Perpetual extraction of work from a nonequilibrium dynamical system under Markovian feedback control

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By treating both control parameters and dynamical variables as probabilistic variables, we develop a succinct theory of perpetual extraction of work from a generic classical nonequilibrium system subject to a heat bath via repeated measurements under a Markovian feedback control. It is demonstrated that a problem for perpetual extraction of work in a nonequilibrium system is reduced to a problem of Markov chain in the higher-dimensional phase space. We derive a version of the detailed fluctuation theorem, which was originally derived for classical nonequilibrium systems by Horowitz and Vaikuntanathan [Phys. Rev. E **82**, 061120 (2010)], in a form suitable for the analyses of perpetual extraction of work. Since our theory is formulated for generic dynamics of probability distribution function in phase space, its application to a physical system is straightforward. As simple applications of the theory, two exactly solvable models are analyzed. The one is a nonequilibrium two-state system and the other is a particle confined to a one-dimensional harmonic potential in thermal equilibrium. For the former example, it is demonstrated that the observer on the transitory steps to the stationary state can lose energy and that work larger than that achieved in the stationary state can be extracted. For the latter example, it is demonstrated that the optimal protocol for the extraction of work via repeated measurements can differ from that via a single measurement. The validity of our version of the detailed fluctuation theorem, which determines the upper bound of the expected work in the stationary state, is also confirmed for both examples. These observations provide useful insights into exploration for realistic modeling of a machine that extracts work from its environment.

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I. INTRODUCTION

The importance of extraction of work from a thermodynamic system subject to a heat bath under a feedback control has become more widely known in physics sdue to recent experimental realizations [1–4]. In a simplified view of physics, such a system is often modeled as a Brownian particle confined to a potential. The control parameters that characterize the system are tuned by the observer in his own way according to the outcome of the measurement. If the energy of the Brownian particle is lowered, the work has been extracted successfully and the gained energy will be consumed in mechanical or chemical or another form. The feedback control dates back to the well-known Szilárd engine [5] and it has been one of the central topics in nonequilibrium thermodynamics, involving the concepts of information theory [6–9]. There exist comprehensive reviews on the recent progress in the nonequilibrium thermodynamics which focus on the nanoscale feedback controls [10,11]. For a real system having ways for extraction of work from its environment, whether it is artificial or not, one of the most important question is how to realize the maximum extracted work. When we consider such a problem, we should not overlook that the measurements and the controls are performed repeatedly.

Theoretical studies on extraction of work via measurements under feedback controls for classical systems have been done also in the recent literature [12–18]. Cao and Feito [16] examined repeated measurements under feedback control from the viewpoint of entropy reduction. They also analyzed the maximum extracted work within the framework of thermodynamics. Abreu and Seifert [14] examined in detail the extraction of work from a particle in a harmonic potential subject to a heat bath via an imprecise measurement. They revealed the relationship between the extracted work and the protocol of the feedback control, paying attention to the information obtained in the measurement [9,19–22]. Esposito and Schaller [17] examined nonequilibrium systems under feedback control which affects only the energy barriers between system states by extending the traditional local detailed balance.

Horowitz and Vaikuntanathan derived the detailed fluctuation theorem [23] for nonequilibrium classical systems by establishing the definition and interpretation of the reverse process corresponding to a forward process under feedback control. One of its consequence is the second law of thermodynamics for discrete feedback, which imposes upper bound on the extracted work through feedback loops. A version of the detailed fluctuation theorem in a form suitable for the analyses of perpetual extraction of work is derived in the present study, since it provides important insights into the characteristics unique to the perpetual extraction of work.

In the present study, we develop a succinct theory of perpetual extraction of work from a generic classical nonequilibrium system subject to a heat bath via repeated measurements under a Markovian feedback control. We treat the control parameters tuned by the observer as probabilistic variables, as well as the dynamical variables fluctuating in the system. Mandala and Jarzynski [18] has constructed a minimal model of an autonomous Maxwell demon on a similar spirit. The new control parameters are determined by the observer according to the result of a measurement and the control parameters just before the measurement. Thus, even if the distribution of the dynamical variables is that achieved in the thermal equilibrium for certain control parameters, the memory effect, i.e., the influence of previous measurements, remains as the present control parameters. This fact is nothing but the reason for the requirement of the development of a new generic theory for multiple measurements under feedback control. We focus on the dynamics of the probability distribution function (PDF) to derive the expression for extracted work via a measurement under feedback control, together with the condition for perpetual extraction of work. Since we work in the phase space, application of our theory to physical systems is straightforward. We analyze two exactly solvable models as simple applications of our generic theory. The one is a nonequilibrium two-state system and the other is an equilibrium particle confined to a one-dimensional harmonic potential.

II. FORMALISM

A. Generic expressions for nonequilibrium systems

1. Extraction of work and stationarity condition

Let us consider a system subject to a heat bath in which a dynamical variable $\boldsymbol{x}$ changes as time evolves. The system is characterized by the control parameters, which we denote collectively by $\boldsymbol{\lambda}$. An observer performs a measurement on the system to obtain physical quantities, which we denote collectively by $\boldsymbol{f}_{\mathrm{m}}$. We assume that a Markovian feedback control is implemented on this system, that is, the observer changes the control parameter to the new one $\widetilde{\boldsymbol{\lambda}}$ in a deterministic manner according to the control parameter before the measurement and the measured quantity: $\widetilde{\boldsymbol{\lambda}} = \widetilde{\boldsymbol{\lambda}}(\boldsymbol{f}_{\mathrm{m}},\boldsymbol{\lambda})$. The measured value of the physical quantity does not necessarily coincide with its real value since the measurement is not completely precise. The measured value is thus described by a conditional probability as $p_{\mathrm{m}}(\boldsymbol{f}_{\mathrm{m}}|\boldsymbol{x},\boldsymbol{\lambda})$. Since the measured quantity is probabilistic, the control parameter is also probabilistic, even when the feedback control is completely precise. We therefore treat the control parameter $\boldsymbol{\lambda}$ as a probabilistic variable as well as $\boldsymbol{x}$. The system is thus represented by the nonequilibrium PDF $P(\boldsymbol{x},\boldsymbol{\lambda})$ in the higher-dimensional phase space for $\boldsymbol{x}$ and $\boldsymbol{\lambda}$ rather than that only for $\boldsymbol{x}$. We omit the time variable for simplicity unless otherwise stated in what follows.

While feedback control with a finite time lapse has been studied theoretically [24–26], we assume instantaneous change in the control parameter in the present study. In a real system, feedback control can suffer from an error and the intended control parameter is not necessarily realized [16]. We therefore introduce a probability distribution $p_{\mathrm{c}}(\boldsymbol{\lambda};\boldsymbol{\lambda}')$, which represents the probability that the control parameter $\boldsymbol{\lambda}$ is actually realized when the observer intends to realize $\boldsymbol{\lambda}'$. When the feedback control is completely precise, this probability distribution is the δ function: $p_{\mathrm{c}}(\boldsymbol{\lambda};\boldsymbol{\lambda}') = \delta(\boldsymbol{\lambda}-\boldsymbol{\lambda}')$. We can obtain the PDF P_{out} just after the measurement as

$$P_{\mathrm{out}}(\boldsymbol{x},\boldsymbol{\lambda}) = \int d\boldsymbol{\lambda}' G(\boldsymbol{\lambda},\boldsymbol{\lambda}';\boldsymbol{x}) P_{\mathrm{in}}(\boldsymbol{x},\boldsymbol{\lambda}'), \tag{1}$$

where

$$G(\boldsymbol{\lambda},\boldsymbol{\lambda}';\boldsymbol{x}) \equiv \int d\boldsymbol{f}_{\mathrm{m}} p_{\mathrm{c}}(\boldsymbol{\lambda};\widetilde{\boldsymbol{\lambda}}(\boldsymbol{f}_{\mathrm{m}},\boldsymbol{\lambda}')) p_{\mathrm{m}}(\boldsymbol{f}_{\mathrm{m}}|\boldsymbol{x},\boldsymbol{\lambda}') \tag{2}$$

is the propagator of the control parameter. P_{in} is the PDF just before the measurement.

The energy of the system is a function of the dynamical variable, $E(\boldsymbol{x},\ \boldsymbol{\lambda})$. The expected work applied to the system through the instantaneous change in the control parameter is given by the expected difference between the energies after and before the measurement as

$$\begin{aligned}\langle W\rangle &= \int d\boldsymbol{x} d\boldsymbol{\lambda} [P_{\mathrm{out}}(\boldsymbol{x},\boldsymbol{\lambda}) - P_{\mathrm{in}}(\boldsymbol{x},\boldsymbol{\lambda})] E(\boldsymbol{x},\boldsymbol{\lambda}) \\ &= \int d\boldsymbol{x} d\boldsymbol{\lambda} P_{\mathrm{in}}(\boldsymbol{x},\boldsymbol{\lambda}) [\widetilde{E}(\boldsymbol{x},\boldsymbol{\lambda}) - E(\boldsymbol{x},\boldsymbol{\lambda})],\end{aligned} \tag{3}$$

where

$$\widetilde{E}(\boldsymbol{x},\boldsymbol{\lambda}) \equiv \int d\boldsymbol{\lambda}' E(\boldsymbol{x},\boldsymbol{\lambda}') G(\boldsymbol{\lambda}',\boldsymbol{\lambda};\boldsymbol{x}) \tag{4}$$

is the measurement-averaged energy with $\boldsymbol{x}$ and $\boldsymbol{\lambda}$ given before the measurement. $\langle\cdot\rangle$ represents the average over all the possible dynamical variables and the control parameters. With the vectorial representation, Eq. (3) is written as $\langle W\rangle[\widetilde{\boldsymbol{\lambda}},\boldsymbol{P}_{\mathrm{in}}] = \boldsymbol{P}_{\mathrm{in}} \cdot (\widetilde{\boldsymbol{E}}[\widetilde{\boldsymbol{\lambda}}] - \boldsymbol{E})$, a functional of the initial PDF and the protocol. If W is negative, the work has been successfully extracted.

We assume the time translational symmetry of the system. When the observer performs the second measurement after an interval Δt of time, the PDF just before the second measurement has evolved to be

$$P_{\mathrm{out}}(\boldsymbol{x},\boldsymbol{\lambda},t+\Delta t) \equiv \int d\boldsymbol{x}' U(\boldsymbol{x},\boldsymbol{x}';\boldsymbol{\lambda},\Delta t) P_{\mathrm{out}}(\boldsymbol{x}',\boldsymbol{\lambda},t). \tag{5}$$

$U(\boldsymbol{x},\boldsymbol{x}';\boldsymbol{\lambda},\Delta t)$ is the time development operator of the PDF for a control parameter $\boldsymbol{\lambda}$. If the system and the observer are sound as a single machine for perpetual extraction of work from the environment, the input PDF for a measurement and that for the next measurement are expected to be identical. We assume that such a PDF $P_*(\boldsymbol{x},\boldsymbol{\lambda})$ is achieved when a sufficient number of measurements have been performed. From Eqs. (1) and (5), it satisfies the condition

$$\boldsymbol{P}_* = U(\Delta t) G \boldsymbol{P}_*, \tag{6}$$

which mathematically means that $\boldsymbol{P}_*$ is the eigenfunction of the operator $U(\Delta t)G$ belonging to the eigenvalue 1. The problem for the perpetual extraction of work in the nonequilibrium system has been reduced to the problem of Markov chain in the higher-dimensional phase space. $\boldsymbol{P}_*$ is the stationary distribution function in the higher-dimensional phase space. In particular, if it does not depend on the initial PDF, the expected work after the stationary distribution is achieved is given by $\langle W\rangle_*[\widetilde{\boldsymbol{\lambda}},\Delta t] = \boldsymbol{P}_*[\widetilde{\boldsymbol{\lambda}},\Delta t] \cdot (\widetilde{\boldsymbol{E}}[\widetilde{\boldsymbol{\lambda}}] - \boldsymbol{E})$, also independent of the initial PDF.

