ are defined by

$$\frac{1}{\tau} = -A_v(\Delta,0,V_s(\Delta)), \quad \frac{\lambda}{\tau} = A_{\dot{h}}(\Delta,0,V_s(\Delta)), \tag{2.4}$$

and differentiating the definition (2.2) of V_s with respect to Δ leads to

$$V_s'(\Delta) = \tau A_h(\Delta,0,V_s(\Delta)). \tag{2.5}$$

Here and henceforth, primes denote derivatives with respect to Δ, and subscripts denote partial derivatives of A, e.g., $A_h \equiv \partial A/\partial h$, evaluated in the steady state ($h = \Delta, \dot{h} = 0, v = V_s(\Delta)$). We see from the inequalities (2.1) that τ, λ, and V_s' must all be positive (or at least non-negative). The definitions (2.4) agree with the notation of Ou *et al.* [15] for the FVDM, but unlike in that paper λ and τ can be nontrivial functions of the spacing Δ.

With no time delay, the analysis of (2.3) is straightforward and well known. One finds that steady flow is linearly unstable if $V_s'(\Delta)$ is greater than

$$\Omega_c(\Delta) \equiv \frac{1 + 2\lambda(\Delta)}{2\tau(\Delta)}. \tag{2.6}$$

Note that it is possible to have *absolute stability* if V_s' never exceeds Ω_c for any Δ. Otherwise, one finds that as V_s' is increased past Ω_c, the first perturbation to become unstable has wave number $k = 0$. Expanding (2.3) as a power series in k shows that the linear growth rate for small k then obeys $\sigma(k) = ikV_s'(\Delta) + O(k^2)$, so that as the perturbation grows or decays, it moves at a rate (in *cars* per unit time) of $V_s'(\Delta)$ in the upstream direction relative to the cars. Since the cars are a distance Δ apart, at least before the perturbation develops, this is a *distance* per unit time of $\Delta V_s'(\Delta)$ relative to the cars. And since the cars are moving in the downstream direction at $V_s(\Delta)$ relative to the road, the perturbation moves relative to the road at a velocity of approximately $V_s(\Delta) - \Delta V_s'(\Delta)$, which may be positive (downstream) or negative (upstream).

We will show below that a small delay time does not change this picture; on the other hand, a moderately large delay time can change it dramatically. To see an extreme example of the latter, consider the basic stability relation (2.3) for $k = 0$ perturbations. These satisfy either $\sigma = 0$ or

$$(\sigma t_d)e^{\sigma t_d} = -t_d/\tau. \tag{2.7}$$

When there is no time delay, this equation describes a linearly stable perturbation with growth rate $\sigma = -1/\tau$, but it is straightforward to show that it has roots with positive real parts if t_d/τ exceeds $\pi/2$. Thus for a delay this long, steady flow is unstable no matter what the values of V_s' and λ are.

For shorter delay times, we identify the onset of instability of steady flow by first looking for conditions under which a perturbation with a given wave number k is marginally unstable, and then seeking the minimum value of the control parameter V_s'/Ω_c for which *some* perturbation is marginal. Specifically, we first set the linear growth rate in (2.3) to something purely imaginary, $\sigma t_d = i\theta$, then separate the real and imaginary parts to obtain

$$\tan\frac{k}{2} = \frac{\theta\cos\theta}{(1+2\lambda)(t_d/\tau) - \theta\sin\theta}, \tag{2.8a}$$

$$\frac{V_s'}{\Omega_c} = \frac{\tau^2\theta^2 - 2(1+\lambda)t_d\tau\theta\sin\theta + (1+2\lambda)t_d^2}{(1+2\lambda)t_d^2\cos\theta}. \tag{2.8b}$$

These can be regarded as parametric equations for the critical curves: given a θ, and with set values of the delay time t_d and the parameters λ and τ (determined from the traffic spacing Δ), they give the wave number k of the perturbation whose linear growth rate is $-i\theta/t_d$ and the critical value of the control parameter V_s'/Ω_c at which it becomes unstable. A plot of the critical control parameter versus k typically has a minimum which occurs for some θ below $\pi/2$. The value at this minimum marks the onset of linear instability of steady flow: when the control parameter is below it, no perturbation is marginal, but rather all perturbations decay.

For small θ, we find from (2.8a) and (2.8b) that perturbations with small wave numbers,

$$k = \frac{2\tau}{(1+2\lambda)t_d}\theta + O(\theta^3), \tag{2.9}$$

are marginally unstable for

$$\frac{V_s'}{\Omega_c} = 1 + \frac{P\tau^2}{(1+2\lambda)t_d^2}\theta^2 + O(\theta^4), \tag{2.10}$$

where

$$P = 1 - 2(1+\lambda)(t_d/\tau) + \frac{1+2\lambda}{2}(t_d/\tau)^2. \tag{2.11}$$

If P is positive, then the critical control parameter has a local minimum at $\theta = 0$. This occurs provided the delay time is sufficiently short, specifically if it satisfies

$$\frac{t_d}{\tau} < 2\,\frac{1+\lambda-\sqrt{\frac{1}{2}+\lambda+\lambda^2}}{1+2\lambda}. \tag{2.12}$$

In fact, with some algebraic effort we can show that when t_d/τ is below this limit, $\theta = 0$ gives the *global* minimum of the critical control parameter. Thus when (2.12) holds, the results for onset of linear stability are identical to the results with no delay: it occurs at $V_s' = \Omega_c$, with $k = 0$ perturbations being the first to go unstable and the linear growth rates of long-wave perturbations going as $\sigma \approx i\Omega_c k$.

