

Criticality in alternating layered Ising models. II. Exact scaling theory

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Part I of this article studied the specific heats of planar alternating layered Ising models with strips of strong coupling J_1 sandwiched between strips of weak coupling J_2 , to illustrate qualitatively the effects of connectivity, proximity, and enhancement in analogy to those seen in extensive experiments on superfluid helium by Gasparini and co-workers. It was demonstrated graphically that finite-size scaling descriptions hold in a variety of temperature regions including in the vicinity of the two specific heat maxima. Here we provide exact theoretical analyses and asymptotics of the specific heat that support and confirm the graphical findings. Specifically, at the overall or bulk critical point, the anticipated (and always present) logarithmic singularity is shown to vanish exponentially fast as the width of the stronger strips increases.

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The previous paper, Part I [1], considered a range of exactly soluble alternating layered Ising (ALI) models and presented extensive plots of their specific heats. The primary motivation (as explained in Part I) was to illustrate and study theoretically the phase-transition phenomena of “proximity,” “connectivity,” and “enhancement” as highlighted experimentally by recent studies of superfluid helium by Gasparini and co-workers [2–7]. However, the ALI models have intrinsic interest as instructive examples of the general two-dimensional (2D) layered Ising models. The exact solubility of the general layered models was reported in 1969 [8] and noted and developed, independently, in the context of randomly coupled systems by McCoy and Wu [9] and further studied in Refs. [10,11].

Specifically, our work addresses ALI models in which, in a standard infinite two-dimensional square lattice with Ising spins at each site, (i, j) , infinite strips of width m_1 are coupled by nearest-neighbor (NN) energies of strength J_1 in alternation with infinite strips of width $m_2 = sm_1$ and coupling $J_2 = rJ_1$. More explicitly, the NN couplings between spins at sites (i, j) , $(i, j + 1)$, and $(i + 1, j)$ in the same strip are independent of j but equal to J_1 for $i = 1, 2, \dots, m_1$ but to J_2 for $i = m_1 + 1, m_1 + 2, \dots, (m_1 + m_2)$, and so on with overall period $(m_1 + m_2)$. The boundary spins separating layers are thus at $i = 1 + n(m_1 + m_2)$ and at $i = 1 + (n + 1)m_1 + nm_2$ for $n = 0, \pm 1, \pm 2, \dots$

This paper then presents the details of the exact calculations on which the plots (and discussion) of Part I was based. In Sec. I, we present the specific integrals for the free energy without giving detailed derivations, because the ALI models are special cases of the general layered models [10,11] where the details can be found. The explicit forms of the integrals were used to plot the specific heat for the alternating layered systems shown in Figs. 2 and 3 of Part I. Around the overall or unique bulk critical point $T_c(r, s)$ the specific heat diverges logarithmically with an amplitude $A(r, s)$ that is shown in Sec. II to decrease exponentially fast with increasing m_1 . In Sec. III, we examine the behavior of the free energy near T_{1c} and T_{2c} , which are, respectively, the bulk critical temperatures of uniform 2D Ising models with strong couplings J_1 and weak

couplings J_2 , or, otherwise, the limiting temperatures of the specific heat maxima for infinitely wide layers. The conditions for the data collapse shown in Figs. 4 and 5 of Part I are thus verified. In Sec. IV, we show that finite-size scaling holds in both regimes, whenever data collapse occurs. Finally, Sec. V studies the behaviors of the enhancements in the two regimes, while the paper ends with a short summary.

I. THE FREE ENERGY

Layered Ising models were studied earlier in Refs. [8–12]. As the alternating layered model with cyclic boundary conditions imposed in the infinite vertical direction, and free boundary conditions in the horizontal layered direction are special cases, the calculation for the free energy per site, $f(J_1, J_2; m_1, m_2; T)$, is almost identical to that in Refs. [10,11]. Thus for layers of thickness m_1 and couplings J_1 sandwiched between layers of thickness m_2 and couplings J_2 , one has

$$-\frac{f(J_1, J_2; T)}{k_B T} = \frac{\ln[(2S_1)^{m_1}(2S_2)^{m_2}]}{2(m_1 + m_2)} + \frac{1}{m_1 + m_2} \int_0^{\pi/2} \frac{d\theta}{\pi} \ln \frac{1}{2} [W + \sqrt{W^2 - 4}], \quad (1)$$

where, for $i = 1, 2$, we use here and below

$$S_i = \sinh 2K_i, \quad C_i = \cosh 2K_i, \quad (2)$$

with

$$K_1 = J_1/(k_B T), \quad K_2 \equiv rK_1, \quad (3)$$

while the function $W(T; J_1, J_2; m_1, m_2; \theta)$ is given by

$$W = U_1^+ U_2^+ + U_1^- U_2^- + \frac{1}{2}(C_1 C_2 - 1) V_1 V_2, \quad (4)$$

in which for $i = 1, 2$,

$$U_i^\pm = U^\pm(t_i, m_i) = \frac{1}{2}(\alpha_i^{m_i} + \alpha_i^{-m_i}) \pm \frac{1}{2}(\alpha_i^{m_i} - \alpha_i^{-m_i})g_i, \\ V_i = (\alpha_i^{m_i} - \alpha_i^{-m_i})\bar{g}_i, \quad (5)$$

where we have introduced the basic temperature variables t_i via

$$t_i = (1 - S_i)/\sqrt{2S_i} \approx 2K_{ic} - 2K_i \approx 2K_{ic}(T/T_{ic} - 1), \\ 2K_{ic} = \ln(\sqrt{2} + 1), \quad (6)$$