When the numbers of values taken by $\boldsymbol{x}$ and $\boldsymbol{\lambda}$ are finite, the transition matrix $U(\Delta t)G$ is a finite-dimensional stochastic matrix, whose elements are all non-negative and the sum of the elements of each column vector in the matrix is unity. If the matrix is irreducible in addition, the Perron-Frobenius theorem [27] is straightforwardly applicable in this case. The theorem states that $U(\Delta t)G$ has the eigenvalue 1 and only one corresponding eigenvector, and the magnitudes of all the other eigenvalues are smaller than 1. If the input PDF contains any eigenvector belonging to the eigenvalue other than 1, the component coming from that eigenvector decays via the repeated operations of the transition matrix, which indicates that the convergence of the PDF to the stationary distribution is dominated by the eigenvalue ν having the second largest magnitude. The exponential decay to $1/e$ of the initial component thus requires $n_{\mathrm{decay}} \equiv -1/\ln|\nu|$ measurements. n_{decay} is nothing but the mixing time of the Markov chain. Since the several mathematical assumptions introduced above are plausible in practical situations, we can expect in general that the exponentially rapid convergence to the stationary state is achieved in an ensemble of real machines for perpetual extraction of work.

If we consider a more realistic situation, we should take the generalized cost c of each measurement into account. While the thermodynamic cost of measurement has been modeled by focusing on the microscopic states of measurement devices [20,28], we regard c simply as an energy cost needed at each measurement. We assume c depends neither on $\boldsymbol{x}$ nor $\boldsymbol{\lambda}$ for simplicity. The expected work per unit time,

$$\langle w\rangle_*[\widetilde{\boldsymbol{\lambda}},\Delta t] = \frac{\langle W\rangle_*[\widetilde{\boldsymbol{\lambda}},\Delta t]+c}{\Delta t}, \tag{7}$$

should thus be optimized with respect to the protocol and the interval of the measurements. Whether a real system having ways for extraction of work from its environment is artificial or not, it should be designed to attain the largest negative $\langle w\rangle_*$ for repeated measurements, not for a single measurement.

2. Reverse process and detailed fluctuation theorem

One of the important and useful tools for the analysis of perpetual extraction of work is the detailed fluctuation theorem under repeated feedback, which was originally derived for nonequilibrium systems by Horowitz and Vaikuntanathan [23] on the basis of Hamiltonian dynamics under discrete feedback control. We rederive the detailed fluctuation theorem below in a suitable form for the analyses of stationary states for perpetual extraction of work.

Let us consider a situation in which an observer performs N measurements, beginning with an initial control parameter $\boldsymbol{\lambda}_0$ and the initial PDF $P(\boldsymbol{x},\boldsymbol{\lambda},t_0) = P_{\mathrm{in}}(\boldsymbol{x}|\boldsymbol{\lambda}_0)\delta(\boldsymbol{\lambda}-\boldsymbol{\lambda}_0)$. $t_i(i=0,\ldots,N-1)$ represents the time at which the $(i+1)$th measurement is performed. We denote the realized dynamical variables up to the $(i+1)$th measurement collectively by the trajectory $\boldsymbol{X}_{i+1} = \{\boldsymbol{x}_0,\ldots,\boldsymbol{x}_i\}$, where $\boldsymbol{x}_j$ is the dynamical variable realized at the $(j+1)$th measurement. Let $\boldsymbol{\lambda}_i(i=1,\ldots,N-1)$ be the control parameter fixed in the interval between t_{i-1} and t_i, and $\boldsymbol{\lambda}_N$ be that for $t > t_{N-1}$. For a given trajectory $\boldsymbol{X}_N$ of the dynamical variable and that of the control parameter $\boldsymbol{\Lambda}_{N+1} \equiv \{\boldsymbol{\lambda}_0,\ldots,\boldsymbol{\lambda}_N\}$, the total work applied to the system via the N measurements is given by

$$W[\boldsymbol{X}_N,\boldsymbol{\Lambda}_{N+1}] = \sum_{i=0}^{N-1} E(\boldsymbol{x}_i,\boldsymbol{\lambda}_{i+1}) - E(\boldsymbol{x}_i,\boldsymbol{\lambda}_i). \tag{8}$$

As is known in statistical mechanics, the partition function $Z(\boldsymbol{\lambda})$ of the system with a control parameter $\boldsymbol{\lambda}$ is related to the free energy of the system via the relation $e^{-\beta F(\boldsymbol{\lambda})} = Z(\boldsymbol{\lambda})$, where β is the inverse temperature. The difference in the free energy between after the last measurement and before the first measurement $\Delta F(\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_N)$ thus satisfies the relation $e^{-\beta\Delta F} = Z(\boldsymbol{\lambda}_N)/Z(\boldsymbol{\lambda}_0)$.

The probability distribution of $\boldsymbol{X}_N$ for a given trajectory $\boldsymbol{\Lambda}_N$ of the control parameter is calculated by acting the time development operators on the initial PDF successively as

$$P[\boldsymbol{X}_N|\boldsymbol{\Lambda}_N] = \left[\prod_{i=0}^{N-2} U(\boldsymbol{x}_{i+1},\boldsymbol{x}_i;\boldsymbol{\lambda}_{i+1},\Delta t_i)\right] P_{\mathrm{in}}(\boldsymbol{x}_0,t_0|\boldsymbol{\lambda}_0), \tag{9}$$

where $\Delta t_i \equiv t_{i+1}-t_i$.

Remember that the reverse time development operator is defined via the unitarity condition

$$\int d\boldsymbol{y} U(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\lambda},\Delta t)U(\boldsymbol{y},\boldsymbol{x}';\boldsymbol{\lambda},-\Delta t) = \delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{10}$$

where $U(\boldsymbol{x},\boldsymbol{x}';\boldsymbol{\lambda},-\Delta t)$ is the reverse time development operator from $\boldsymbol{x}'$ to $\boldsymbol{x}$ by a time interval Δt under a control parameter $\boldsymbol{\lambda}$. Here we define the following quantity:

$$r(\boldsymbol{x}',\boldsymbol{x};\boldsymbol{\lambda},\Delta t) \equiv \ln\frac{U(\boldsymbol{x}',\boldsymbol{x};\boldsymbol{\lambda},\Delta t)P_{\mathrm{eq}}(\boldsymbol{x}|\boldsymbol{\lambda})}{U(\boldsymbol{x},\boldsymbol{x}';\boldsymbol{\lambda},-\Delta t)P_{\mathrm{eq}}(\boldsymbol{x}'|\boldsymbol{\lambda})}, \tag{11}$$

where $P_{\mathrm{eq}}(\boldsymbol{x}|\boldsymbol{\lambda}) \equiv \exp[-\beta E(\boldsymbol{x},\boldsymbol{\lambda})]/Z(\boldsymbol{\lambda})$ is the canonical distribution function. The argument of the logarithm on the right-hand side of Eq. (11) is the ratio of the two probabilities in the thermal equilibrium. The one in the numerator is the probability that the dynamical variable is $\boldsymbol{x}$ and after an interval Δt of forward time development it is $\boldsymbol{x}'$, while the other in the denominator is the probability that the dynamical variable is $\boldsymbol{x}'$ and after an interval Δt of reverse time development it is $\boldsymbol{x}$. When these two probabilities are equal, $r(\boldsymbol{x}',\boldsymbol{x};\boldsymbol{\lambda},\Delta t)$ vanishes. We thus can interpret it as a microscopic-irreversibility indicator. Using Eq. (11), we introduce the microscopic-irreversibility indicator of the whole process as

$$R[\boldsymbol{X}_N;\boldsymbol{\Lambda}_N] \equiv \sum_{i=0}^{N-2} r(\boldsymbol{x}_{i+1},\boldsymbol{x}_i;\boldsymbol{\lambda}_{i+1},\Delta t_i). \tag{12}$$

We define the following probability distribution for the given trajectory of the control parameter by acting the reverse time development operators on the canonical distribution function for $\boldsymbol{\lambda}_N$ successively as

$$\widetilde{P}_{\mathrm{dyn}}[\boldsymbol{X}_N|\boldsymbol{\Lambda}_{N+1}] \equiv P_{\mathrm{eq}}(\boldsymbol{x}_N|\boldsymbol{\lambda}_N)\prod_{i=0}^{N-2} U(\boldsymbol{x}_i,\boldsymbol{x}_{i+1};\boldsymbol{\lambda}_{i+1},-\Delta t_i), \tag{13}$$

which is determined only by the dynamics of the PDF in the forward time development, regardless of the feedback protocol.

The relative entropy [29] between two probability distributions f and g is defined as $\int dx f(x)\ln[f(x)/g(x)]$. Here we define the trajectory-dependent relative-entropy density of the initial PDF with respect to the corresponding canonical distribution as

$$d_{\mathrm{in}}(\boldsymbol{x}_0;\boldsymbol{\lambda}_0) \equiv \ln\frac{P_{\mathrm{in}}(\boldsymbol{x}_0,t_0|\boldsymbol{\lambda}_0)}{P_{\mathrm{eq}}(\boldsymbol{x}_0|\boldsymbol{\lambda}_0)}. \tag{14}$$

By using Eqs. (8), (9), (12), (13), and (14), we obtain the relation

$$\frac{P[\boldsymbol{X}_N|\boldsymbol{\Lambda}_N]}{\widetilde{P}_{\mathrm{dyn}}[\boldsymbol{X}_N|\boldsymbol{\Lambda}_{N+1}]} = \exp[\beta\{W[\boldsymbol{X}_N,\boldsymbol{\Lambda}_{N+1}]-\Delta F(\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_N)\} + R[\boldsymbol{X}_N;\boldsymbol{\Lambda}_N] + d_{\mathrm{in}}(\boldsymbol{x}_0;\boldsymbol{\lambda}_0)] \tag{15}$$

for the fixed trajectory $\boldsymbol{\Lambda}_{N+1}$ of the control parameter, that is, $\boldsymbol{\lambda}_i$'s are the free parameters in this relation.