Otherwise, we find the θ at which the critical control parameter is minimized by setting the θ derivative of (2.8b) to zero, which yields

$$0 = (2\theta\cos\theta + \theta^2\sin\theta) - 2(1+\lambda)(t_d/\tau)(\theta + \cos\theta)\sin\theta + (1+2\lambda)(t_d/\tau)^2\sin\theta. \tag{2.13}$$

Our results for the onset of instability are shown in Fig. 1, which plots the parameters λ on the horizontal axis and t_d/τ on the vertical axis. Solid and dotted lines show, respectively, values of the control parameter above which steady flow is linearly unstable for given λ and t_d/τ, and the wave numbers at which the first instability appears.

The lowest solid curve marks the upper boundary of the region given by (2.12), in which instability sets in at $V_s' = \Omega_c$ and $k = 0$. Higher solid curves are found by solving (2.8b) and (2.13) simultaneously for λ and t_d/τ with successively smaller values of V_s'/Ω. Along the topmost curve, the minimum critical control parameter is zero, and above this curve it would be negative. Since both V_s' and Ω_c must be positive, above this

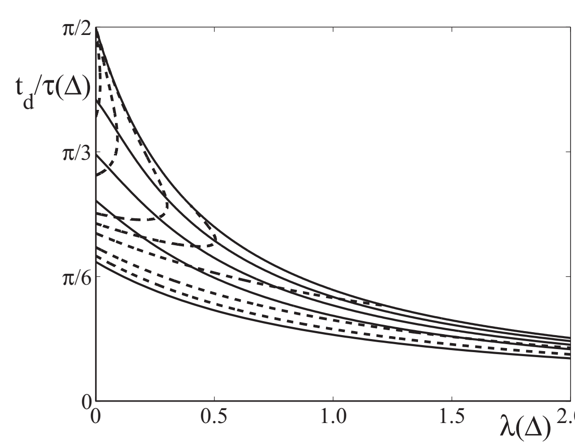

FIG. 1. Stability diagram for steady traffic flow with a uniform spacing Δ. The delay time is t_d and the parameters $\tau(\Delta)$ and $\lambda(\Delta)$ are defined in (2.4). Solid curves mark parameter values for which steady flow first becomes linearly unstable when the control parameter $V_s'(\Delta)/\Omega_c(\Delta)$ equals (upwards from the bottom) 1.0, 0.75, 0.50, 0.25, and 0.0. Dashed curves mark parameter values for which the first mode to become unstable has wave number (upwards and left from the bottom) $\pi/3$, $\pi/2$, $2\pi/3$, $3\pi/4$, $\tan^{-1}(-2/\pi) = 0.819\,546\pi$, π, and $7\pi/6$. For all parameter values below the lowest solid curve, instability sets in at $V_s'(\Delta)/\Omega_c(\Delta) = 1$ with wave number 0. Above the highest solid curve, steady flow is always linearly unstable.

topmost curve steady flow would be *absolutely unstable*: it would be linearly unstable no matter how small V_s' is.

The dotted curves mark points in parameter space at which the first instability occurs at a given wave number, found by solving (2.8a) and (2.13) simultaneously for λ and t_d/τ. For $k \to 0$ the curve coincides with the lowest solid curve, and for higher k the curves shift upward and to the left. Each curve crosses the absolute instability boundary $V_s' = 0$, up to $k = \tan^{-1}(-2/\pi) = 0.819\,546\,\pi$. From this wave number up to $k = \tan^{-1}(\pi/2) = 1.319\,546\,\pi$, the dotted curves approach the point $\lambda = 0, t_d/\tau = \pi/2$ asymptotically as $\theta \to \pi/2$. Note that there is nothing special about the "pairing" mode $k = \pi$.

We see, then, that simple car-following models like the FVDM [15] do capture the important results of linear stability analysis for the much more general model embodied in (1.1), provided the delay time satisfies (2.12): steady traffic flow with constant spacing Δ is linearly stable when $V_s'(\Delta)$ is less than $\Omega_c(\Delta)$, defined by (2.6) and (2.4); at the onset of instability it is the long-wavelength ($k = 0$) perturbations that first become unstable; at onset these perturbations move upstream through the line of cars at a rate of $V_s'(\Delta)$. This is not too surprising, since the FVDM is essentially the general model (albeit with no time delay) expanded to linear order in the car velocities. What is different here, however, is the possibility that the stability limit Ω_c can be a nontrivial function of Δ. In particular, the criterion for absolute stability—in which steady flow is linearly stable for all Δ—is that the maximum value of $V_s'(\Delta) - \Omega_c(\Delta)$ be negative, *not* that the slope of V_s at its *inflection point* be less than Ω_c. This will turn out to have important consequences for the reduced equations that approximate the full model, both near the onset of stability and near the threshold of absolute stability.

III. LONG-WAVE PERTURBATIONS

As we saw from the linear stability analysis, when traffic conditions are such that steady flow is just on the verge of instability and the delay time is not too large, it is the long-wavelength perturbations that are the first to grow. It is therefore of interest to approximate the full model in a way that focuses on these perturbations. To this end, we assume that the deviation of the cars' actual positions from their steady-flow positions varies slowly from one car to the next. By expanding (2.3) for small wave number k, we find that the corresponding linear growth rate is given by

$$\sigma(k) = V_s'(\Delta)[ik + (V_s' - \Omega_c)\tau k^2 + O(k^3)], \tag{3.1}$$

so that each mode grows or decays at a rate proportional to k^2, while moving relative to traffic at a rate (in cars per unit time) of $-V_s'(\Delta)$. Motivated by these considerations we write

$$x_n = n\Delta + V_s(\Delta)t + f(z,T), \tag{3.2}$$

where

$$z \equiv \epsilon[n + V_s'(\Delta)t], \quad T \equiv \epsilon^2 t. \tag{3.3}$$

Here, ϵ is a small but otherwise arbitrary parameter and f and its derivatives are taken to be of order unity. Thus in order for f to *change* by an amount of order unity, the car number n must change by an amount of order ϵ^{-1}, so that ϵ measures the slowness of the variation of the position deviation f along the line of cars. This ansatz amounts to assuming that there is a pattern of deviations of headway from the uniform spacing Δ, that this pattern propagates upstream through the line of cars at a rate $V_s'(\Delta)$, and that its shape changes slowly as it propagates.