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which are identical to the variables used in Refs. [13,14], but differ from symbols t_1 and t_2 , defined in (I.13) and (I.18) of Part I although by only a constant factor when close to T_{1c} and T_{2c} , respectively. The amplitude functions in (5) are then

$$\begin{aligned} g_i &= g_i(t_i; \omega) \\ &= [t_i \sqrt{1 + t_i^2(1 - \omega^2)} + \omega^2 \sqrt{(1 + t_i^2)(2 + t_i^2)}] / Y_i, \\ \bar{g}_i &= \bar{g}_i(t_i; \omega) = \omega \sqrt{1 - \omega^2} / Y_i, \\ Y_i &= Y(t_i; \omega) = \sqrt{(\omega^2 + t_i^2)(1 + \omega^2 + t_i^2)}, \end{aligned} \quad (7)$$

while $\omega = \sin \theta$, and the layer spacings m_1 and m_2 , enter through

$$\begin{aligned} \alpha_i &= \alpha(t_i) = c_i + \sqrt{c_i^2 - 1}, \\ c_i &= c(t_i) = 2t_i^2 + 2\omega^2 + 1. \end{aligned} \quad (8)$$

We note that by comparing with Eqs. (2.5) in Ref. [13] one finds $c(t) = c_\ell$ with $1 - 2\omega^2 = \cos(\pi \ell / n)$. It is also easily seen that $2Y_i = \sqrt{c_i^2 - 1}$. Conversely, we also have the relations

$$\begin{aligned} C_i &= \sqrt{1 + t_i^2} [\sqrt{2 + t_i^2} - t_i], \\ S_i &= 1 + t_i^2 - t_i \sqrt{2 + t_i^2}. \end{aligned} \quad (9)$$

The terms $U_i^+ = U^+(t_i, m_i)$ in (4) are related to the free energy $f^\infty(m_i; J_i; T)$ of an infinite strip of width m_i with coupling energy J_i , which is [14]

$$-\frac{f^\infty(m_i; J_i; T)}{k_B T} = \frac{\ln(2S_i)}{2} + \frac{1}{\pi m_i} \int_0^{\pi/2} d\theta \ln U^+(t_i, m_i). \quad (10)$$

The remaining terms in (4) are related to the interaction between the strips. If $J_2 \rightarrow 0$, so that the system becomes uncoupled, the relations (6) yield $t_2 \rightarrow (2S_2)^{-1/2} \rightarrow \infty$, which is used in (8) to give $\alpha_2 \rightarrow 4t_2^2 \rightarrow 2/S_2$. Consequently, from (5) we have $U_2^+ = \frac{1}{2}\alpha_2^{m_2} = 2^{m_2-1}/(S_2)^{m_2}$, and from (7) we find $g_2 \rightarrow 1$ and $\bar{g}_2 \rightarrow 0$. These results establish $U_2^- = 0$ and $V_2 = 0$. In this limit, the free energy in (1) becomes

$$\begin{aligned} -\frac{f(J_1, 0; T)}{k_B T} &= \frac{m_1 \ln(2S_1) + 2m_2 \ln 2}{2(m_1 + m_2)} \\ &+ \frac{1}{\pi(m_1 + m_2)} \int_0^{\pi/2} d\theta \ln U^+(t_1, m_1), \end{aligned} \quad (11)$$

which is the free energy per site for infinite strips of width m_1 , coupling J_1 , separated by empty infinite strips of width m_2 . This is identical to the result in (10), except that the factor $1/m_1$ in (10) is replaced by $1/(m_1 + m_2)$, while the additional

constant term $2m_2 \ln 2/(m_1 + m_2)$ does not contribute to the specific heat.

For completeness, we also let $J_1 = 0$ to find

$$\begin{aligned} -\frac{f(0, J_2; T)}{k_B T} &= \frac{m_2 \ln(2S_2) + 2m_1 \ln 2}{2(m_1 + m_2)} \\ &+ \frac{1}{\pi(m_1 + m_2)} \int_0^{\pi/2} d\theta \ln U^+(t_2, m_2). \end{aligned} \quad (12)$$

For future purposes [entailed in establishing relations (I.19), and (I.21)], we recall the modified temperature $\check{T}(T)$, introduced in (I.20), and then define

$$\begin{aligned} -\frac{f(0, J_2; \check{T})}{k_B T} &= \frac{m_2 \ln(2S_2) + 2m_1 \ln 2}{2(m_1 + m_2)} \\ &+ \frac{1}{\pi(m_1 + m_2)} \int_0^{\pi/2} d\theta \ln U^+(-t_2, m_2). \end{aligned} \quad (13)$$

Using (5), we may rewrite (4) as

$$\begin{aligned} W &= \frac{1}{2}(\alpha_1^{m_1} + \alpha_1^{-m_1})(\alpha_2^{m_2} + \alpha_2^{-m_2}) \\ &+ \frac{1}{2}(\alpha_1^{m_1} - \alpha_1^{-m_1})(\alpha_2^{m_2} - \alpha_2^{-m_2})G(t_1, t_2; \omega), \end{aligned} \quad (14)$$

in which we have

$$\begin{aligned} (Y_1 Y_2)G(t_1, t_2; \omega) &= (Y_1 Y_2)[g_1 g_2 + (C_1 C_2 - 1)\bar{g}_1 \bar{g}_2] \\ &= [t_1 t_2 \sqrt{(1 + t_1^2)(1 + t_2^2) - \omega^2}](1 - \omega^2) \\ &+ \omega^2 \sqrt{(1 + t_1^2)(1 + t_2^2)(2 + t_1^2)(2 + t_2^2)}. \end{aligned} \quad (15)$$

For the uniform Ising model the ratio $r = J_2/J_1$ is unity, so that $t_1 = t_2$ and $\alpha_1 = \alpha_2$. We now use (7) and (9) to show that G in (15) reduces to $G = g_1^2 + S_1^2 \bar{g}_1^2 = 1$. As a result (14) simplifies to

$$\begin{aligned} W &= \frac{1}{2}(\alpha_1^{m_1} + \alpha_1^{-m_1})(\alpha_1^{m_2} + \alpha_1^{-m_2}) \\ &+ \frac{1}{2}(\alpha_1^{m_1} - \alpha_1^{-m_1})(\alpha_1^{m_2} - \alpha_1^{-m_2}) \\ &= (\alpha_1^{m_1+m_2} + \alpha_1^{-m_1-m_2}). \end{aligned} \quad (16)$$

Consequently, the free energy in (1) becomes

$$-\frac{f(J_1, J_1; T)}{k_B T} = \frac{1}{2} \ln(2S_1) + \frac{1}{\pi} \int_0^{\pi/2} d\theta \ln \alpha_1, \quad (17)$$

which is the same as the free energy of the uniform Ising model [9].