We denote the observed physical quantities collectively by the trajectory $\boldsymbol{F}_{\mathrm{m}}^{i+1} = \{\boldsymbol{f}_{\mathrm{m}}^0, \ldots, \boldsymbol{f}_{\mathrm{m}}^i\}$. The joint probability distribution of $\boldsymbol{X}_N$ and $\boldsymbol{F}_{\mathrm{m}}^N$ is written as

$$\mathcal{P}[\boldsymbol{X}_N, \boldsymbol{F}_{\mathrm{m}}^N] = P_{\mathrm{m}}[\boldsymbol{F}_{\mathrm{m}}^N | \boldsymbol{X}_N, \widetilde{\boldsymbol{\Lambda}}_N] P[\boldsymbol{X}_N | \widetilde{\boldsymbol{\Lambda}}_N], \tag{16}$$

where $\widetilde{\boldsymbol{\Lambda}}_{i+1} \equiv \{\boldsymbol{\lambda}_0, \widetilde{\boldsymbol{\lambda}}_1, \ldots, \widetilde{\boldsymbol{\lambda}}_i\}$ with $\widetilde{\boldsymbol{\lambda}}_j \equiv \widetilde{\boldsymbol{\lambda}}(\boldsymbol{f}_{\mathrm{m}}^{j-1}, \widetilde{\boldsymbol{\lambda}}_{j-1})$ is the trajectory of the control parameter implied by the trajectory $\boldsymbol{F}_{\mathrm{m}}^i$ of the measured quantities.

$$P_{\mathrm{m}}[\boldsymbol{F}_{\mathrm{m}}^N | \boldsymbol{X}_N, \boldsymbol{\Lambda}_N] \equiv \prod_{i=0}^{N-1} p_{\mathrm{m}}(\boldsymbol{f}_{\mathrm{m}}^i | \boldsymbol{x}_i, \boldsymbol{\lambda}_i) \tag{17}$$

is the probability distribution of the trajectory of measured quantities via the N measurements with fixed trajectories of the dynamical variable and the control parameter.

Following Horowitz and Vaikuntanathan [23], we define the mutual information density for the fixed trajectories $\boldsymbol{X}_N$ and $\boldsymbol{F}_{\mathrm{m}}^N$ as

$$\begin{aligned} I[\boldsymbol{X}_N, \boldsymbol{F}_{\mathrm{m}}^N] &\equiv \ln \frac{\mathcal{P}[\boldsymbol{X}_N, \boldsymbol{F}_{\mathrm{m}}^N]}{P[\boldsymbol{X}_N | \widetilde{\boldsymbol{\Lambda}}_N] P[\boldsymbol{F}_{\mathrm{m}}^N | \widetilde{\boldsymbol{\Lambda}}_N]} \\ &= \ln \frac{P_{\mathrm{m}}[\boldsymbol{F}_{\mathrm{m}}^N | \boldsymbol{X}_N, \widetilde{\boldsymbol{\Lambda}}_N]}{P[\boldsymbol{F}_{\mathrm{m}}^N | \widetilde{\boldsymbol{\Lambda}}_N]}, \end{aligned} \tag{18}$$

where Eq. (16) has been used. $I[\boldsymbol{X}_N, \boldsymbol{F}_{\mathrm{m}}^N]$ represents the change in the uncertainty of the observer's knowledge on the microscopic state of the system upon making the N measurements.

Remembering that Eq. (15) holds for an arbitrary $\boldsymbol{\Lambda}_{N+1}$, we obtain, by using Eqs. (16) and (18), the detailed fluctuation theorem [23],

$$\begin{aligned} &\frac{\mathcal{P}[\boldsymbol{X}_N, \boldsymbol{F}_{\mathrm{m}}^N]}{\widetilde{\mathcal{P}}[\boldsymbol{X}_N, \boldsymbol{F}_{\mathrm{m}}^N]} \\ &\quad = \exp\big[\beta\{W[\boldsymbol{X}_N, \widetilde{\boldsymbol{\Lambda}}_{N+1}] - \Delta F(\boldsymbol{\lambda}_0, \widetilde{\boldsymbol{\lambda}}_N)\} + I[\boldsymbol{X}_N, \boldsymbol{F}_{\mathrm{m}}^N] \\ &\qquad + R[\boldsymbol{X}_N; \widetilde{\boldsymbol{\Lambda}}_N] + d_{\mathrm{in}}(\boldsymbol{x}_0; \boldsymbol{\lambda}_0)\big], \end{aligned} \tag{19}$$

where we have *defined* the probability distribution of the $\boldsymbol{x}_i$'s and the $\boldsymbol{f}_{\mathrm{m}}^i$'s in the reverse process as

$$\widetilde{\mathcal{P}}[\boldsymbol{X}_N, \boldsymbol{F}_{\mathrm{m}}^N] \equiv \widetilde{P}_{\mathrm{dyn}}[\boldsymbol{X}_N | \widetilde{\boldsymbol{\Lambda}}_{N+1}] P[\boldsymbol{F}_{\mathrm{m}}^N | \widetilde{\boldsymbol{\Lambda}}_N]. \tag{20}$$

For plausible definition and interpretation of the reverse process, the factor $P[\boldsymbol{F}_{\mathrm{m}}^N | \widetilde{\boldsymbol{\Lambda}}_N]$ on the right-hand side of the definition above should not be associated with a measurement in the reverse time development, as will be explained later.

By multiplying the inverse of the both sides of Eq. (19) by $\mathcal{P}[\boldsymbol{X}_N, \boldsymbol{F}_{\mathrm{m}}^N]$ and integrating them with respect to $\boldsymbol{X}_N$ and $\boldsymbol{F}_{\mathrm{m}}^N$, we obtain the generalized Jarzynski's equality [23,30–32] for the given $\boldsymbol{\lambda}_0$,

$$\overline{e^{-\beta(W-\Delta F)-I-R-d_{\mathrm{in}}}} = 1, \tag{21}$$

where the overline indicates the average over all possible trajectories for the given $\boldsymbol{\lambda}_0$. Similarly, by taking the average of the logarithms of the both sides of Eq. (19) with respect to $\boldsymbol{X}_N$ and $\boldsymbol{F}_{\mathrm{m}}^N$, we obtain another relation. We obtain the relative-entropy work relation [19,23,33] for the given $\boldsymbol{\lambda}_0$ as

$$D[\mathcal{P}|\widetilde{\mathcal{P}}] = \beta(\overline{W} - \overline{\Delta F}) + \overline{I} + \overline{R} + D[P_{\mathrm{in}}|P_{\mathrm{eq}}], \tag{22}$$

where the left-hand side is the relative entropy [29] of $\mathcal{P}[\boldsymbol{X}_N, \boldsymbol{F}_{\mathrm{m}}^N]$ with respect to $\widetilde{\mathcal{P}}[\boldsymbol{X}_N, \boldsymbol{F}_{\mathrm{m}}^N]$. $D[P_{\mathrm{in}}|P_{\mathrm{eq}}]$ is the relative entropy of the initial PDF with respect to the canonical distribution function, that is, the average of $d_{\mathrm{in}}(\boldsymbol{x}_0; \boldsymbol{\lambda}_0)$ [see Eq. (14)]. $D[\mathcal{P}|\widetilde{\mathcal{P}}]$ measures the distinguishability between the forward and the reverse processes. It is interpreted to measure microscopically the intensity of the arrow of time [19,23,34,35]. For any non-negative probability distributions f and g, their relative entropy is non-negative: $D[f|g] \geqslant 0$, whose equality holds if and only if f is equal to g everywhere f is nonzero [29]. It should be stressed here that $\widetilde{\mathcal{P}}[\boldsymbol{X}_N, \boldsymbol{F}_{\mathrm{m}}^N]$ is not necessarily non-negative since it contains the reverse time development operator [see Eqs. (13) and (20)]. Although $U(\boldsymbol{x}, \boldsymbol{x}'; \boldsymbol{\lambda}, -\Delta t)$ should represent physically the transition from $\boldsymbol{x}'$ to $\boldsymbol{x}$ in the reverse time development, it can be unphysically negative, as demonstrated later for the example of a nonequilibrium two-state system. Even in such a case, the condition Eq. (10) ensures the conservation of probability mathematically and, hence, the detailed fluctuation theorem, Eq. (19), holds.

For a case in which the measurement is performed only once ($N = 1$), the reverse time development operator is not involved in $\widetilde{\mathcal{P}}[\boldsymbol{X}_N, \boldsymbol{F}_{\mathrm{m}}^N]$ and $R[\boldsymbol{X}_N; \widetilde{\boldsymbol{\Lambda}}_N]$ vanishes, and, hence, non-negative relative entropy is ensured in Eq. (22): $D[\mathcal{P}|\widetilde{\mathcal{P}}] \geqslant 0$, whose equality with the fixed $\boldsymbol{\lambda}_0$ holds if and only if $\mathcal{P}[\boldsymbol{X}_1, \boldsymbol{F}_{\mathrm{m}}^1] = \widetilde{\mathcal{P}}[\boldsymbol{X}_1, \boldsymbol{F}_{\mathrm{m}}^1]$ for all $\boldsymbol{X}_1$'s and $\boldsymbol{F}_{\mathrm{m}}^1$'s giving nonzero $\mathcal{P}[\boldsymbol{X}_1, \boldsymbol{F}_{\mathrm{m}}^1]$. Since $D[P_{\mathrm{in}}|P_{\mathrm{eq}}] \geqslant 0$, the deviation of the initial PDF from the canonical distribution function raises the upper bound of the extracted work.

The detailed fluctuation theorem in the form of Eq. (19) can be rewritten to a more implicative form. Let us consider the moment in which the dynamical variable $\boldsymbol{x}_i$ is realized at the $(i+1)$th measurement and the control parameter has been changed to $\boldsymbol{\lambda}_{i+1}$. The PDF of $\boldsymbol{x}$ just after the $(i+1)$th measurement is thus $P_+(\boldsymbol{x}, t_i) \equiv \delta(\boldsymbol{x} - \boldsymbol{x}_i)$. The PDF just before the next measurement is given by $P_+(\boldsymbol{x}, t_{i+1}) \equiv U(\boldsymbol{x}, \boldsymbol{x}_i; \boldsymbol{\lambda}_{i+1}, \Delta t_i)$. The difference in the entropy between the PDFs at t_{i+1} and t_i in the forward process is calculated as

$$\begin{aligned} \Delta S_+(t_i \to t_{i+1}) &\equiv S[P_+(\boldsymbol{x}, t_{i+1})] - S[P_+(\boldsymbol{x}, t_i)] \\ &= -\int d\boldsymbol{x} U(\boldsymbol{x}, \boldsymbol{x}_i; \boldsymbol{\lambda}_{i+1}, \Delta t_i) \\ &\quad \times [\ln U(\boldsymbol{x}, \boldsymbol{x}_i; \boldsymbol{\lambda}_{i+1}, \Delta t_i) + C], \end{aligned} \tag{23}$$

where we have defined the divergent integral

$$C \equiv -\int d\boldsymbol{x} \delta(\boldsymbol{x}) \ln \delta(\boldsymbol{x}). \tag{24}$$

Similarly, the difference in the entropy between the PDFs at t_i and t_{i+1} in the reverse process is calculated as

$$\begin{aligned} \Delta S_-(t_i \leftarrow t_{i+1}) &\equiv S[P_-(\boldsymbol{x}, t_i)] - S[P_-(\boldsymbol{x}, t_{i+1})] \\ &= -\int d\boldsymbol{x} U(\boldsymbol{x}, \boldsymbol{x}_{i+1}; \boldsymbol{\lambda}_{i+1}, -\Delta t_i) \\ &\quad \cdot [\ln U(\boldsymbol{x}, \boldsymbol{x}_{i+1}; \boldsymbol{\lambda}_{i+1}, -\Delta t_i) + C]. \end{aligned} \tag{25}$$