If f is of order unity, then the velocities of the cars and the spacings between them deviate from their steady-flow values by order ϵ,

$$v_n = V_s(\Delta) + \epsilon V_s'(\Delta) f_z + \epsilon^2 f_T, \tag{3.4}$$

$$x_{n+1} - x_n = \Delta + \epsilon f_z + \tfrac{1}{2}\epsilon^2 f_{zz} + \cdots, \tag{3.5}$$

where subscripts again denote partial derivatives. As we see, the function

$$g(z,T) \equiv f_z(z,T) \tag{3.6}$$

then gives the leading-order deviation in both headway and velocity from the exact steady state; we will have occasion to express many of our results in terms of it.

To derive an evolution equation for f, we substitute the ansatz (3.2) and (3.3) into the equation of motion (1.1) and expand in powers of ϵ. The calculation is lengthy but straightforward; carrying it out to fourth order yields (after canceling an overall factor ϵ^2)

$$\begin{aligned} f_T = {} & D f_{zz} + \tfrac{1}{2}V_s'' f_z^2 + \epsilon\big[C_{11T} f_{zT} + C_{11} f_{zzz} + C_{12T} f_z f_T \\ & + C_{12} f_z f_{zz} + C_{13} f_z^3\big] + \epsilon^2\big[C_{21} f_{zzzz} + C_{22a} f_z f_{zzz} \\ & + C_{22b} f_{zz}^2 + C_{23} f_z^2 f_{zz} + C_{24} f_z^4 + \cdots\big] + O(\epsilon^3). \end{aligned} \tag{3.7}$$

The omitted terms in the ϵ^2 correction all involve T derivatives of f, and turn out not to affect any of our analysis. The various coefficients are given by

$$D = (\Omega_c - V_s')\tau V_s', \tag{3.8a}$$

$$C_{11T} = \lambda - 2\tau V_s', \tag{3.8b}$$

$$C_{11} = \left(\frac{1+3\lambda}{6} - \tau t_d V_s'^2\right) V_s', \tag{3.8c}$$

$$C_{12T} = \tau(A_{hv} + A_{vv} V_s') = \frac{\tau'}{\tau}, \tag{3.8d}$$

$$\begin{aligned} C_{12} &= \frac{\tau}{2}\big[A_{hh} + (A_{hv} + 2A_{h\dot h})V_s' + 2A_{\dot h v} V_s'^2\big] \\ &= \frac{1}{2}(V_s'' + 2\tau V_s' \Omega_c'), \end{aligned} \tag{3.8e}$$

$$\begin{aligned} C_{13} &= \frac{\tau}{6}\big(A_{hhh} + 3A_{hhv} V_s' + 3A_{hvv} V_s'^2 + A_{vvv} V_s'^3\big) \\ &= \frac{1}{6}\left(V_s''' - \frac{3\tau'}{\tau} V_s''\right), \end{aligned} \tag{3.8f}$$

$$C_{21} = \left(\frac{1+4\lambda}{24} - \frac{1}{2}\tau t_d^2 V_s'^3\right) V_s', \tag{3.8g}$$

$$\begin{aligned} C_{22a} &= \frac{\tau}{6}\big[A_{hh} + (A_{hv} + 3A_{h\dot h})V_s' + 3A_{\dot h v} V_s'^2\big] \\ &= \frac{1}{6}\left[V_s'' + \tau V_s' \left(\frac{1+3\lambda}{\tau}\right)'\right], \end{aligned} \tag{3.8h}$$

$$\begin{aligned} C_{22b} &= \frac{\tau}{8}\big(A_{hh} + 4A_{h\dot h} V_s' + 4A_{\dot h\dot h} V_s'^2\big) \\ &= \frac{1}{8}\big[V_s'' + 4\tau V_s' \Omega_c' + \tau V_s'^2 (A_{vv} - 4A_{\dot h v} + 4A_{\dot h\dot h})\big], \end{aligned} \tag{3.8i}$$

$$C_{23} = \frac{\tau}{4}\big[A_{hhh} + 2(A_{hhv} + A_{hh\dot{h}})V_s' + (A_{hvv} + 4A_{h\dot{h}v})V_s'^2 + 2A_{\dot{h}vv}V_s'^3\big] = \frac{1}{4}\left[V_s''' - \frac{3\tau'}{\tau}V_s'' + 2\tau V_s'\Omega_c'' + \tau V_s'V_s''(A_{vv} - 2A_{\dot{h}v})\right], \tag{3.8j}$$

$$C_{24} = \frac{\tau}{24}\big(A_{hhhh} + 4A_{hhhv}V_s' + 6A_{hhvv}V_s'^2 + 4A_{hvvv}V_s'^3 + A_{vvvv}V_s'^4\big) = \frac{1}{24}\left[V_s'''' - \frac{4\tau'}{\tau}V_s''' + 6\tau\left(\frac{1}{\tau}\right)''V_s'' + 3\tau A_{vv}V_s''^2\right], \tag{3.8k}$$

where all derivatives of A are evaluated in the steady state, and derivatives of V_s, τ, λ, and $\Omega_c = (1+2\lambda)/2\tau$ are evaluated at Δ. In the OVM and FVDM, the acceleration function $A(h,\dot{h},v)$ is taken to be linear in the velocities $\dot{h}$ and v, with coefficients that are independent of h. For these models, then, τ, λ, and Ω_c are constants, and many of the coefficients above simplify considerably. As we will soon see, the fact that C_{12} then reduces to $V_s''/2$ will prove particularly consequential. On the other hand, features that arise due to nonzero Δ derivatives of τ, λ, and Ω_c will be novel.