The specific heat of the alternating layered model, which is the second derivative of the free energy in (1), is thus given by

$$\begin{aligned} \frac{C(J_1, J_2; m_1, m_2; T)}{k_B} &= K_1^2 \frac{d^2}{dK_1^2} \left[-\frac{f(J_1, J_2; T)}{k_B T} \right] \\ &= \frac{-2m_1 K_1^2}{(m_1 + m_2) S_1^2} - \frac{2m_2 (r K_1)^2}{(m_1 + m_2) S_2^2} \\ &+ \frac{K_1^2}{\pi(m_1 + m_2)} \int_0^{\pi/2} d\theta \left[\left(\frac{d^2 W}{dK_1^2} \right) / (W^2 - 4)^{1/2} - W \left(\frac{dW}{dK_1} \right)^2 / (W^2 - 4)^{3/2} \right]. \end{aligned} \quad (18)$$

These are the formulas used to plot the specific heats in the figures in the previous paper, Part I [1].

In considering the expression (I.1) for the specific heat near the bulk critical point T_c , it is natural, having dealt with the amplitude $A(r,s)$ of the logarithmic singularity, to inquire as to the leading background term $B(r,s)$. As our notation suggests, this is expected, on the grounds of duality [15], to be continuous through T_c so that there is no discontinuity associated with bulk criticality. However, the calculation of the dependence of $B(r,s)$ on m_1 proves not straightforward and has not been attempted (although the continuity is surely supported by the numerics reported in Part I).

II. AMPLITUDE FOR THE LOGARITHMIC DIVERGENCE

The amplitude of the logarithmic divergence in (I.1) is obtained [10,11] by expanding the term inside the square root in (1) as

$$1 - 4/W^2 = A_1^2 (J_1/k_B)^2 [(1/T) - (1/T_c)]^2 + A_2^2 \theta^2 + \dots, \quad (19)$$

where the coefficient A_1 is given by Hamm in (1.8) of Ref. [11] as

$$\begin{aligned} A_1 &= 2m_1(1 + S_{1c}^{-1}) + 2m_2r(1 + S_{2c}^{-1}), \\ S_{1c} &= \sinh 2K_c, \quad S_{2c} = \sinh(2rK_c). \end{aligned} \quad (20)$$

The integration over θ around the origin yields

$$\begin{aligned} C(T)/k_B &= -A(r,s) \ln |1 - (T/T_c)| + O(1), \\ r &= J_2/J_1, \quad s = m_2/m_1, \\ A(r,s) &= \frac{A_1^2 K_c^2}{2\pi A_2(m_1 + m_2)}, \quad K_c = \frac{J_1}{k_B T_c}. \end{aligned} \quad (21)$$

Since we only have two kinds of bonds, the sum in (1.9) of Ref. [11] can be evaluated to obtain

$$\begin{aligned} A_2^2 &= (\epsilon_{1c}^{m_1} - \epsilon_{1c}^{-m_1})^2 \left[\frac{S_{1c}^2}{(\epsilon_{1c} - \epsilon_{1c}^{-1})^2} + \frac{S_{2c}^2}{(\epsilon_{2c} - \epsilon_{2c}^{-1})^2} \right. \\ &\quad \left. - \frac{S_{1c} S_{2c} (z_{1c} z_{2c}^{-1} + z_{2c} z_{1c}^{-1})}{(\epsilon_{1c} - \epsilon_{1c}^{-1})(\epsilon_{2c} - \epsilon_{2c}^{-1})} \right], \end{aligned} \quad (22)$$

where the temperature dependent parameters are

$$\begin{aligned} z_i &= z_i(T) = \tanh(J_i/k_B T), \\ \epsilon_i(T) &= z_i e^{2(J_i/k_B T)}, \end{aligned} \quad (23)$$

while $z_{ic} = z_i(T_c)$ and $\epsilon_{ic} = \epsilon_i(T_c)$. Notice that $\epsilon_i(T)$ depends only on J_i , and at the critical temperature T_{ic} of a uniform planar Ising model whose coupling energy is J_i , we have

$$\epsilon_i(T_{ic}) = 1 \Rightarrow K_{ic} = J_i/(k_B T_{ic}) = \frac{1}{2} \ln(\sqrt{2} + 1), \quad (24)$$

which is equivalent to (I.5) [1]. The general critical temperature expression (I.4) is equivalent to

$$\epsilon_1^{m_1}(T_c) \epsilon_2^{m_2}(T_c) = 1 \quad \text{or} \quad \epsilon_{1c} \epsilon_{2c}^s = 1. \quad (25)$$

From (24) and (25), we find $T_c < T_{1c}$, and $T_c \rightarrow T_{1c}$ either as $s \rightarrow 0$ or as $r \rightarrow 1$. For $r \neq 1$ and $s \neq 0$, we find $\epsilon_{1c} > 1$ and $\epsilon_{2c} < 1$, so that $\epsilon_{1c}^{m_1} = \epsilon_{2c}^{-m_2} \rightarrow \infty$ in the limit $m_1 \rightarrow \infty$. We shall consider the scaling behavior for the two cases separately.