Seeing Eqs. (23) and (25), it is reasonable to define the trajectory-dependent entropy production densities in the forward and reverse processes as

$$\Delta s_+(\boldsymbol{x}', \boldsymbol{x}; \boldsymbol{\lambda}, \Delta t) \equiv -\ln U(\boldsymbol{x}', \boldsymbol{x}; \boldsymbol{\lambda}, \Delta t), \tag{26}$$

$$\Delta s_-(\boldsymbol{x}, \boldsymbol{x}'; \boldsymbol{\lambda}, -\Delta t) \equiv -\ln U(\boldsymbol{x}, \boldsymbol{x}'; \boldsymbol{\lambda}, -\Delta t), \tag{27}$$

respectively. We do not include C in these definitions since it would vanish later if included. By using these quantities, the microscopic-irreversibility indicator, Eq. (11), is written as

$$r(\boldsymbol{x}',\boldsymbol{x};\boldsymbol{\lambda},\Delta t) = -\Delta s_+(\boldsymbol{x}',\boldsymbol{x};\boldsymbol{\lambda},\Delta t) + \Delta s_-(\boldsymbol{x},\boldsymbol{x}';\boldsymbol{\lambda},-\Delta t) + \beta[E(\boldsymbol{x}',\boldsymbol{\lambda}) - E(\boldsymbol{x},\boldsymbol{\lambda})]. \tag{28}$$

Substitution of this into the total microscopic-irreversibility indicator, Eq. (12), leads to

$$R[\boldsymbol{X}_N;\boldsymbol{\Lambda}_N] = -s_+[\boldsymbol{X}_N;\boldsymbol{\Lambda}_N] + s_-[\boldsymbol{X}_N;\boldsymbol{\Lambda}_N] + \beta Q[\boldsymbol{X}_N;\boldsymbol{\Lambda}_N], \tag{29}$$

where

$$s_+[\boldsymbol{X}_N;\boldsymbol{\Lambda}_N] \equiv \sum_{i=0}^{N-2} \Delta s_+(\boldsymbol{x}_{i+1},\boldsymbol{x}_i;\boldsymbol{\lambda}_{i+1},\Delta t_i), \tag{30}$$

$$s_-[\boldsymbol{X}_N;\boldsymbol{\Lambda}_N] \equiv \sum_{i=0}^{N-2} \Delta s_-(\boldsymbol{x}_i,\boldsymbol{x}_{i+1};\boldsymbol{\lambda}_{i+1},-\Delta t_i), \tag{31}$$

$$Q[\boldsymbol{X}_N;\boldsymbol{\Lambda}_N] \equiv \sum_{i=0}^{N-2} E(\boldsymbol{x}_{i+1},\boldsymbol{\lambda}_{i+1}) - E(\boldsymbol{x}_i,\boldsymbol{\lambda}_{i+1}). \tag{32}$$

s_+ is the net entropy production density in the forward time development from t_0 to t_{N-1}, while s_- is that in the reverse time development from t_{N-1} to t_0. Q is nothing but the net heat that has flown into the system in the intervals between the measurements. We should keep in mind that s_- can be imaginary unphysically since s_- contains the reverse time development operator [see Eq. (27)]. By substituting Eq. (29) into Eq. (19), we obtain the detailed fluctuation theorem in another form,

$$\frac{\mathcal{P}[\boldsymbol{X}_N,\boldsymbol{F}_{\mathrm{m}}^N]}{\widetilde{\mathcal{P}}[\boldsymbol{X}_N,\boldsymbol{F}_{\mathrm{m}}^N]} = e^{\beta(W+Q-\Delta F)+I-s_++s_-+d_{\mathrm{in}}}. \tag{33}$$

The relation in this form looks more fascinating than Eq. (19) since the former involves explicitly the fundamental quantities in thermodynamics, the heat and the produced entropy. Although what the relation Eq. (33) in conjunction with information theory alludes for nonequilibrium thermodynamics and microscopic reversibility might be interesting, further examination is not done in the present study.

3. *Interpretation of reverse process*

Here we try to establish the interpretation of the reverse process of the forward process represented by $\mathcal{P}[\boldsymbol{X}_N,\boldsymbol{F}_{\mathrm{m}}^N]$, Eq. (16). The interpretation should be consistent with the definition of $\widetilde{\mathcal{P}}[\boldsymbol{X}_N,\boldsymbol{F}_{\mathrm{m}}^N]$, Eq. (20). Let us imagine that a man who knows only the trajectory of the control parameters $\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_N$ corresponding to $\boldsymbol{F}_{\mathrm{m}}^N$ is walking on the time axis in the negative direction. We call him the reverse observer. He comes from $t=\infty$ to reach t_{N-1} first, before the arrival at which the control parameter is $\boldsymbol{\lambda}_N$. As the first condition for the reverse process, we require that a randomly selected trajectory of $\boldsymbol{f}_0,\ldots,\boldsymbol{f}_{N-1}$ from the set of trajectories corresponding to $\widetilde{\boldsymbol{\Lambda}}_N(\boldsymbol{F}_{\mathrm{m}}^N)$ coincide with $\boldsymbol{F}_{\mathrm{m}}^N$. Such coincidence is represented by the probability $P[\boldsymbol{F}_{\mathrm{m}}^N|\widetilde{\boldsymbol{\Lambda}}_N]$. As the second condition for the reverse process, we require that the dynamical variable realized at t_{N-1} be the one in the forward process in the thermal

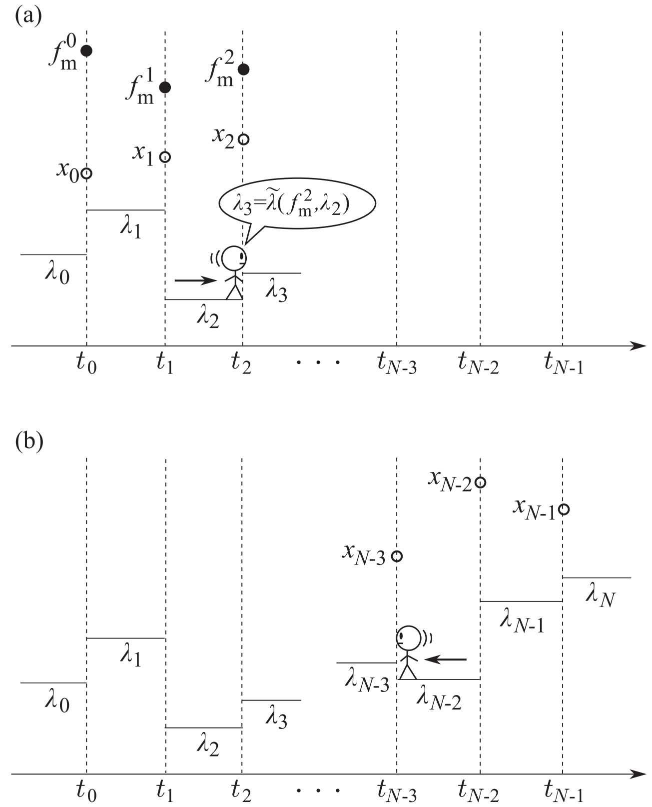

FIG. 1. Schematic illustration of a forward process under feedback control and its corresponding reverse process based on the interpretation proposed in the present study. (a) The trajectories of $\boldsymbol{x}_i$'s and $\boldsymbol{f}_{\mathrm{m}}^i$'s in the forward process. The forward observer is walking on the time axis in the positive direction. Measurements are performed at t_i's ($i=0,\ldots,N-1$) and the control parameters (shown as steps) are changed according to the measured quantities $\boldsymbol{f}_{\mathrm{m}}^i$'s (filled circles) and the protocol. The realization of $\boldsymbol{x}_i$'s (open circles) depends on the dynamics of the system subject to the thermal fluctuation. (b) The trajectory of $\boldsymbol{x}_i$'s in the reverse process corresponding to the forward process illustrated above. The reverse observer is walking on the time axis in the negative direction. The control parameters are fixed at those implied by the forward process. He does not perform any measurement. The realization of $\boldsymbol{x}_i$'s depends on the reverse dynamics of the system, defined via the relation Eq. (10).

equilibrium with $\boldsymbol{\lambda}_N$ and the series of dynamical variables $\boldsymbol{x}_{N-2},\ldots,\boldsymbol{x}_0$ be realized at $t_{N-2},\ldots,t_0$ via the reverse time development of the PDF. This condition is represented by the probability $\widetilde{P}_{\mathrm{dyn}}[\boldsymbol{X}_N|\boldsymbol{\Lambda}_{N+1}]$, as defined in Eq. (13). The product of these two probabilities is nothing but the right-hand side of Eq. (20), which implies that it is reasonable to interpret the reverse process as the situation characterized by the two conditions introduced above. The schematic illustration of the forward process and the corresponding reverse process is shown in Fig. 1. It is noted here that the reverse observer does not perform any measurement in our interpretation. This is reasonable since the probability $P[\boldsymbol{F}_{\mathrm{m}}^N|\widetilde{\boldsymbol{\Lambda}}_N]$ does not involve $\boldsymbol{x}_i$'s explicitly and hence it should not be associated with measurements in the reverse process. The disappearance of the measurements in the reverse process might seem curious; however, it is necessary for the plausible interpretation. Roughly speaking, this subtlety comes from the fact that the feedback control is causal, that is, the control parameter is changed after a measurement is carried out. If the measurements were involved in the reverse process, each control parameter seen by the reverse observer would precede its corresponding measurement [23,35]. Such an acausal situation is undesirable for a plausible interpretation. We do not go further in the present study for more satisfactory establishment of the definition and the interpretation of a reverse process.

4. Upper bound of extracted work in stationary state

Here we derive a relation for the expected work and the mutual information when the nonequilibrium stationary state is achieved. When the stationarity condition, Eq. (6), is achieved, the distribution function of the control parameter is unchanged via the feedback control. The average difference between the free energies after and before a measurement thus vanishes: $\langle \Delta F \rangle_* = 0$, where $\langle \cdot \rangle_*$ represents the average with respect to the stationary state, $P_*(\boldsymbol{x},\boldsymbol{\lambda})$. Let us consider the work fluctuation relation with a given $\boldsymbol{\lambda}_0$, Eq. (22), for a single measurement ($N = 1$), in which $\overline{R}$ vanishes. By averaging Eq. (22) over $\boldsymbol{\lambda}_0$ for the stationary state, we obtain

$$\beta \langle W \rangle_* + \langle I \rangle_* + \langle D[P_* | P_{\rm eq}] \rangle_* \geqslant 0. \tag{34}$$

This relation means that the upper bound of the expected applied work $\langle W \rangle_*$ is determined only by the expected mutual information $\langle I \rangle_*$ and the expected relative entropy $\langle D[P_*|P_{\rm eq}] \rangle_*$, even when the expected difference $\overline{\Delta F}$ between the free energies for a fixed initial control parameter is nonzero.