According to (3.7), the position deviation f obeys a Burgers equation with corrections,

$$f_T = Df_{zz} + \tfrac{1}{2}V_s''f_z^2 + O(\epsilon), \tag{3.9}$$

or, after differentiating with respect to z and using the definition (3.6),

$$g_T = Dg_{zz} + V_s''gg_z + O(\epsilon). \tag{3.10}$$

This result has been obtained previously for the OVM [14], the FVDM [15], and the COVM [7]; the present calculation shows that the derivation of the Burgers equation as well as the identification of the coefficients in terms of the steady velocity function V_s continue to be valid even at the level of generality of (1.1).

Near the onset of instability of steady traffic, D is small. Thus, in this regime, the neglected order-ϵ corrections might have effects comparable in size to those of the linear term in the Burgers equation. We turn next to an examination of this situation.

IV. NEAR THE ONSET OF INSTABILITY

Suppose the delay time is short enough to satisfy (2.12), and parameters are near the onset of instability. Specifically, we write

$$V_s'(\Delta) = \Omega_c(\Delta) + \epsilon^2\delta, \tag{4.1}$$

where ϵ is small and δ is arbitrary; we could (but will not) choose δ to be $+1$ when steady flow is slightly unstable or -1 when it is slightly stable, which would then give a specific meaning to the previously arbitrary small parameter ϵ. The unstable perturbations then have wave numbers of order ϵ, and from the expansion (3.1) we see that their amplitudes grow at rates of order ϵ^3. Thus we retain the definition of z in (3.3), but replace the definition of T by

$$T \equiv \epsilon^3 t. \tag{4.2}$$

We see from (3.8a) that D is of order ϵ^2, so in (3.7), either f must grow without bound or the f_z^2 term must eventually be balanced by the correction terms; in the latter case, the size of f must then be of order ϵ. Thus we modify the original ansatz (3.2) to read

$$x_n = n\Delta + V_s(\Delta)t + \epsilon f(z,T). \tag{4.3}$$

We may now substitute this new ansatz into the equation of motion (1.1) and expand to fifth order in ϵ. Equivalently, however, we can simply note that the new scalings have the effect of multiplying each f in (3.7) by an additional factor of ϵ and also multiplying each T derivative by an additional ϵ. After canceling an overall factor of ϵ^2, this then gives, to order ϵ,

$$f_T = C_{11}f_{zzz} + \frac{1}{2}V_s''f_z^2 + \epsilon\left[-\frac{1+2\lambda}{2}\delta f_{zz} + C_{11T}f_{zT} + C_{12}f_zf_{zz} + C_{21}f_{zzzz}\right]. \tag{4.4}$$

The leading order of this evolution equation is related to the Korteweg–de Vries equation. To put it in the standard KdV form, we first take its z derivative and rewrite it in terms of $g \equiv f_z$. In the corrections, we then replace g_T with the leading-order terms, substitute the explicit expressions for the coefficients, and use the fact that V_s' is close to Ω_c. This gives

$$g_T = C_{11}g_{zzz} + V_s''gg_z - \frac{1+2\lambda}{2}\epsilon \times\left[g\delta + \frac{1+2\lambda}{4}PV_s'g_{zz} + \frac{1}{2}(V_s'' - \Omega_c')g^2\right]_{zz}, \tag{4.5}$$

where P, given by (2.11), is positive for delay times short enough to satisfy (2.12); one can show that C_{11} is then positive as well.

One-soliton solutions of the KdV equation—the leading order of (4.5)—describe local increases or decreases in the car density that preserve their form while propagating along the line of cars. There is a one-parameter family of these solutions, given explicitly by

$$g^{(0)}(z,T;q) = Mq^2\text{sech}^2(qz + q^3uT), \tag{4.6}$$

where the soliton parameter q is free, and

$$M = \frac{12C_{11}}{V_s''}, \quad u = 4C_{11}. \tag{4.7}$$

Recall that z specifies the location of the soliton in a reference frame moving upstream at a rate (cars per unit time) of V_s', so the effect of the u in this expression is to increase that rate by ϵ^2q^2u. The amplitude M is positive if V_s'' is positive; in this case the soliton represents a local rarefaction of traffic, with both the headway and speed being higher in the center of the sech^2 lump. On the other hand, if V_s'' is negative, then the soliton describes a local jam, a region where cars move more slowly and are closer together than average. For larger q the disturbance is larger in amplitude but smaller in spatial extent, and moves more rapidly through the line of traffic. The arbitrariness of q reflects the invariance of the KdV equation under the simultaneous rescalings $z \to qz$,

$T \to q^3 T,\ g \to q^2 g$. This invariance, in turn, reflects the arbitrariness of the small parameter ϵ: replacing ϵ with $q\epsilon$ implements the rescalings.

The correction term $\epsilon g_{zz}\delta$ in (4.5) breaks the invariance of the equation under rescaling, and so one might expect that the corrections may break the continuous family of solutions (4.6) down to a discrete set, or possibly a single solution. That is, we expect that only a discrete subset of the family of $\epsilon = 0$ solutions actually represents the $\epsilon \to 0$ limits of solutions of the full equation (4.5). Typically, one derives a solvability condition by linearizing (4.5) about the zero-order solution, and the surviving solutions have q values that satisfy that solvability condition. However, it is more instructive to calculate instead how the correction terms cause the value of q to evolve in time [28,29]. The solvability condition gives no indication of whether the "selected" q is stable or unstable, while this multiple-time-scales approach does have something to say about this.