(1) Now consider the scaling limit for $r \neq 1$, $m_2 = sm_1$ fixed, and $m_1 \rightarrow \infty$ such that $s \rightarrow 0$. We find from (24) and (25)

$$\ln \epsilon_1(T_c) - \ln \epsilon_1(T_{1c}) = -s \ln \epsilon_2(T_c) \approx -s \ln \epsilon_2(T_{1c}). \quad (26)$$

Now we substitute (23) into this relation to find

$$\begin{aligned} \frac{2J_1}{k_B} \left[\frac{1}{T_c} - \frac{1}{T_{1c}} \right] + \ln \left[\frac{\tanh(J_1/k_B T_c)}{\tanh(J_1/k_B T_{1c})} \right] \\ \approx s \ln \left[\frac{1 + e^{-2rJ_1/k_B T_{1c}}}{e^{2rJ_1/k_B T_{1c}} - 1} \right]. \end{aligned} \quad (27)$$

After expanding the left-hand side around T_{1c} and using (24) on the right we find

$$\begin{aligned} 4K_{1c}[(T_{1c}/T_c) - 1] &\approx s p_r, \\ p_r &= \ln[(\sqrt{2} - 1)^r + 1] - \ln[(\sqrt{2} + 1)^r - 1]. \end{aligned} \quad (28)$$

Consequently for $s \rightarrow 0$, we have

$$\begin{aligned} A_1 &= 4m_1[1 + O(s)], \\ A_2 &= \frac{\epsilon_{1c}^{m_1} - \epsilon_{1c}^{-m_1}}{\epsilon_{1c} - \epsilon_{1c}^{-1}} + O(1) \approx \frac{\sinh m_2 p_r}{s p_r}. \end{aligned} \quad (29)$$

The amplitude of the logarithmic divergence scales as

$$A(r,s) \approx \frac{8K_{1c}^2 p_r m_2}{\pi \sinh(p_r m_2)} + O(s). \quad (30)$$

For $r = 1$, we have $p_r = 0$, which reproduces the original Onsager result [9,13,14]. For $m_2 \rightarrow 0$, we find $\sinh(p_r m_2) \rightarrow p_r m_2$, so that (30) again reproduces the Onsager result. In the opposite limit $m_2 \rightarrow \infty$, we find the amplitude decays exponentially fast as

$$A(r,s) = (8K_{1c}^2/\pi) p_r m_2 e^{-p_r m_2}. \quad (31)$$

(2) In order to have a nonvanishing logarithmic amplitude for fixed $s = m_2/m_1$, with $m_1 \rightarrow \infty$, one must let $r \rightarrow 1$. Accordingly, we study the amplitude of the logarithmic singularity in the scaling limit that $(1-r)m_1$ is fixed. In similar fashion to our derivation of (27), we use (24) and (25) to find

$$\begin{aligned} \frac{1}{T_c} &\approx \frac{1}{T_{1c}} \left[1 + \frac{s(1-r)}{1+s} \right], \\ \frac{r}{T_c} &\approx \frac{1}{T_{1c}} \left[1 - \frac{1-r}{1+s} \right]. \end{aligned} \quad (32)$$

Expanding terms in (20) and (22) as a series in $1-r$, and keeping only the leading two terms, we obtain

$$\begin{aligned} S_{1c} &\approx 1 + \frac{2\sqrt{2}K_{1c}s(1-r)}{1+s}, \\ S_{2c} &\approx 1 - \frac{2\sqrt{2}K_{1c}(1-r)}{1+s}, \\ z_{1c}/z_{2c} &\approx 1 + 2K_{1c}(1-r), \\ z_{2c}/z_{1c} &\approx 1 - 2K_{1c}(1-r), \\ \epsilon_{1c} &\approx 1 + 4K_{1c}s(1-r)/(1+s) \\ &\approx e^{4K_{1c}s(1-r)/(1+s)}, \\ \epsilon_{2c} &\approx 1 - 4K_{1c}(1-r)/(1+s) \\ &\approx e^{-4K_{1c}(1-r)/(1+s)}. \end{aligned} \quad (33)$$

Substituting these asymptotic relations into (20) and (22), we find

$$A_1 \approx 4m_1(1+s),$$

$$A_2 \approx \frac{(1+s)^2}{4K_{1c}s(1-r)} \sinh \left[\frac{4K_{1c}sm_1(1-r)}{(1+s)} \right]. \quad (34)$$

Consequently, the scaling form of the amplitude of the logarithmic divergence of the specific heat is

$$A(r,s) \approx \frac{16K_{1c}^2sq}{\pi(s+1) \sinh[2sq/(1+s)]},$$

$$q = 2K_{1c}(1-r)m_1. \quad (35)$$

For $r = 1$, one has $q = 0$, and this expression again reproduces the original Onsager result. For $r < 1$, with $m_1 \rightarrow \infty$, we have $q \gg 1$ so the denominator is exponentially large, which means the amplitude is exponentially small. This and (31) are central results that explain why the logarithmic singularity at T_c becomes essentially unobservable in Figs. 2–4 of Part I.

III. BEHAVIOR NEAR T_{1c} AND T_{2c}

Near T_{1c} , the specific heat of the weaker strip is small, and so in Part I we introduced a net contribution

$$C_1(J_1, J_2; T) = (1+s)[C(J_1, J_2; T) - C(0, J_2; T)]. \quad (36)$$

In Fig. 5 of Part I [1], the scaling plots of C_1 reveal that the behavior becomes independent of m_2 for T near T_{1c} . In this section, we examine the condition for such behavior to hold. We define the free energy corresponding to $C_1(J_1, J_2; T)$ as

$$f_1(J_1, J_2; T) = (1+s)[f(J_1, J_2; T) - f(0, J_2; T)]. \quad (37)$$

For fixed weakness ratio $r = J_2/J_1$, we can see that from (8) and (6) that $\alpha_1 \simeq 1$ for $t_1, \omega \simeq 0$. Because $r \neq 1$, for $T \simeq T_{1c}$ ($t_1 \simeq 0$), we have $t_2 \neq 0$ and $\alpha_2 > 1$. Thus for m_2 sufficiently large, $\alpha_2^{m_2} \gg \alpha_2^{-m_2}$, and we may drop terms involving $\alpha_2^{-m_2}$ in (14) to arrive at the form

$$W \approx \alpha_2^{m_2} W_1(t_1, t_2; m_1) = \alpha_2^{m_2} \left[\frac{1}{2}(\alpha_1^{m_1} + \alpha_1^{-m_1}) + \frac{1}{2}(\alpha_1^{m_1} - \alpha_1^{-m_1})G(t_1, t_2; \omega) \right], \quad (38)$$

in which $G(t_1, t_2; \omega)$ was defined in (15). Similarly we find

$$U^+(\pm t_2, m_2) = \alpha_2^{m_2} [1 + g_2(\pm t_2)] + O(\alpha_2^{-m_2}). \quad (39)$$

Consequently the free energy introduced in (37) becomes

$$-f_1(J_1, J_2; T)/(k_B T) = \frac{1}{2} \ln(S_1/2) + (\pi m_1)^{-1} \int_0^{\pi/2} d\theta$$

$$\times [\mathcal{I}_1(t_1, t_2; m_1) + O(\alpha_2^{-m_2})],$$

$$\mathcal{I}_1(t_1, t_2; m_1) = \ln W_1(t_1, t_2; m_1) - \ln \frac{1}{2} [1 + g_2(t_2)]. \quad (40)$$

It is easy to see from this that f_1 is essentially independent of m_2 . Therefore, in the temperature range where $\alpha_2^{m_2} \gg \alpha_2^{-m_2}$, which is the case for those plots in Fig. 5 of Part I, the specific heat difference $C_1(T)$ defined in (36) becomes, indeed, independent of m_2 .