We define the efficiency of the perpetual extraction of work as

$$\eta_* \equiv \frac{|\beta \langle W \rangle_*|}{\langle I \rangle_* + \langle D[P_*|P_{\rm eq}] \rangle_*} \tag{35}$$

for a negative $\langle W \rangle_*$. η_* measures how efficiently the information acquired by the observer is used for the extraction of work in the stationary state. When it is smaller than unity, some of the information acquired in each measurement is not used for the perpetual extraction of work. η_* is equal to unity if and only if the equality of the relation Eq. (34) holds.

B. Expressions for systems in thermal equilibrium

For a case in which the time intervals between the consecutive measurements on a nonequilibrium system is much longer than the relaxation time of the system, we can regard that the thermal equilibrium corresponding to the control parameter being realized is achieved just before each measurement. The expressions derived above can be simplified for such a low measurement rate regime. Completely precise feedback controls are assumed for simplicity in this subsection. In this case, the initial PDF is written as

$$P_{\rm in}(\boldsymbol{x},\boldsymbol{\lambda}) = P_{\rm eq}(\boldsymbol{x}|\boldsymbol{\lambda}) P_{\rm in}(\boldsymbol{\lambda}), \tag{36}$$

where $P_{\rm in}(\boldsymbol{\lambda}) = \int d\boldsymbol{x} P_{\rm in}(\boldsymbol{x},\boldsymbol{\lambda})$ is the initial distribution of $\boldsymbol{\lambda}$. Remember that the system does not undergo any change in the control parameter between two consecutive measurements. The expression of the distribution of $\boldsymbol{\lambda}$ after a measurement is thus calculated by inserting Eq. (36) into Eq. (1) and integrating both sides with respect to $\boldsymbol{x}$ as

$$P_{\rm out}^{\Delta t}(\boldsymbol{\lambda}) = P_{\rm out}(\boldsymbol{\lambda}) = \int d\boldsymbol{\lambda}' G(\boldsymbol{\lambda},\boldsymbol{\lambda}') P_{\rm in}(\boldsymbol{\lambda}'), \tag{37}$$

where

$$G(\boldsymbol{\lambda},\boldsymbol{\lambda}') \equiv \int d\boldsymbol{x} G(\boldsymbol{\lambda},\boldsymbol{\lambda}';\boldsymbol{x}) P_{\rm eq}(\boldsymbol{x}|\boldsymbol{\lambda}') \tag{38}$$

is the reduced propagator. The expected work applied to the system per measurement is calculated by substituting Eq. (36) into Eq. (3). Equation (37) indicates that the stationary distribution for a thermal equilibrium system is obtained by considering only the phase space of the control parameter. The stationarity condition for the repeated measurements on a system in thermal equilibrium is thus written as

$$\boldsymbol{P}_* = G \boldsymbol{P}_*. \tag{39}$$

It is noted here that the condition for a nonequilibrium case, Eq. (6), is for the PDF of $\boldsymbol{x}$ and $\boldsymbol{\lambda}$, while that for an equilibrium case, Eq. (39), is for the PDF only of $\boldsymbol{\lambda}$. The PDF involving $\boldsymbol{x}$, Eq. (36), in the latter case thus is given by $P_*(\boldsymbol{x},\boldsymbol{\lambda}) = P_{\rm eq}(\boldsymbol{x}|\boldsymbol{\lambda}) P_*(\boldsymbol{\lambda})$.

The detailed fluctuation theorem in this case can be derived without introducing the microscopic-irreversibility indicator, Eq. (11). The generalized Jarzynski's equality with a given initial control parameter $\boldsymbol{\lambda}_0$ for a nonequilibrium system, Eq. (21), is written in this case as

$$\overline{e^{-\beta(W - \Delta F) - I}} = 1. \tag{40}$$

Due to the absence of the reverse time development operator in this case, the relative-entropy work relation for the nonequilibrium system, Eq. (22), is written in this case as

$$\beta(\overline{W} - \overline{\Delta F}) + \overline{I} \geqslant 0. \tag{41}$$

The relative-entropy work relation for the nonequilibrium stationary state, Eq. (34), is written in this case as

$$\beta \langle W \rangle_* + \langle I \rangle_* \geqslant 0. \tag{42}$$

III. EXAMPLES

As simple applications of the theory formulated above, let us analyze two exactly solvable models. Completely precise feedback controls are assumed for simplicity in what follows.

A. Nonequilibrium two-state system

1. Setup and dynamics

As an example of the perpetual extraction of work from a nonequilibrium system, here we consider a two-state system in which the state variable x takes 0 or 1. The control parameter λ takes 0 or 1 as well. The energy of the system is $\varepsilon > 0$ when $x = 1$ and $\lambda = 1$ and otherwise zero: $E(x,\lambda) = \varepsilon \delta_{x1} \delta_{\lambda 1}$. We assume the following situation. When the observer performs a measurement of x on the system with $\lambda = 0$, he obtains the accurate result at a probability $1/2$, which means that the outcome of the measurement and the realized x are uncorrelated. For the system with $\lambda = 1$, on the other hand, an accurate result at a probability p is obtained. This situation is

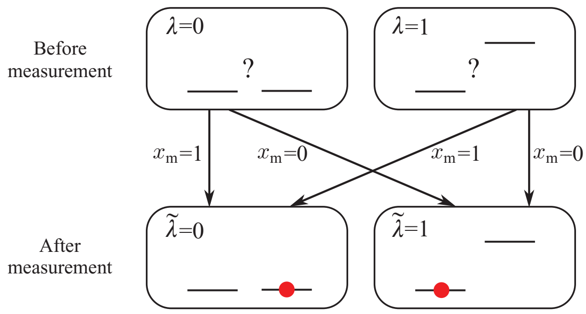

FIG. 2. (Color online) Schematic illustration of a measurement and the feedback control on the two-state system. The observer sets the control parameter λ to 0 if the measured state $x_{\rm m}$ is 1, while λ is set to 1 if the measured state is 0.

described by the conditional probability

$$p_{\rm m}(x_{\rm m}|x,\lambda) = \frac{\delta_{\lambda 0}}{2} + \delta_{\lambda 1}[p\delta_{x_{\rm m}x} + (1-p)(1-\delta_{x_{\rm m}x})], \quad (43)$$

where $\bar{p} \equiv 1 - p$. For extracting work, λ is set to 0 if the measured state is 1, while λ is set to 1 if the measured state is 0, as illustrated in Fig. 2. This feedback protocol is described by $\widetilde{\lambda}(x_{\rm m}) = \delta_{x_{\rm m}0}$. The propagator of the control parameter, Eq. (2), is calculated as

$$G(\lambda,\lambda';x) = \delta_{\lambda 1}\, p_{\rm m}(0|x,\lambda') + \delta_{\lambda 0}\, p_{\rm m}(1|x,\lambda'). \quad (44)$$

The measurement-averaged energy, Eq. (4), is thus calculated as $\widetilde{E}(x,\lambda) = \varepsilon\delta_{x1}[\delta_{\lambda 0}/2 + \delta_{\lambda 1}(1-p)]$. We assume that the PDF for a fixed λ evolves with the relaxation time τ as time develops to achieve the canonical distribution

$$P_{\rm eq}(x|\lambda) = \frac{\alpha^{\lambda x}}{1+\alpha^{\lambda}}, \quad (45)$$

where $\alpha \equiv e^{-\beta\varepsilon}$ is the Boltzmann factor. The time evolution equation of the PDF between two consecutive measurements is thus governed by the following differential equation:

$$\frac{\partial P(x,\lambda,t)}{\partial t} = \frac{1}{\tau}[P_{\rm eq}(x|\lambda)P(\lambda) - P(x,\lambda,t)], \quad (46)$$

where $P(\lambda) \equiv \sum_x P(x,\lambda,t)$ does not depend on t since no measurement is performed in the interval concerned. The time-dependent PDF for an interval without measurements is obtained by integrating Eq. (46) as

$$P_{\rm out}(x,\lambda,t) = P_{\rm eq}(x|\lambda)P_{\rm out}(\lambda) + [P_{\rm out}(x,\lambda,0) - P_{\rm eq}(x|\lambda)P_{\rm out}(\lambda)]e^{-t/\tau}. \quad (47)$$

The origin of time has been set to the last measurement. $P_{\rm out}(\lambda)$ is the distribution of the control parameter after the last measurement. The forward time development operator $U(x,x';\lambda,\Delta t)$ in matrix representation thus is written as

$$U(x,x';0,\Delta t) = \begin{pmatrix} (1+\gamma)/2 & (1-\gamma)/2 \\ (1-\gamma)/2 & (1+\gamma)/2 \end{pmatrix}, \quad (48)$$

$$U(x,x';1,\Delta t) = \frac{1}{1+\alpha}\begin{pmatrix} 1+\alpha\gamma & 1-\gamma \\ \alpha(1-\gamma) & \alpha+\gamma \end{pmatrix}. \quad (49)$$

Using Eqs. (44), (48), and (49), the PDF just before the next measurement is calculated from that just before the last measurement as

$$\begin{pmatrix} P_{\rm out}^{\Delta t}(0,0) \\ P_{\rm out}^{\Delta t}(0,1) \\ P_{\rm out}^{\Delta t}(1,0) \\ P_{\rm out}^{\Delta t}(1,1) \end{pmatrix} = \begin{pmatrix} \frac{1+\gamma}{4} & \frac{(1+\gamma)(1-p)}{2} & \frac{1-\gamma}{4} & \frac{(1-\gamma)p}{2} \\ \frac{1+\alpha\gamma}{2(1+\alpha)} & \frac{(1+\alpha\gamma)p}{1+\alpha} & \frac{1-\gamma}{2(1+\alpha)} & \frac{(1-\gamma)(1-p)}{1+\alpha} \\ \frac{1-\gamma}{4} & \frac{(1-\gamma)(1-p)}{2} & \frac{1+\gamma}{4} & \frac{(1+\gamma)p}{2} \\ \frac{\alpha(1-\gamma)}{2(1+\alpha)} & \frac{\alpha(1-\gamma)p}{1+\alpha} & \frac{\alpha+\gamma}{2(1+\alpha)} & \frac{(\alpha+\gamma)(1-p)}{1+\alpha} \end{pmatrix} \times \begin{pmatrix} P_{\rm in}(0,0) \\ P_{\rm in}(0,1) \\ P_{\rm in}(1,0) \\ P_{\rm in}(1,1) \end{pmatrix}, \quad (50)$$

where $\gamma \equiv e^{-\Delta t/\tau}$.

The reverse time development operator is determined by the unitarity condition, Eq. (10), as

$$U(x,x';0,-\Delta t) = \frac{1}{2\gamma}\begin{pmatrix} 1+\gamma & -1+\gamma \\ -1+\gamma & 1+\gamma \end{pmatrix}, \quad (51)$$

$$U(x,x';1,-\Delta t) = \frac{1}{\gamma(1+\alpha)}\begin{pmatrix} \alpha+\gamma & -1+\gamma \\ \alpha(-1+\gamma) & 1+\alpha\gamma \end{pmatrix}. \quad (52)$$

The microscopic-irreversibility indicator $r(x',x;\lambda,\Delta t)$, defined in Eq. (11), is calculated as

$$r(x',x;0,\Delta t) = \begin{cases} \ln\gamma & (x = x') \\ \ln(-\gamma) = \ln\gamma + i\pi & (x \neq x') \end{cases}, \quad (53)$$

$$r(x',x;1,\Delta t) = \begin{cases} \ln\frac{\gamma(1+\alpha\gamma)}{\alpha+\gamma} & (x = x' = 0) \\ \ln\frac{\gamma(\alpha+\gamma)}{1+\alpha\gamma} & (x = x' = 1) \\ \ln\gamma + i\pi & (x \neq x') \end{cases}. \quad (54)$$

r can be complex, which comes from the negative elements in the reverse time development operator [see Eqs. (51) and (52)]. This situation is physically bizarre; however, it is mathematically correct for the generalized Jarzynski's equality to hold.