To begin the calculation, we assume that we can write the first-order solution to (4.5) in the form

$$g(z,T) = g^{(0)}(z,T;q(\epsilon T)) + \epsilon g^{(1)}(z,T). \tag{4.8}$$

That is, we allow the soliton parameter q to evolve slowly in time. Substituting this into (4.5) and expanding to first order gives

$$\mathcal{L}g^{(1)} + \frac{\partial g^{(0)}}{\partial q}\dot{q} = -\frac{1+2\lambda}{2}\bigg[g^{(0)}\delta + \frac{1+2\lambda}{4}PV_s' g^{(0)}_{zz} + \frac{1}{2}(V_s'' - \Omega_c')g^{(0)2}\bigg]_{zz}, \tag{4.9}$$

where $\dot{q}$ denotes the derivative of q with respect to the slow time variable ϵT, and the linear operator $\mathcal{L}$ is given by

$$\mathcal{L} = \frac{\partial}{\partial T} - C_{11}\frac{\partial^3}{\partial z^3} - V_s''\frac{\partial}{\partial z}g^{(0)}. \tag{4.10}$$

The fact that there is a continuous family of solutions $g^{(0)}$ of the zero-order problem suggests that there will be a function that is annihilated by the adjoint operator $\mathcal{L}^\dagger$, and in fact this function turns out to be $g^{(0)}$ itself. Explicitly, with an inner product that consists of integrating over all z and averaging over all T, the adjoint operator is

$$\mathcal{L}^\dagger = -\frac{\partial}{\partial T} + C_{11}\frac{\partial^3}{\partial z^3} + g^{(0)}V_s''\frac{\partial}{\partial z}. \tag{4.11}$$

Applying this to $g^{(0)}$ gives zero because $g^{(0)}$ satisfies (4.5) to leading order. Thus if we take the inner product of $g^{(0)}$ with both sides of (4.9), the term involving $g^{(1)}$ vanishes; evaluating the remaining inner products then yields

$$\dot{q} = \frac{4[1+2\lambda(\Delta)]}{15}q^3\delta + \frac{4[1+2\lambda(\Delta)]}{105} \times \bigg\{-5[1+2\lambda(\Delta)]PV_s'(\Delta) + 48C_{11}\frac{V_s''(\Delta)-\Omega_c'(\Delta)}{V_s''(\Delta)}\bigg\}q^5. \tag{4.12}$$

The usual solvability condition is simply the right hand side of this set equal to zero; it locates the "selected" q values which are not moved by the correction terms. Keeping the $\dot{q}$ term on the left reveals whether q's near those values are driven toward or away from them. If we set $\Omega_c' = 0$ and $t_d = 0$ in this equation, then the selected q agrees with the solvability result of Ou *et al.* [15] for the FVDM.

Note that V_s'' drops out of (4.12) completely if Ω_c' vanishes. This is because for $\Omega_c' = 0$, both the quadratic terms in (4.5), and none of the linear terms, carry factors of V_s''. As a result, V_s'' can be scaled out of the equation completely. For nonzero Ω_c', on the other hand, there is a role for V_s'' to play.

In the OVM, the FVDM, and the COVM, Ω_c is a constant, so $\Omega_c' = 0$ and the coefficient of q^5 on the right side of (4.12) can be shown to be positive when the delay time satisfies (2.12). Thus for $\delta > 0$—the range for which steady traffic flow is linearly unstable—(4.12) always drives the value of q higher and higher, in fact diverging in finite time. This means that the actual pattern of traffic flow evolves out of the regime in which the KdV equation (4.5) is valid. On the other hand, for $\delta < 0$ there is a nontrivial fixed point of (4.12) with $q \propto |\delta|^{1/2}$. If q starts below this, then it decreases, decaying to zero as $T^{-1/2}$, while if q starts above the fixed point, then it is again driven to infinity. Thus the "selected" q actually marks the threshold of a *finite-amplitude* instability of steady traffic, even when steady flow is *linearly* stable.

New possibilities arise when Ω_c' is nonzero, because the coefficient of q^5 in (4.12) can be negative. Should that be the case, there would be a nontrivial fixed point when δ is positive. Thus when steady traffic flow is (slightly) linearly unstable, the instability would lead to small-amplitude jams described by soliton solutions of the KdV equation (4.5). When steady flow is linearly stable, so $\delta < 0$, then (4.12) would have q decay to zero no matter where it started; there would be no hint of a finite-amplitude instability in this case.

Note that at the threshold of *absolute* stability we have $V_s'' = \Omega_c'$. In this case the coefficient of q^5 in (4.12) is manifestly negative. In general, the calculation above then predicts that traffic will be in the regime where small-amplitude jams are seen. This fails, however, if the threshold of absolute stability is at or near an inflection point of V_s, as is the case in the OVM, the FVDM, and the COVM, since one of the coefficients in the leading-order equation would then be small.

V. NEAR AN INFLECTION POINT

The picture changes dramatically if, in addition to being near the onset of instability, the traffic spacing Δ is near an inflection point Δ_i of the steady-state velocity function V_s. Suppose $V_s'(\Delta_i)$ is within order ϵ^2 of the instability limit,

$$V_s'(\Delta_i) = \Omega_c(\Delta_i) + \epsilon^2\delta_i. \tag{5.1}$$

Let Δ itself be within order ϵ^2 of the inflection point, say

$$\Delta = \Delta_i + \epsilon^2\beta. \tag{5.2}$$

It follows that $V_s''(\Delta)$ is also small, specifically

$$V_s''(\Delta) = \epsilon^2 V_s'''(\Delta_i)\beta. \tag{5.3}$$

Note that if V_s' has a *maximum* at Δ_i, then $V_s'''(\Delta)$ is negative. Since $V_s''(\Delta_i)$ vanishes, it follows that $V_s'(\Delta)$ differs from $V_s'(\Delta_i)$ only by order ϵ^4. From (3.8a) we then find

$$D = -\frac{1+2\lambda}{2}\epsilon^2\delta \quad \text{with} \quad \delta = \delta_i - \Omega_c'\beta \tag{5.4}$$

with corrections of order ϵ^4. As before, steady flow is linearly unstable for positive δ, stable for negative δ.