To understand the behavior of the integral in (40), we compare it with the free energy of uncoupled strips when the horizontal coupling, say \bar{J}_2 , is set to be identically zero in the weaker strips while the vertical couplings J_2 remain nonzero. Since the original papers [10,11] on layered Ising systems address the more general case with vertical couplings unequal to the horizontal couplings, we may easily obtain the free energy for such a case as

$$-f^{(u)}(J_1; J_2; T)/(k_B T) = [2(m_1 + m_2)]^{-1} \left[m_1 \ln(2S_1) + m_2 [\ln 2 + \ln(1 + C_2)] + \frac{2}{\pi} \int_0^{\pi/2} d\theta \ln W^{(u)} \right], \quad (41)$$

where, in the integrand, we now have

$$W^{(u)} = \frac{1}{2}(\alpha_1^{m_1} + \alpha_1^{-m_1}) + \frac{1}{2}(\alpha_1^{m_1} - \alpha_1^{-m_1})g^{(u)}(t_1; J_2), \quad (42)$$

$$g^{(u)}(t_1; J_2; \omega) = g_1(t_1) + \frac{\omega^2(1-\omega^2)[-t_1\sqrt{1+t_1^2} + \sqrt{(1+t_1^2)(2+t_1^2)} - 1]}{(\omega^2 - \frac{1}{2} - \frac{1}{2}C_2/S_2)Y_1},$$

where we recall that $Y_1 = Y(t_1; \omega)$ is defined in (7). In $\mathcal{I}_1(t_1, t_2; m_1)$ of (40), the second term is related to the surface free energy, while the first term is very similar to the free energy of uncoupled infinite strips of width m_1 , each with one of its boundary columns having vertical couplings J_2 . This can be seen for $t_2 > 0$ by rewriting the function G given by (15) as

$$G(t_1, t_2; \omega) = g_1(t_1)$$

$$+ \omega^2(1-\omega^2)Y_1^{-1}[-t_1\sqrt{(1+t_1^2)}R_1(t_2; \omega)$$

$$+ \sqrt{(1+t_1^2)(2+t_1^2)}R_2(t_2; \omega) - Y_2^{-1}], \quad (43)$$

in which we have

$$R_1(t_2; \omega) = \frac{1 + 2t_2^2 + \omega^2}{Y_2(Y_2 + t_2\sqrt{1+t_2^2})}, \quad (44)$$

$$R_2(t_2; \omega) = \frac{2 + 2t_2^2 + \omega^2}{Y_2[Y_2 + \sqrt{(1+t_2^2)(2+t_2^2)}]}.$$

When $r \neq 1$, for $t_1 \simeq 0$ we find that t_2 is large and positive, so that $R_1(t_2; \omega)$ and $R_2(t_2; \omega)$ are not singular. It is easily seen from (43) and (42) that though the functions G and $g^{(u)}$ are different, both differ from $g_1(t)$ by factors which are of the order ω^2/Y_1 , which do not contribute to the scaling function as shall be shown later in Sec. IV.

Similarly, for $T \simeq T_{2c}$, so that $\alpha_2 \simeq 1$, but $\alpha_1 > 1$, we see that whenever $\alpha_1^{m_1} \gg \alpha_1^{-m_1}$, we may drop the terms $\alpha_1^{-m_1}$ entering W in (14), to find

$$\begin{aligned} W &= \alpha_1^{m_1} W_2(t_1, t_2; m_2) + O(\alpha_1^{-m_1}), \\ W_2(t_1, t_2; m_2) &= \frac{1}{2}(\alpha_2^{m_2} + \alpha_2^{-m_2}) + \frac{1}{2}(\alpha_2^{m_2} - \alpha_2^{-m_2})G(t_1, t_2), \\ U^+(t_1, m_1) &= \alpha_1^{m_1}[1 + g_1(t_1)] + O(\alpha_1^{-m_1}). \end{aligned} \quad (45)$$

As a consequence we find from (1) and (12) that

$$-\frac{f_2(J_1, J_2; T)}{k_B T} = \frac{m_1 + m_2}{m_2 k_B T} [-f(J_1, J_2; T) + f(J_1, 0; T)] \quad (46)$$

$$\begin{aligned} &= \frac{1}{2} \ln(S_2/2) + \int_0^{\pi/2} \frac{d\theta}{\pi m_2} \left[\ln W_2(t_1, t_2; m_2) \right. \\ &\quad \left. - \ln \frac{1}{2}(1 + g_1) + O(\alpha_1^{-m_1}) \right]. \end{aligned} \quad (47)$$

Since f_2 is independent of m_1 , the plots of $C_2(T)$ in Fig. 6 of Part I [1] for different m_1 lie on the same curve demonstrating the data collapse.

From (8), we find for $\omega \sim 0$ the results

$$\begin{aligned} \alpha_1^{-m_1} &\approx e^{-2|t_1|m_1} \propto e^{-2m_1/\xi_1(T)}, \\ \alpha_2^{-m_2} &\approx e^{-2|t_2|m_2} \propto e^{-2m_2/\xi_2(T)}, \end{aligned} \quad (48)$$

where $\xi_i(T)$ is the bulk correlation length of the uniform Ising model with couplings J_i . This means that if r increases, so that t_2 becomes closer to t_1 , then for (38) to hold, so that data collapse occurs as shown in Fig. 5 of Part I [1], we must have m_2 large. Likewise as r increases, one sees that relations (45) still are valid provided m_1 is large with the consequence that data collapse still occurs near T_{2c} .