2. *Confirmation of the generalized Jarzynski's equality*

Before proceeding to the analysis of perpetual extraction of work, let us confirm the validity of the our version of the generalized Jarzynski's equality, Eq. (21), by performing numerical simulations. For the nonequilibrium two-state system with the parameters $p = 0.8$, $\alpha = 0.7$, and $\gamma = 0.5$, we generated 10^7 trajectories of x and $x_{\rm m}$ for each of $N = 2, 3,$ and 4 using random numbers. The typical distributions of $-\beta(W - \Delta F) - I - R - d_{\rm in}$ on the complex plane and the histograms of $\exp[-\beta(W - \Delta F) - I - R - d_{\rm in}]$ occurring in the feedback processes each of which includes N measurements are shown in Fig. 3. The initial control parameter $\lambda_0 = 0$ is used in all the cases. The initial PDF deviating from the canonical distribution function is prepared as $P_{\rm in}(x = 0|\lambda_0) = 0.8$ and $P_{\rm in}(x = 1|\lambda_0) = 0.2$ so nonzero $d_{\rm in}$ appears. The numbers of different complex values of $-\beta(W - \Delta F) - I - R - d_{\rm in}$ for $N = 2, 3,$ and 4 were found to be 16, 60, and 221, respectively. We found that the number of trajectories is large enough for all the N's, so the averages of $\exp[-\beta(W - \Delta F) - I - R - d_{\rm in}]$ were confirmed to converge to unity regardless of N.

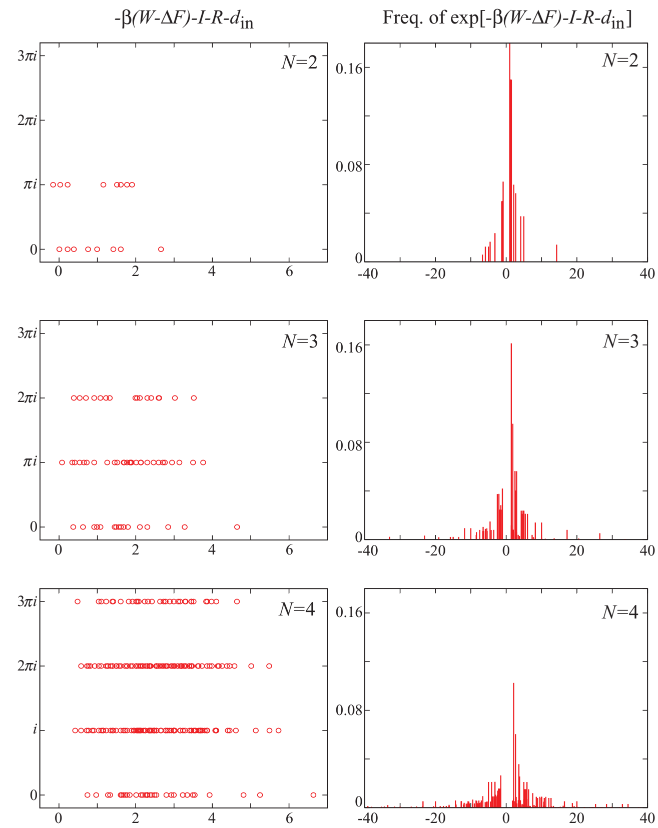

FIG. 3. (Color online) For the nonequilibrium two-state system with the parameters $p = 0.8$, $\alpha = 0.7$, and $\gamma = 0.5$, the typical distributions of $-\beta(W - \Delta F) - I - R - d_{\text{in}}$ on the complex plane (left panels) and the histograms of $\exp[-\beta(W - \Delta F) - I - R - d_{\text{in}}]$ (right panels) occurring in the feedback processes each of which includes N measurements are shown for $N = 2$, 3, and 4. The initial control parameter $\lambda_0 = 0$ is used in all the cases. The initial PDF deviating from the canonical distribution function is prepared as $P_{\text{in}}(x = 0|\lambda_0) = 0.8$ and $P_{\text{in}}(x = 1|\lambda_0) = 0.2$.

3. Confirmation of relative-entropy work relation

Let us consider a case for a single measurement ($N = 1$). Using Eqs. (43) and (45) and the relation $p(x,x_{\text{m}}|\lambda) = p_{\text{m}}(x_{\text{m}}|x,\lambda)P_{\text{in}}(x|\lambda)$, we obtain the joint probability distribution at the time of the measurement as

$$p(x,x_{\text{m}}|\lambda_0) = \begin{cases} \frac{1}{2}P_{\text{in}}(x|0) & (\lambda_0 = 0) \\ [p\delta_{x_{\text{m}}x} + \bar{p}(1 - \delta_{x_{\text{m}}x})]P_{in}(x|1) & (\lambda_0 = 1) \end{cases}. \tag{55}$$

The marginalized distribution $p(x_{\text{m}}|\lambda_0) = \sum_x p(x,x_{\text{m}}|\lambda_0)$ is thus calculated as

$$p(x_{\text{m}}|0) = \tfrac{1}{2}, \quad p(0|1) = pq + \bar{p}\bar{q}, \quad p(1|1) = \bar{p}q + p\bar{q}, \tag{56}$$

where $q \equiv P_{\text{in}}(0|\lambda_0), \bar{q} \equiv 1 - q = P_{\text{in}}(1|\lambda_0)$. For fixed trajectories of x and x_{m} and the initial control parameter λ_0, the work applied to the system via the measurement is given by

$$W[x,x_{\text{m}}|\lambda_0] = E(x,\widetilde{\lambda}(x_{\text{m}})) - E(x,\lambda_0) = \varepsilon\delta_{x1}(\delta_{x_{\text{m}}0} - \delta_{\lambda_0 1}). \tag{57}$$

The applied work with the fixed λ_0 averaged over the trajectories of x and x_{m} is thus given by

$$\overline{W} = \sum_{x,x_{\text{m}}} p(x,x_{\text{m}}|\lambda_0)W[x,x_{\text{m}}|\lambda_0] = \begin{cases} \varepsilon\bar{q}/2 & (\lambda_0 = 0) \\ -\varepsilon p\bar{q} & (\lambda_0 = 1) \end{cases}. \tag{58}$$

The free energy of the system with a control parameter λ is given by

$$F(\lambda) = \begin{cases} -\frac{1}{\beta}\ln 2 & (\lambda = 0) \\ -\frac{1}{\beta}\ln(1+\alpha) & (\lambda = 1) \end{cases}. \tag{59}$$

The difference between the free energies after and before the measurement with the fixed λ_0 averaged over the trajectories of x and x_{m} is thus given by

$$\begin{aligned} \overline{\Delta F} &= \sum_{x,x_{\text{m}}} p(x,x_{\text{m}}|\lambda_0)[F(\widetilde{\lambda}(x_{\text{m}})) - F(\lambda_0)] \\ &= \begin{cases} \frac{1}{2\beta}\ln\frac{2}{1+\alpha} & (\lambda_0 = 0) \\ -[\bar{p}q + p\bar{q}]\frac{1}{\beta}\ln\frac{2}{1+\alpha} & (\lambda_0 = 1) \end{cases}. \end{aligned} \tag{60}$$

The mutual information density for the initial control parameter λ_0 is calculated from Eq. (18) for $N = 1$ as

$$\begin{aligned} &I[x,x_{\text{m}}|\lambda_0] \\ &= \ln\frac{p(x,x_{\text{m}}|\lambda_0)}{P_{\text{in}}(x|\lambda_0)p(x_{\text{m}}|\lambda_0)} \\ &= \begin{cases} 0 & (\lambda_0 = 0) \\ \ln[(p\delta_{x0} + \bar{p}\delta_{x1})/(pq + \bar{p}\bar{q})] & (\lambda_0 = 1, x_{\text{m}} = 0) \\ \ln[(p\delta_{x1} + \bar{p}\delta_{x0})/(\bar{p}q + p\bar{q})] & (\lambda_0 = 1, x_{\text{m}} = 1) \end{cases}. \end{aligned} \tag{61}$$

The mutual information averaged over the trajectory of x and x_{m} with the fixed λ_0 is given by

$$\begin{aligned} \overline{I} &= \sum_{x,x_{\text{m}}} p(x,x_{\text{m}}|\lambda_0)I[x,x_{\text{m}}|\lambda_0] \\ &= \begin{cases} 0 & (\lambda_0 = 0) \\ p\ln p + \bar{p}\ln\bar{p} & \\ -(pq + \bar{p}\bar{q})\ln(pq + \bar{p}\bar{q}) & \\ -(\bar{p}q + p\bar{q})\ln(\bar{p}q + p\bar{q}) & (\lambda_0 = 1) \end{cases}. \end{aligned} \tag{62}$$

When $\lambda_0 = 1$, $\overline{W}$ and $\overline{\Delta F}$ are nonpositive for arbitrary ε and p and $\overline{I}$ vanishes for $p = 1/2$ regardless of ε.

The relative entropy of the initial PDF with respect to the canonical distribution function is calculated as

$$\begin{aligned} D[P_{\text{in}}|P_{\text{eq}}] &= \sum_x P_{\text{in}}(x|\lambda_0)\ln\frac{P_{\text{in}}(x|\lambda_0)}{P_{\text{eq}}(x|\lambda_0)} \\ &= \ln(1 + \alpha^{\lambda_0}) + q\ln q + \bar{q}\ln\frac{\bar{q}}{\alpha^{\lambda_0}}. \end{aligned} \tag{63}$$

The quantities calculated above satisfy the relation Eq. (22) of non-negative relative entropy, $\beta(\overline{W} - \overline{\Delta F}) + \overline{I} + D[P_{\text{in}}|P_{\text{eq}}] \geqslant 0$ for an arbitrary λ_0. In Fig. 4, $\beta\overline{W}$, $\beta\overline{\Delta F}$, $\overline{I}$, $D[P_{\text{in}}|P_{\text{eq}}]$, and $\beta(\overline{W} - \overline{\Delta F}) + \overline{I} + D[P_{\text{in}}|P_{\text{eq}}]$ for $\lambda_0 = 1$ are plotted for $\alpha = 0.8$, 0.4, and 0.1 as functions of p. For the lowermost panel, q is set to 0.5 so $D[P_{\text{in}}|P_{\text{eq}}]$ is nonzero. For the other three panels, q's are set to $1/(1+\alpha)$, corresponding