From the expansion (3.7) we see that if both D and V_s'' are of order ϵ^2, then f_T is of order ϵ, and the leading-order terms are the three order-ϵ terms on the right side that do not contain T derivatives. It is appropriate, then, to return to the original ansatz (3.2) for x_n, but retain the new definition (4.2) of the slow time T. As in the preceding section, we may substitute all this into the equation of motion (1.1) and expand (to fifth order in ϵ), or we may simply take the general expansion (3.7) and multiply each T derivative by an additional factor ϵ. After substituting (5.4) for D and (5.3) for V_s'' and canceling an overall factor of ϵ, this yields, to first order,

$$f_T = C_{11} f_{zzz} + C_{12} f_z f_{zz} + C_{13} f_z^3 + \epsilon\bigg[-\frac{1+2\lambda}{2} f_{zz}\delta + \frac{\beta}{2} V_s''' f_z^2 + C_{11T} f_{zT} + C_{12T} f_z f_T + C_{21} f_{zzzz} + C_{22a} f_z f_{zzz} + C_{22b} f_{zz}^2 + C_{23} f_z^2 f_{zz} + C_{24} f_z^4 \bigg]. \tag{5.5}$$

To simplify the corrections, we again substitute the leading-order terms for each f_T in the correction. We then differentiate with respect to z and write the resulting equation in terms of g to obtain

$$g_T = C_{11} g_{zzz} + \frac{1+2\lambda}{4}\Omega_c'(g^2)_{zz} + \frac{1}{6}V_s'''(g^3)_z + \epsilon\bigg[-\frac{1+2\lambda}{2} g_z\delta + \frac{\beta}{2}V_s''' g^2 - \frac{(1+2\lambda)^2}{8} P V_s' g_{zzz} + \tilde{C}_{22a} g g_{zz} + \tilde{C}_{22b} g_z^2 + \tilde{C}_{23} g^2 g_z + \frac{1}{24} V_s'''' g^4 \bigg]_z, \tag{5.6}$$

with the new coefficients given by

$$\tilde{C}_{22a} = C_{22a} + C_{12T} C_{11} + C_{11T} C_{12} = -\frac{(1+2\lambda)^2}{4}\left(\frac{1+\lambda}{\tau}\right)' - \Omega_c^3 t_d \tau', \tag{5.7a}$$

$$\tilde{C}_{22b} = C_{22b} + C_{11T} C_{12} = \frac{(1+2\lambda)^2}{4}\left[-\Omega_c' + \frac{A_{vv} - 4A_{vh} + 4A_{hh}}{8\tau} \right], \tag{5.7b}$$

$$\tilde{C}_{23} = C_{23} + 3C_{11T} C_{13} + C_{12T} C_{12} = \frac{1+2\lambda}{4}\left[(\Omega_c - V_s')'' + 2\frac{\tau'}{\tau}\Omega_c' \right]. \tag{5.7c}$$

Note that $\tilde{C}_{22a}$ and $\tilde{C}_{22b}$ vanish under the assumptions underlying the OVM and FVDM.

The leading order of (5.6) was derived by Hayakawa and Nakanishi [11] for a model that includes look-back. Note that it reduces to the modified Korteweg–de Vries equation *only* if Ω_c' vanishes or is at most of order ϵ. Thus, the mKdV equation only appears when *both* V_s'' and Ω_c' are small for a given Δ. That is, the mKdV equation is *not* a generic feature of car-following models. Rather, it appears only in those cases for which the threshold of absolute stability occurs at or near an inflection point of the steady-state velocity function. If the inflection point and the threshold of absolute stability are not close together, then we obtain a KdV equation near the former (as we found in the preceding section) and the more general Hayakawa-Nakanishi equation (5.6) near the latter. Equation (5.6) does, however, share some important features of the mKdV equation.

In those cases where the mKdV equation does apply, its hyperbolic-tangent "kink" solutions are of particular interest. In the context of traffic, a one-kink solution describes a "domain wall" separating regions of uniform traffic flow with different speeds and densities. The mKdV equation has a one-parameter family of such solutions, and one can carry out a solvability calculation to determine which are preserved by the correction terms. Like the mKdV equation, the Hayakawa-Nakanishi equation (5.6) also admits a one-parameter family of kink solutions. Remarkably, in the general case it is also possible to carry out the perturbation calculation to find members of the family that persist when the order-ϵ corrections are included [11].

One-kink solutions of the leading order of (5.6) are given by

$$g^{(0)}(z,T;q) = Mq\tanh(qz + q^3 uT), \tag{5.8}$$

where M and u are given by

$$V_s''' M^2 - 3(1+2\lambda)\Omega_c' M + 12C_{11} = 0, \quad u = \tfrac{1}{6}V_s''' M^2. \tag{5.9}$$

Since V_s''' is negative and C_{11} is positive, there are two solutions for M, one positive and one negative. With u being negative, both move relative to traffic slightly more slowly than V_s'. As was the case for the KdV solitons, the various powers of q in (5.8) reflect the invariance of the leading-order evolution equation under the simultaneous rescalings $z \to qz$, $T \to q^3 T$, $g \to q^{-1} g$, which in turn reflects the arbitrariness of the small parameter ϵ.