Even though (45) in (47) looks similar to (38) in (40), there are significant differences. In the regime $T \simeq T_{1c}$ the deviation t_2 is large and positive, while for $T \simeq T_{2c}$, one finds that t_1 is a large negative number, so that instead of (43), G in (15) for $T \simeq T_{2c}$ behaves as

$$\begin{aligned} G(t_1, t_2; \omega) &= g_2(-t_2) \\ &\quad + \omega^2(1 - \omega^2)Y_2^{-1} \left[t_2 \sqrt{(1 + t_2^2)} R_1(|t_1|; \omega) \right. \\ &\quad \left. + \sqrt{(1 + t_2^2)(2 + t_2^2)} R_2(t_1; \omega) - Y_1^{-1} \right], \end{aligned} \quad (49)$$

where $R_1(|t_1|; \omega)$ and $R_2(t_1; \omega)$ are seen from (44) to be nonsingular for t_1 large. Comparing this relation with (43), the flipping of the sign of t_2 in g_2 is the reason that the rounded peak at $T_{1\max}$ is below T_{1c} , while $T_{2\max}$ is above T_{2c} . This then sets the stage for what otherwise might be regarded as a purely phenomenological introduction of the modified temperature variable $\check{T}(T) = T_{2c} - (T - T_{2c})$ in (I.20).

IV. SCALING FUNCTIONS

We now consider f_1 in (40) in the scaling limit $m_1 \rightarrow \infty$ and $T \rightarrow T_{1c}$, and show that its scaling function is identical to that in (10) for a infinite strip of width m_1 and couplings J_1 [14]. In fact we shall show that when the differences in the integrands are of the order ω^2/Y_1 , as in (43) or in (42), the

scaling functions remain unchanged. We shall outline now the steps used to obtain the scaling function.

Step 1. We first change the integration variable in (40) to $\omega = \sin \theta$, and then split the interval of integration over ω into two parts, namely, $[0, 1] \rightarrow [0, c/m_1] + [c/m_1, 1]$, where here and below we take $c = \ln m_1$. Then we will approximate the integrand differently in the two distinct intervals.

Step 2. In the interval $[0, c/m_1]$, ω and t_1 are small, so we make the approximation

$$\begin{aligned} g_2 &\approx 1, \quad \alpha_1^{m_1} = e^{2m_1 \arcsin \sqrt{t_1^2 + \omega^2}} \approx e^{2X_1}, \\ X_1 &= \sqrt{\tau_1^2 + \phi^2}, \quad \tau_1 = m_1 t_1, \quad \phi = m_1 \omega, \\ G(t_1, t_2) &= g_1 + O(\omega^2/Y_1) \\ &\approx t_1 / \sqrt{t_1^2 + \omega^2} = \tau_1 / X_1. \end{aligned} \quad (50)$$

Note especially the introduction of the scaling variable τ_1 ; this is used in order to conform to the convention of the previous papers [13,14] in place of the scaling variable x_1 used in Part I [1]. But, as seen from (6), the two variables are related simply by a constant, i.e., $\tau_1 = 2K_{1c}x_1$ with $2K_{1c} = \ln(\sqrt{2} + 1)$. Using (50), the integrand in (40) can now be written as

$$\mathcal{I}_1 \approx \ln W_1 \approx \mathcal{H}(\tau_1, \phi) = \ln[\cosh 2X_1 + \sinh 2X_1(\tau_1/X_1)]. \quad (51)$$

After changing the variable of integration from $\omega \rightarrow \phi = m_1 \omega$, we split the interval of integration of ϕ to $[0, c] = [0, 1] + [1, c]$.

Step 3. In the interval $\omega \in [c/m_1, 1]$, we find, in (38), $\alpha_1^{m_1} \gg \alpha_1^{-m_1}$, so that $\alpha_1^{-m_1}$ can be dropped in W_1 , and the integrand in (40) becomes

$$\mathcal{I}_1(\omega) \approx \ln \left\{ \alpha_1^{m_1} \frac{1}{2} [1 + G(t_1, t_2; \omega)] \right\} - \ln \frac{1}{2} [1 + g_2(t_2; \omega)]. \quad (52)$$

Step 4. The integration over ω in the interval for the integrand in (52) is then split into two parts $[c/m_1, 1] = [1/m_1, 1] - [1/m_1, c/m_1]$. We denote the integrals over $[1/m_1, 1]$ by

$$\begin{aligned} \Sigma_1 &= \frac{1}{\pi} \int_{1/m_1}^1 \frac{d\omega}{\sqrt{1 - \omega^2}} \ln \alpha_1, \\ \Omega_1 &= \frac{1}{\pi m_1} \int_{1/m_1}^1 \frac{d\omega}{\sqrt{1 - \omega^2}} [\ln(1 + G) - \ln(1 + g_2)]. \end{aligned} \quad (53)$$

In the interval $\omega \in [1/m_1, c/m_1]$, we use (50) for the integrand in (52), so that

$$\mathcal{I}_1 \approx \mathcal{H}'(\tau_1, \phi) = \ln \left[e^{2X_1} \frac{1}{2} (1 + \tau_1/X_1) \right]. \quad (54)$$

Changing the variable of integration $\omega \rightarrow \phi = \omega m_1$, and combining it with the integral over the interval $[1, c]$ of the integrand in (51) in step 2, we obtain

$$\begin{aligned} \delta \mathcal{H}(\tau_1, \phi) &= \mathcal{H}(\tau_1, \phi) - \mathcal{H}'(\tau_1, \phi) \\ &= \ln[1 + e^{-2X_1}(X_1 - \tau_1)/(X_1 + \tau_1)]. \end{aligned} \quad (55)$$

For $\phi \geq c$, we find

$$\delta \mathcal{H}(\tau_1, \phi) \approx e^{-2X_1}(X_1 - \tau_1)/(X_1 + \tau_1) \ll 1. \quad (56)$$

Thus, the interval of integration $[1, c]$ can be extended to $[1, \infty]$ with negligible error.