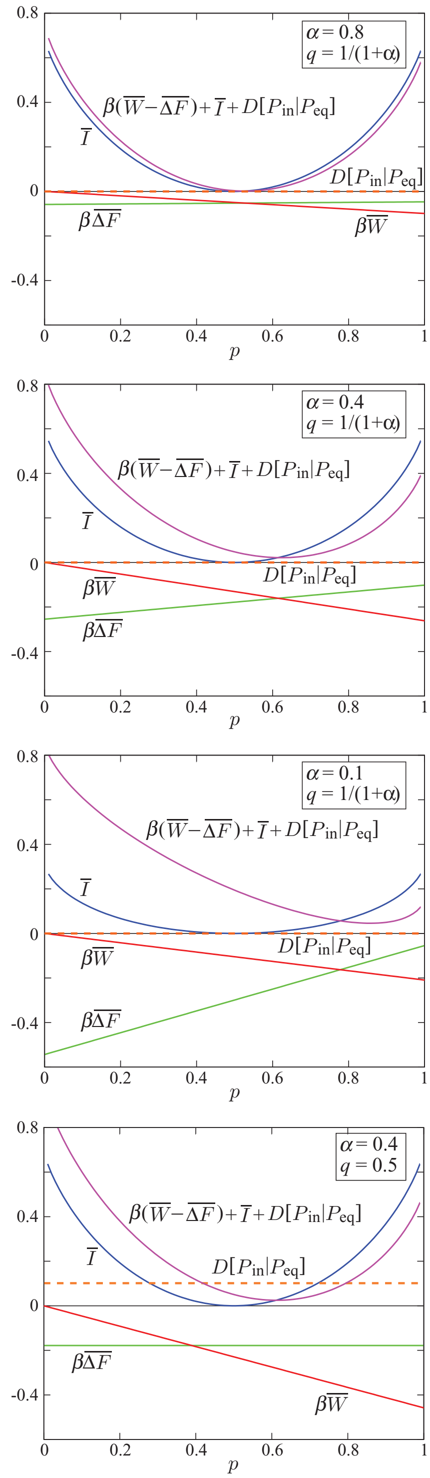

FIG. 4. (Color online) For the nonequilibrium two-state system with $\lambda_0 = 1$, the expected work $\beta\overline{W}$ applied to the system via a single measurement, the expected change in the free energy $\beta\overline{\Delta F}$, the expected mutual information $\overline{I}$, the relative entropy of the initial PDF $D[P_{\rm in}|P_{\rm eq}]$, and $\beta(\overline{W} - \overline{\Delta F}) + \overline{I} + D[P_{\rm in}|P_{\rm eq}]$ are plotted for $\alpha = 0.8$, 0.4, and 0.1 as functions of p. For the lowermost panel, q is set to 0.5 so $D[P_{\rm in}|P_{\rm eq}]$ is nonzero. For the other three panels, q is set to the values corresponding to the canonical distribution functions.

to the canonical distribution functions so $D[P_{\rm in}|P_{\rm eq}]$ vanishes. It is seen that the inequality $\beta(\overline{W} - \overline{\Delta F}) + \overline{I} + D[P_{\rm in}|P_{\rm eq}] \geqslant 0$ correctly holds. It should be noticed that the relative entropy of the initial PDF must be taken into account for the inequality to hold.

4. Stationary state

From the stationarity condition, Eq. (6), and the normalization condition, the stationary distribution $P_*(x,\lambda)$ is uniquely given by

$$P_*(0,0) = N(p)[-(1+\alpha)\gamma p^2 + (-1+\alpha+\gamma)p + 1], \quad (64)$$

$$P_*(0,1) = N(p)\frac{(1+\alpha)\gamma p + 2 - \gamma}{2}, \quad (65)$$

$$P_*(1,0) = N(p)[-(1+\alpha)\gamma p^2 + [-1+\alpha+(2+\alpha)\gamma]p + 1 - \gamma], \quad (66)$$

$$P_*(1,1) = N(p)\frac{-(1+\alpha)\gamma p + 2\alpha + \gamma}{2}, \quad (67)$$

where $N(p) \equiv [-2(1+\alpha)\gamma p^2 + \{-2 + 2\alpha + (3+\alpha)\gamma\}p + 3 + \alpha - \gamma]^{-1}$ is positive within the range $0 \leqslant p \leqslant 1$. We obtain the expected work for the stationary distribution, from Eq. (3),

$$\langle W\rangle_* = \varepsilon\frac{N(p)}{2}(1-\gamma)[1-(1+\alpha)p]. \quad (68)$$

$\langle W\rangle_*$ is a monotonic decreasing function of p. To see the typical behavior of the nonequilibrium PDF, we plotted the time development of the PDF with $p = 0.8$ for $\alpha = 0.6$ and $\gamma = 0.4$ and that for $\alpha = 0.7$ and $\gamma = 0.5$ in Figs. 5(a) and 5(b), respectively. The expected work $\langle W\rangle$ via each measurement is also plotted. The canonical distribution, Eq. (45), was adopted as their initial PDFs. In each of the two cases, the PDF approaches rapidly the stationary distribution via the measurements. It has almost reached the stationary distribution just after the sixth measurement ($t/\Delta t = 5$) and thus $\langle W\rangle$ is almost equal to $\langle W\rangle_*$ for both $\lambda_0 = 0$ and 1. We can draw two lessons from Figs. 5(a) and 5(b) as follows: (i) the observer, on the way to the stationary distribution, can fail to extract work, that is, he or she can lose energy, and (ii) the observer can extract work larger than that in the stationary distribution.

$\langle W\rangle_*$ as functions of p for various combinations of α and γ are plotted in Fig. 6(a). For the extraction of work, the condition $p > (1+\alpha)^{-1}$ must be satisfied. This result indicates that a smaller probability of the realization of the higher-energy state (a smaller α) requires a higher precision of the measurement (a larger p) for the extraction of work. For completely precise measurements ($p = 1$), the cost-incorporated expected work per unit time, Eq. (7), is given by

$$\langle w\rangle_* = \varepsilon\left[\frac{1-\gamma}{2(\gamma - 3 - \alpha^{-1})} + \frac{c}{\varepsilon}\right]\frac{1}{\Delta t}, \quad (69)$$

plotted in Fig. 6(b) as functions of Δt for various combinations of α and c. It is observed that a larger α attains a lower $\langle w\rangle_*$ when c is fixed, which is reasonable, since a larger α means a more frequent transition to the higher-energy state. It is also interesting to see that a too-large cost can prevent the observer from extracting the work regardless of Δt, despite the completely precise measurement. Figure 6(c) shows the numerically found optimal time interval $\Delta t_{\rm opt}$

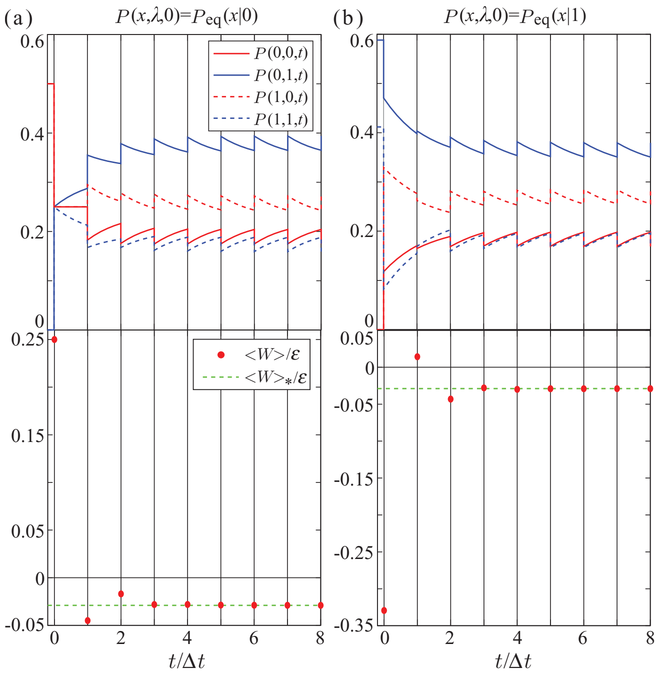

FIG. 5. (Color online) [(a) and (b)] Time development of the nonequilibrium probability distribution functions with $p = 0.8$ (upper panels) and the expected work $\langle W \rangle$ applied to the system at each measurement (lower panels). (a) For $\alpha = 0.6$ and $\gamma = 0.4$ starting from $P_{\rm eq}(x|0)$ and (b) for $\alpha = 0.7$ and $\gamma = 0.5$, starting from $P_{\rm eq}(x|1)$. The origins of time are set to the respective first measurements. The red (brighter) and the blue (darker) curves correspond to $\lambda = 0$ and 1, respectively, in the upper panels. The dashed lines in the lower panels represent the expected work in the respective stationary distributions.

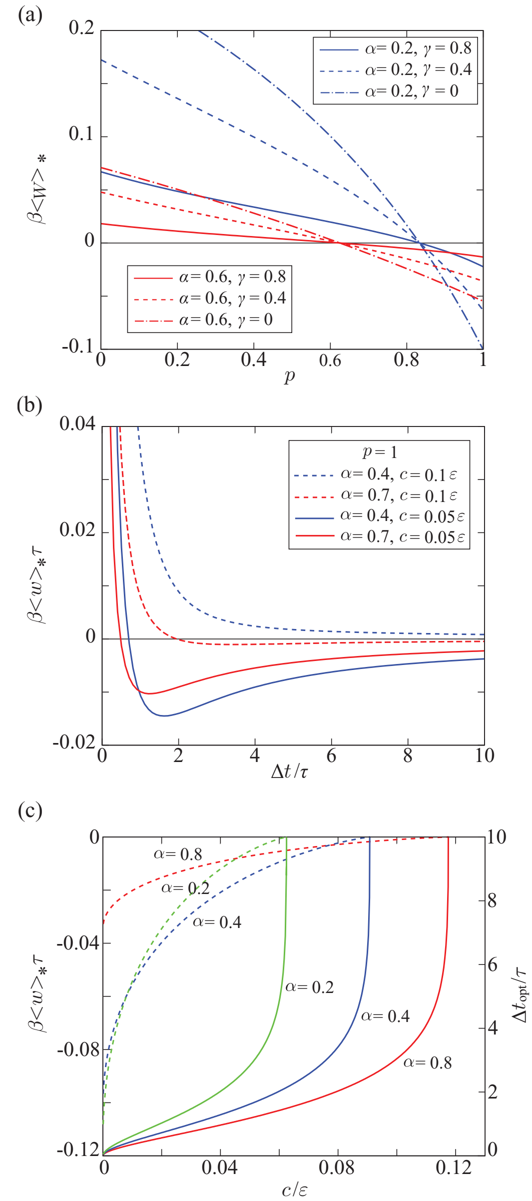

FIG. 6. (Color online) (a) Expected work $\langle W \rangle_*$ applied to the system in the stationary state as functions of p for various combinations of α and γ. The red (brighter) curves correspond to $\alpha = 0.6$, while the blue (darker) ones to $\alpha = 0.2$. (b) Cost-incorporated expected work $\langle w \rangle_*$ per unit time applied to the system for $p = 1$ as functions of Δt for various combinations of α and c. The red (brighter) curves correspond to $\alpha = 0.7$, while the blue (darker) ones to $\alpha = 0.4$. (c) Optimal time interval $\Delta t_{\rm opt}$ (solid curves) between measurements and the corresponding largest negative $\langle w \rangle_*$ (dashed curves) as functions of the cost for $p = 1$ and $\alpha = 0.2$, 0.4, and 0.8.

between measurements and the corresponding largest negative $\langle w \rangle_*$ as functions of the cost for $p = 1$ and $\alpha = 0.2$, 0.4, and 0.8. It is found that there exists for each α a critical cost above which negative $\langle w \rangle_*$ is not attained for any Δt. It is also found that a smaller cost enables the observer to extract a larger work for a fixed α, as expected.