To see the effect of the order-ϵ corrections on the kink solutions, we would like to carry out a perturbation calculation analogous to the one in the preceding section. That is, we write the first-order solution as

$$g(z,T) = g^{(0)}(z,T;q(\epsilon T)) + \epsilon g^{(1)}(z,T), \tag{5.10}$$

allowing the parameter q to evolve slowly in time, and substitute into (5.6). The linear operator $\mathcal{L}$ acting on $g^{(1)}$ is now

$$\mathcal{L} = \frac{\partial}{\partial T} - C_{11}\frac{\partial^3}{\partial z^3} - \frac{(1+2\lambda)}{2}\Omega_c'\frac{\partial^2}{\partial z^2}g^{(0)} - \frac{1}{2}V_s'''\frac{\partial}{\partial z}g^{(0)2}. \tag{5.11}$$

Next we seek the function $s(z,T)$ which is annihilated by the adjoint operator $\mathcal{L}^\dagger$,

$$0 = \mathcal{L}^\dagger s = -s_T + C_{11}s_{zzz} - \frac{(1+2\lambda)}{2}\Omega_c' g^{(0)} s_{zz} + \frac{1}{2}V_s''' g^{(0)2} s_z. \tag{5.12}$$

This equation can, in fact, be solved using standard techniques; we find that s is given by

$$s = s(qz + q^3 uT) \quad \text{with} \quad \frac{ds(y)}{dy} = \operatorname{sech}^{2+p} y, \tag{5.13}$$

the exponent p being given by

$$p = -\frac{M(1+2\lambda)\Omega_c'}{2C_{11}}. \tag{5.14}$$

Note that for $\Omega_c' = 0$ we have simply $p = 0$, so s reduces to $g^{(0)}$ as in the preceding section.

We now expand (5.6) about the one-kink solution and take the inner product of the resulting equation with s to obtain the evolution equation for the parameter q. To evaluate the various inner products, we change variables to $y = qz + q^3uT$ and integrate by parts, noticing that the boundary terms do not vanish because neither s nor the $g^{(0)2}$ and $g^{(0)4}$ terms in the correction vanish at infinity. The resulting integrals can be evaluated explicitly in terms of gamma functions. Unfortunately, the inner product of $\partial g^{(0)}/\partial q$ with s diverges—whether or not Ω_c' vanishes. Thus the best we can do is write down the solvability condition, which identifies the single q value for which the $g^{(0)}$ given by (5.8) is the $\epsilon \to 0$ limit of the perturbed solution. Specifically, this yields

$$(1+2\lambda)\delta + V_s''' M\beta = \frac{q^2}{5+2p}\left\{(1+2\lambda)^2(1+p)V_s' + 4[(2+p)\tilde{C}_{22b} - \tilde{C}_{22a}]M + 2\tilde{C}_{23}M^2 - \frac{1}{6}(3+p)V_s'''' M^3\right\}. \tag{5.15}$$

For the FVDM, p, $\tilde{C}_{22a}$, and $\tilde{C}_{22b}$ all vanish, and $\tilde{C}_{23}$ is equal to $-(1+2\lambda)V_s'''/4$. The result above then agrees with that of Ou *et al.* [15], provided we set β and V_s'''' to zero, as they do (explicitly for β and implicitly for V_s'''').

For q values which do not satisfy the solvability equation, the above perturbation calculation is unable to determine what effect the correction terms have on the solution. Thus the solvability approach is silent on the question of how, and indeed whether, the "selected" kink is established.

VI. DISCUSSION

We have examined a very general car-following model embodied in (1.1), in which each car's present acceleration is some general function of its speed, the headway between it and the next car ahead of it, and the rate of change of the headway, all evaluated at some time t_d before the present moment. The model is subject to monotonicity assumptions (2.1) that rule out unreasonable descriptions of driver behavior, and an assumption that sufficiently many derivatives exist. This form subsumes many of the specific car-following models that have been studied in the past, but allows for features that are seldom accounted for. For example, the acceleration function is usually taken to be *linear* in the car velocities, with coefficients that are *independent* of headway [4,6]; the model here relaxes these assumptions.

Our aim has been to investigate the extent to which the results which have been obtained from specific car-following models in the past are generic; that is, which of these results continue to hold at the level of generality of (1.1). In particular, a number of results have been obtained by reducing these models to Korteweg–de Vries (KdV) or modified Korteweg–de Vries (mKdV) equations in particularly important parameter ranges. The reductions, however, proceed from rather long perturbation series, which may well be sensitive to features such as nonlinear velocity dependence that have been left out of the models.

We find that a number of the results of specific car-following models are in fact universal, provided the delay time t_d is not too large. The full velocity difference model (FVDM) of Jiang *et al.* [6,15] is sufficiently general to capture many of these results; no further car-following models within the framework of (1.1) can reach conclusions different than these:

(1) Flow with a uniform spacing Δ between cars can proceed at a steady speed $V_s(\Delta)$, which in general is defined implicitly from the acceleration function via (2.2).

(2) Uniform flow is linearly stable provided the steady-state velocity function $V_s(\Delta)$ satisfies $V_s'(\Delta) < \Omega_c(\Delta) \equiv (1+2\lambda)/2\tau$, where the Δ-dependent parameters λ and τ are given by (2.4). It is then possible to have "absolute stability," where steady flow is stable, at least linearly, for all Δ. Otherwise, at the onset of instability, where $V_s' = \Omega_c$, the marginal perturbation has wave number $k = 0$ and moves backward through the line of traffic at a rate, in cars per unit time, of $V_s'(\Delta)$.

(3) On a long spatial scale, the model can be reduced to a Burgers equation, with a diffusion coefficient proportional to $\Omega_c - V_s'$ and a quadratic coefficient proportional to $V_s''(\Delta)$.

(4) Near the onset of instability, when $V_s'(\Delta)$ is close to $\Omega_c(\Delta)$, the model can be reduced further to a KdV equation plus higher-order corrections. If we restrict our attention to the family of one-soliton solutions of the KdV equation, we find that the correction terms drive an evolution of the soliton parameter, with at most a single member of the family being a fixed point of that evolution.