Step 5. Combining all the steps, we find

$$f_1(J_1, J_2; T) \approx \mathcal{F}_1(\tau_1) + \Sigma_1 + \Omega_1, \quad (57)$$

where with $\mathcal{H}(\tau_1, \phi)$ defined in (51) and $\delta\mathcal{H}(\tau_1, \phi)$ defined in (55), we have

$$\mathcal{F}_1(\tau_1) = \frac{1}{m_1^2 \pi} \left[\int_0^1 d\phi \mathcal{H}(\tau_1, \phi) + \int_1^\infty d\phi \delta\mathcal{H}(\tau_1, \phi) \right]. \quad (58)$$

From (43), we find that for $\omega \in [0, c/m_1]$ or $[1/m_1, c/m_1]$, the terms of the order of ω^2/Y_1 may be dropped; thence the scaling function for an infinite strip of finite width in (10) and the scaling function for (40) can differ only through the term in Ω_1 introduced in (53). Since $T \simeq T_{1c}$, we find that $g_2(\pm t_2)$ is not singular, while its contribution is of order $1/m_1$; hence it does not contribute to the scaling function.

Step 6. The integrals for the derivatives of Σ_1 in (53) can be calculated explicitly. After keeping only the scaling terms we find

$$\begin{aligned} K_1^2 \frac{d^2 \Sigma_1}{dK_1^2} &= \frac{8K_1^2}{\pi} \int_{1/m_1}^1 \frac{d\omega}{\sqrt{1-\omega^2}} \frac{\omega^2(1+\omega^2+t_1^2) - t_1^2(t^2+\omega^2)}{[(1+\omega^2+t_1^2)(t^2+\omega^2)]^{3/2}} \\ &= (8K_{1c}^2/\pi) \left[\ln m_1 + \frac{3}{2} \ln 2 - \ln(1 + \sqrt{\tau_1^2 + 1}) - 1 + \frac{1}{\sqrt{\tau_1^2 + 1}} \right] + O\left(\frac{\ln m_1}{m_1}\right). \end{aligned} \quad (59)$$

The explicit calculation of the second derivatives of Ω_1 is very messy. However, it is easy to see that only the lower limit of the integration at $1/m_1$ can contribute to the scaling function. For $\omega \sim 1/m_1$, the integrand can be expanded as a series in terms of t_1 and ω with the results, on keeping only the leading terms,

$$K_1^2 \frac{d^2 \Omega_1}{dK_1^2} \approx \frac{-K_1^2}{\pi m_1} \int_{1/m_1}^1 d\omega \left[\frac{4t_1}{(t_1^2 + \omega^2)^{3/2}} + \frac{4}{t_1^2 + \omega^2} - \frac{8t_1^2}{(t_1^2 + \omega^2)^2} \right] \approx -\frac{4K_{1c}^2}{\pi} \left[\frac{1}{\tau_1} \left(1 - \frac{1}{\sqrt{\tau_1^2 + 1}} \right) + \frac{1}{1 + \tau_1^2} \right]. \quad (60)$$

As a cross-check, we have also verified that this agrees with the tedious explicit calculations. As the difference between G and g_1 are of the order of ω^2/Y_1 , we find that by replacing G by g_1 in Ω_1 does not change the scaling function. This means that near T_{1c} , the net specific heat $C_1(J_1, J_2; T)$ defined in (I.11) has the *same scaling behavior* as an infinite strip of width m_1 and couplings J_1 . Specifically, we find

$$C_1(J_1, J_2; T) \approx A_0 \ln m_1 + Q(\tau_1) \approx C^\infty(J_1; T), \quad A_0 = 8K_{1c}^2/\pi = 2[\ln(\sqrt{2} + 1)]^2/\pi, \quad (61)$$

where

$$\begin{aligned} Q(\tau_1) &= \frac{1}{2} A_0 \left[\int_0^1 d\phi \frac{d^2}{d\tau_1^2} \mathcal{H}(\tau_1, \phi) + \int_1^\infty d\phi \frac{d^2}{d\tau_1^2} \delta\mathcal{H}(\tau_1, \phi) \right. \\ &\quad + 3 \ln 2 - 2 \ln(1 + \sqrt{\tau_1^2 + 1}) \\ &\quad \left. - \left(2 + \frac{1}{\tau_1} \right) \left(1 - \frac{1}{\sqrt{\tau_1^2 + 1}} \right) - \frac{1}{1 + \tau_1^2} \right]. \end{aligned} \quad (62)$$

Letting $\sigma = 0$ in (2.62) of Ref. [14] we find that the scaling function given there is almost identical to this result; however, the difference term $-A_0\pi/4$ in (2.62) of Ref. [14] turns out to be a slip [16]. More recently the finite-size scaling functions for the Ising model have been shown [17] to be of universal character.

Now to study the specific heat near the lower special region $T \simeq T_{2c}$, we may use the same steps to analyze the integral in (47). Because of (49), for $\phi = m_2\omega \in [0, 1]$, we find that (45) becomes

$$W_2 \approx \mathcal{H}(-\tau_2, \phi) = \ln[\cosh 2X_2 - \sinh 2X_2(\tau_2/X_2)], \quad (63)$$

where

$$\tau_2 = m_2 t_2, \quad X_2 = \sqrt{\tau_2^2 + \phi^2}; \quad (64)$$

for $\phi \in [1, \infty]$, the integrand is approximated by

$$\delta\mathcal{H}(-\tau_2, \phi) = \ln[1 + e^{-2X_2}(X_2 + \tau_2)/(X_2 - \tau_2)]. \quad (65)$$