All these conclusions, as well as the new ones discussed below, hold when there is no time delay, or when the delay time is shorter than the limit given in (2.12). For delay times longer than this, we have given a full accounting of the onset of linear instability. Figure 1 shows the value of the control parameter V_s'/Ω_c at which steady uniform flow becomes linearly unstable, and the wave number of the marginally unstable mode, for given delay time and values of the parameters λ and τ. The curves are found parametrically from (2.8a), (2.8b), and (2.13). This analysis complements the results of Orosz *et al.* [27]. We find in particular that if the delay time is too *long*, then we can have *absolute instability*, where steady uniform flow is always linearly unstable no matter how small $V_s'(\Delta)$ is. For a specific spacing, the delay time above which this occurs is Δ dependent, but is at most $\pi\tau(\Delta)/2$.

In addition to showing that the conclusions above are generic at the level of (1.1), we have found a number of new possibilities that are made possible by the greater generality of the model. Most of these results are consequences of the fact that the linear stability parameters λ and τ can be nontrivial functions of the spacing Δ. Most models in the literature are special cases of (1.1) which make these parameters constant as part of the formulation of the model, so they have been incapable of producing the results we describe.

Near the onset of instability, the possibility of Ω_c having a nontrivial Δ dependence leads to a crucial change in the coefficient of one of the correction terms to the KdV equation. When Ω_c is a constant, the "selected" one-soliton solution exists only when steady flow is linearly stable, and the correction terms drive the soliton parameter away from its "selected" value—toward zero if it starts out lower, toward infinity if it starts higher. Thus the selected soliton marks the threshold of a finite-amplitude instability of the *linearly* stable steady flow. When Ω_c'/V_s'' is large enough, on the other hand, we have the reverse situation: the selected soliton exists only when steady flow is linearly *unstable*, and the corrections drive the soliton parameter *toward* its selected value. In such a case the onset of linear instability of steady flow is a forward bifurcation to a state which has small-amplitude jams. Again, this will not be seen in models in which the linear stability parameters are constants independent of spacing.

In this connection it is worth pointing out that a multiple-time-scales approach [28,29] can be more illuminating than a simple solvability calculation. The latter detects the selected soliton, but the former also gives information about the evolution of other, nonselected solitons and directly speaks to whether the selected soliton is stable or unstable. Note also that the rate at which the soliton parameter evolves is proportional to the *square* of $V_s' - \Omega_c$. Thus, even if a soliton is in the slow process of decaying to zero, it can *appear* to be a persistent feature of the traffic flow.

Absolute stability, i.e., linear stability of all uniform steady-flow solutions, occurs if $V_s'(\Delta) - \Omega_c(\Delta)$ is negative for all Δ, and the threshold of absolute stability is where the maximum value of $V_s'(\Delta) - \Omega_c(\Delta)$ is zero. If the linear stability parameters are constants, then this is where the value of V_s' at an inflection point of V_s is equal to the presumed-constant Ω_c, and near this point the model can then be reduced to a mKdV equation. In the more general case, however, the threshold of absolute stability does *not* coincide with an inflection point of V_s. When the two differ, we find that the mKdV equation does not apply near either point. Instead, near the threshold of absolute stability the model continues to reduce to a KdV equation, with parameters automatically in the range that leads to stable, small-amplitude jams. Near the inflection point, on the other hand, there is a different reduced equation, which is the mKdV equation plus an extra quadratic term. This equation has been found by Hayakawa and Nakanishi [11] in a model that includes look-back. Again, the fact that most models in the literature make Ω_c a constant has led to the misleading appearance that the mKdV equation is a universal description of traffic near the threshold of absolute stability.

Like the mKdV equation, the Hayakawa-Nakanishi equation admits a one-parameter family of kink solutions, and the correction terms select a single one of these solutions that is preserved by the dynamics. When applying the correction terms to either equation, however, the multiple-time-scales approach fails, so it is not straightforward to see how evolution happens within the family of one-kink solutions. This comes about because the slow time derivative of the kink parameter is multiplied by a coefficient that turns out to diverge. This divergence, in turn, seems to come from the fact that changing the kink parameter involves changing the asymptotic values of the kink, i.e., changing the traffic spacing infinitely far from the location of the kink itself. This, and other aspects of the mKdV equation, will be the focus of future work.

It is rather remarkable that although the model allows all sorts of effects that have been left out of car-following models in the past, it turns out that almost all the generality of the reductions of the model to Burgers, KdV, and mKdV-like equations can be captured by merely allowing the linear stability parameters λ and τ to depend on the spacing Δ. That is, all the coefficients in the Burgers equation, the KdV equation and its first-order corrections, and the Hayakawa-Nakanishi equation can be written in terms of these parameters and the steady-state velocity V_s, and their derivatives with respect to Δ. The only exception is that one of the correction terms to the Hayakawa-Nakanishi equation involves nonlinear velocity dependencies, i.e., second derivatives of the acceleration function in (1.1) with respect to the velocities of the cars, that do not combine to form Δ derivatives of the linear stability parameters.

In conclusion, we wish to point out that although the *forms* of the terms appearing in reduced equations like the KdV and mKdV are determined only by the scalings that go into their derivation, the *coefficients* of those terms are sensitive to details of the model. The behavior of the solutions of those reduced equations can be sensitive to the values, and especially to the signs, of those coefficients. Thus *implicit* assumptions made when formulating the model can constrain the coefficients (often constraining them to be zero!), and this can then have important quantitative and qualitative effects on the kind of behavior predicted by the resulting reduced equations. Without analyzing a general model like (1.1) it is not clear which assumptions have important effects, and common results coming from specific models may misleadingly appear universal.

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