Consequently, the integral in (47) becomes

$$f_2(J_1, J_2; T) \approx \mathcal{F}_2(-\tau_2) + \Sigma_2 + \Omega_2, \quad (66)$$

where

$$\mathcal{F}_2(-\tau_2) = \frac{1}{m_2^2 \pi} \left[\int_0^1 d\phi \mathcal{H}(-\tau_2, \phi) + \int_1^\infty d\phi \delta\mathcal{H}(-\tau_2, \phi) \right], \quad (67)$$

$$\Sigma_2 = \frac{1}{\pi} \int_{1/m_2}^1 \frac{d\omega}{\sqrt{1-\omega^2}} \ln \alpha_2, \quad (68)$$

$$\Omega_2 = \frac{1}{\pi m_2} \int_{1/m_2}^1 \frac{d\omega}{\sqrt{1-\omega^2}} [\ln(1 + G) - \ln(1 + g_1)]. \quad (69)$$

Again the derivatives of Σ_2 and Ω_2 can be evaluated, with results which can be obtained from (59) and (60) by replacing τ_1 by $-\tau_2$, and m_1 by m_2 . The second derivative of $\mathcal{F}_2(-\tau_2)$ can also be evaluated to find [18] for $T \sim T_{2c}$,

$$C_2(J_1, J_2; T) \approx A_0 \ln m_2 + Q(-\tau_2), \quad \tau_2 = t_2 m_2. \quad (70)$$

Finally, comparing the free energy in (13) with U^+ given by (5) with (46) with W_2 given in (45), and then using (49), we find

that for $T \sim T_{2c}$,

$$(1 + s^{-1})C(0, J_2; \check{T}) = C_2(J_1, J_2; T) + O(\ln m_2/m_2) \\ \approx A_0 \ln m_2 + Q(-\tau_2), \quad (71)$$

where \check{T} is defined in (I.19) relating to $t_2 \rightarrow -t_2$.

V. ENHANCEMENT

Since the lower maxima of the specific heats $C_2(J_1, J_2; T)$ of the coupled system are above T_{2c} , while the maxima of the specific heats $C(0, J_2; T)$ of the uncoupled system are below T_{2c} , we have introduced in the specific heats $C(0, J_2; \check{T})$ whose free energy is defined in (13) and which has the same behavior as $C_2(J_1, J_2; T)$ for $T \sim T_{2c}$ as shown in (71). We have also defined in Part I [1] the net *enhancement* of the specific heat as

$$\mathcal{E}(J_1, J_2; m_1, m_2; T) \\ = C(J_1, J_2; T) - C(J_1, 0; T) - C(0, J_2; \check{T}(r)). \quad (72)$$

Near T_{1c} , we find that $C(0, J_2; \check{T})$ is similar to $C(0, J_2; T)$ in that it is relatively small and nonsingular and, in fact, does not contribute to the scaling function. We may use (36), (10), and (11) to rewrite the enhancement as

$$\mathcal{E}(J_1, J_2; m_1, m_2; T) = \frac{C_1(T) - C^\infty(T)}{1 + s} + \delta C, \quad (73)$$

where we define the difference δC as

$$\delta C = C(0, J_2; T) - C(0, J_2; \check{T}(r)) \simeq 0. \quad (74)$$

Indeed for $e^{-2m_2/\xi_2(T)} \ll 1$ we find that $C_1(T)$ has the same scaling behavior as $C^\infty(T)$. From (61), we thus find that the enhancement is of the order of a *correction to scaling*. As (59) gives the magnitude of the corrections to scaling, we find that (73) becomes

$$\mathcal{E}(J_1, J_2; m_1, m_2; T) \approx \frac{B_0(r) \ln m_1 + B(r, \tau_1)}{m_1 + m_2}, \quad (75)$$

where $B_0(r)$ and $B(r, \tau_1)$ are functions of order unity whose forms can be gauged from Figs. 9–11 of Part I. On the other hand, we find from (71) the corresponding result

$$\mathcal{E}(J_1, J_2; m_1, m_2; T) \approx \frac{\hat{B}_0(r) \ln m_2 + \hat{B}(r, \tau_2)}{m_1 + m_2}, \quad (76)$$

for T near T_{2c} , when $e^{-2m_1/\xi_1(T)} \ll 1$, with $\hat{B}_0(r)$ and $\hat{B}(r, \tau_2)$ appropriate functions of order unity. As the relative strength r increases, T_{2c} and T_c approach T_{1c} , because $T_{2c} = rT_{1c}$ and $T_{2c} < T_c < T_{1c}$. This also means that the regimes in which (61) or (71) are valid shrink. The explicit form of these corrections to scaling and the functions $B_0(r)$, $B(r, \tau_1)$, $\hat{B}_0(r)$, and $\hat{B}(r, \tau_2)$ are not easy to obtain exactly and the computations have not been attempted. Comparing with (I.23), we have $B_0(r) = \mathcal{B}^+(r)$ and $\hat{B}_0(r) = \mathcal{B}^-(r)$.

VI. SUMMARY

For the alternating layered Ising model, we show there exists a well defined critical temperature, at which, the specific heat diverges according to (21). However, for fixed relative strength $r = J_2/J_1 \neq 1$, and $s = m_2/m_1 \neq 0$, we find the amplitude $A(r, s)$ decreases exponentially fast in m_2 . For large enough m_1 and m_2 the specific heat also has two distinct maxima satisfying the relations $T_c < T_{\max 1} < T_{1c}$ and $T_{2c} < T_{\max 2} < T_c$. These general results agree with the experiments on superfluid helium by Gasparini and co-workers [2–7].

Near T_{1c} , we find the net specific heat, $C_1(T)$ defined in (36), obeys finite-size scaling as established in (61) when $e^{-2m_2/\xi_2(T)}$ is negligible. On the other hand, near T_{2c} , the lower maximum, we find the corresponding $C_2(T)$, whose free energy is defined in (46), obeys the finite-size scaling given by (71) when $e^{-2m_1/\xi_1(T)}$ is small; remarkably, the sign of the appropriate scaled temperature deviation $T - T_{2c}$ is then *reversed* from that for an infinite strip of finite width. However, this corresponds qualitatively to the observed enhancement in the experiments induced by the proximity effects of ordered regions below the true bulk critical point.

It should be noted, however, that modeling the experimental systems would be improved by using three spatial dimensions and, furthermore, Ising spins would better be replaced by XY spins.

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