Let us examine the relative-entropy work relation for the stationary state. In Fig. 7, $\beta\langle W \rangle_*$, $\langle I \rangle_*$, $\langle D[P_*|P_{\rm eq}] \rangle_*$, and $\beta\langle W \rangle_* + \langle I \rangle_* + \langle D[P_*|P_{\rm eq}] \rangle_*$ are plotted for various combinations of α and γ as functions of p. It is seen that the inequality Eq. (34) correctly holds. Figure 7 also shows the efficiency η_* of the perpetual extraction of work, defined in Eq. (35). The plotted efficiency has its maximum at a p in the allowed range for $\langle W \rangle_* < 0$, indicating that a larger amount of extracted work does not necessarily mean more efficient use of the information in the present cases. There are two criteria for the optimal value of p, depending on whether $|\langle W \rangle_*|$ or η_* should be maximized. If we want to maximize the former, 1 is the optimal value of p. If we want to maximize the latter, however, the optimal value is smaller than 1.

B. Particle confined to harmonic potential in thermal equilibrium

1. Setup

As an example for the extraction of work from a system for which the time interval between measurements is much longer than the relaxation time of the system, here we consider a one-dimensional system in which a particle is confined to a harmonic potential. We can regard in this case the thermal equilibrium is achieved just before each measurement and use the reduced expressions derived in Sec. II B. We neglect its kinetic energy and thus the energy of the system with the particle at x is given by $E(x,\lambda) = \frac{k}{2}(x-\lambda)^2$, where k is the stiffness of the harmonic potential and λ is the center of the potential as the control parameter. The observer measures

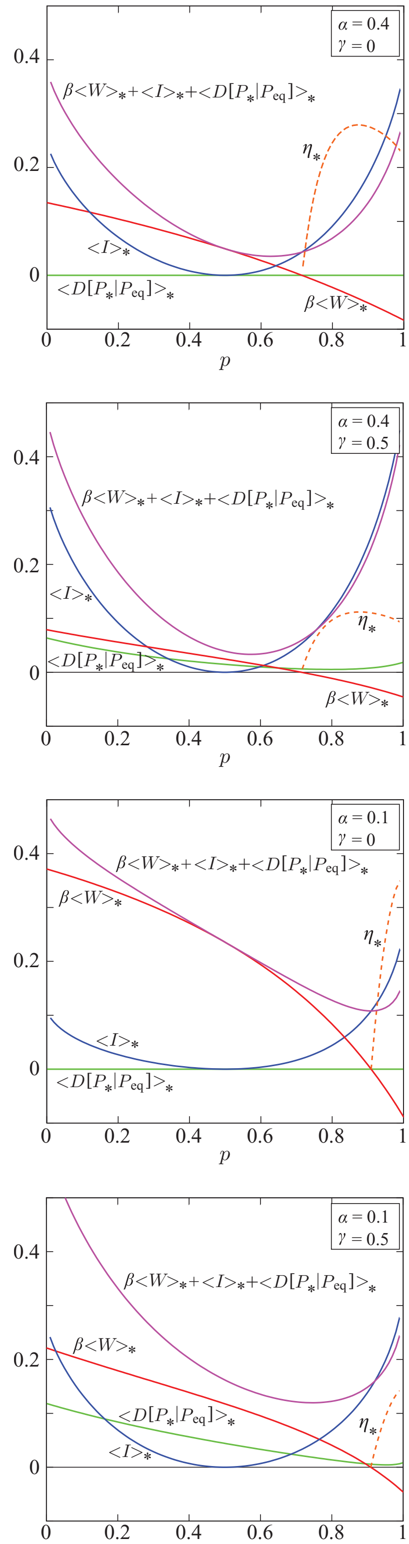

FIG. 7. (Color online) For the nonequilibrium two-state system in the stationary state with various combinations of α and γ, the expected work $\beta\langle W\rangle_*$ applied to the system via a single measurement, the expected mutual information associated with each measurement $\langle I\rangle_*$, the expected relative entropy of the input PDF for each measurement $\langle D[P_*|P_{\rm eq}]\rangle_*$, and $\beta\langle W\rangle_* + \langle I\rangle_* + \langle D[P_*|P_{\rm eq}]\rangle_*$ are plotted as functions of p. The efficiency η_* of the extraction of work is also plotted as dashed curves.

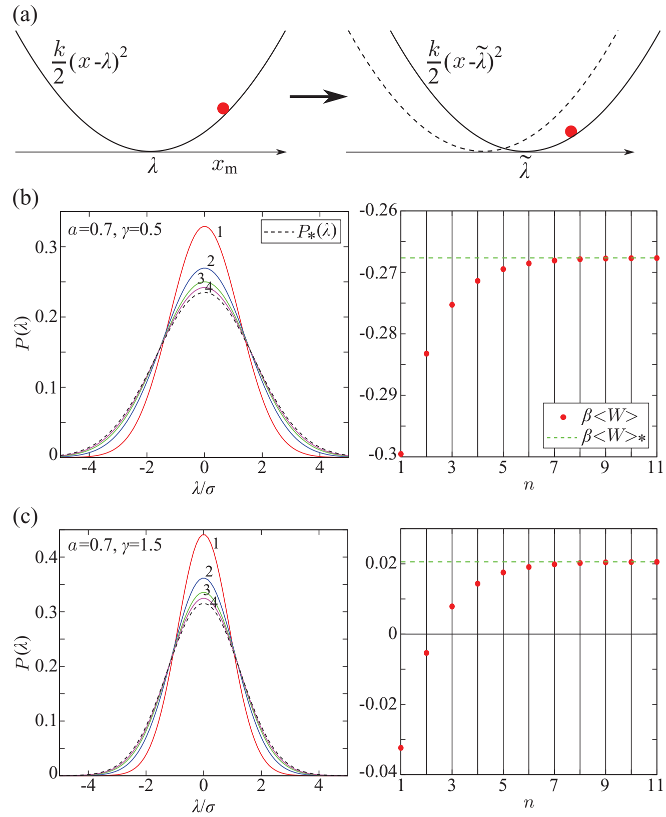

FIG. 8. (Color online) (a) Schematic illustration of a measurement of the particle's position confined to a harmonic potential and the feedback control. Given a measured position $x_{\rm m}$, the observer changes instantaneously the center of the potential from λ to $\tilde{\lambda} = ax_{\rm m}$. (b) Distribution function of λ (left panel) and the expected work $\langle W\rangle$ applied to the system (right panel) via the nth measurement for $a = 0.7$ and $\nu = 0.5$. Left: n is indicated close to the solid curve, while the distribution in the stationary distribution is represented by the dashed curve. Panels of (c) are for $a = 0.7$ and $\nu = 1.5$. For both (b) and (c), the initial distributions were set to $P_{\rm in}(\lambda) = \delta(\lambda)$.

the particle's position $x_{\rm m}$ and changes instantaneously the center of the potential according to the protocol

$$\tilde{\lambda}(x_{\rm m}) = ax_{\rm m} \tag{70}$$

with $a > 0$, as illustrated in Fig. 8(a). We assume that the conditional probability of the measured particle's position is represented by a Gaussian function of width σ [see Eq. (A1)] $p_{\rm m}(x_{\rm m}|x,\lambda) = \mathcal{N}(x_{\rm m};x,\sigma)$ independent of λ. A smaller Gaussian width σ corresponds to a more precise measurement [14]. The canonical distribution for a given λ is $P_{\rm eq}(x|\lambda) = \mathcal{N}(x;\lambda,1/\sqrt{\beta k})$.

2. *Confirmation of relative-entropy work relation*

The joint probability of x and $x_{\rm m}$ for a given λ_0 is t calculated by using the product formula Eq. (A2) as

$$\begin{aligned} p(x,x_{\rm m}|\lambda_0) &= p_{\rm m}(x_{\rm m}|x,\lambda_0)P_{\rm eq}(x|\lambda) \\ &= \mathcal{N}(x_{\rm m};\lambda_0,1/\sqrt{2\zeta})\mathcal{N}(x;B,C), \end{aligned} \tag{71}$$

where $\nu \equiv \beta k \sigma^2$,

$$\zeta \equiv \frac{\beta k}{2(1+\nu)}, \tag{72}$$

$$B \equiv \frac{x_\mathrm{m} + \lambda_0 \nu}{1+\nu}, \quad C \equiv \frac{\sigma}{\sqrt{1+\nu}}. \tag{73}$$

The probability distribution of x_m for the λ_0 is thus given by

$$p(x_\mathrm{m}|\lambda_0) = \int dx\, p(x,x_\mathrm{m}|\lambda_0) = \mathcal{N}(x_\mathrm{m};\lambda_0,1/\sqrt{2\zeta}). \tag{74}$$

For fixed trajectories of x and x_m and the initial control parameter λ_0, the work applied to the system via the measurement is given by $W[x,x_\mathrm{m}|\lambda_0] = \frac{k}{2}[(x-ax_\mathrm{m})^2 - (x-\lambda_0)^2]$. The applied work with the fixed λ_0 averaged over the trajectory of x and x_m is thus given by

$$\begin{aligned}\beta\overline{W} &= \int dx dx_\mathrm{m}\, p(x,x_\mathrm{m}|\lambda_0) W[x,x_\mathrm{m}|\lambda_0] \\ &= \frac{1}{2}[(1-a)^2\{\nu(\lambda_0/\sigma)^2+1\} + a^2\nu - 1],\end{aligned} \tag{75}$$

where we have used the integral formula Eq. (A6).

The mutual information density for the initial control parameter λ_0 is calculated from Eq. (18) for $N=1$ as

$$\begin{aligned}I[x,x_\mathrm{m}|\lambda_0] &= \ln\frac{p(x,x_\mathrm{m}|\lambda_0)}{P_\mathrm{eq}(x|\lambda_0)p(x_\mathrm{m}|\lambda_0)} = \ln\frac{\mathcal{N}(x;B,C)}{\mathcal{N}(x;\lambda_0,1/\sqrt{\beta k})} \\ &= \frac{1}{2}\ln\left(1+\frac{1}{\nu}\right) - \frac{(x-B)^2}{2C^2} + \frac{\beta k}{2}(x-\lambda_0)^2.\end{aligned} \tag{76}$$

The mutual information averaged over the trajectory of x and x_m with the fixed λ_0 is obtained by using the integral formula Eq. (A6) as

$$\overline{I} = \int dx dx_\mathrm{m}\, p(x,x_\mathrm{m}|\lambda_0) I[x,x_\mathrm{m}|\lambda_0] = \frac{1}{2}\ln\left(1+\frac{1}{\nu}\right), \tag{77}$$

independent of λ_0.

From Eqs. (75) and (77), we obtain

$$\beta\overline{W} + \overline{I} \geqslant -\frac{1}{2(1+\nu)} + \frac{1}{2}\ln\left(1+\frac{1}{\nu}\right) > 0 \tag{78}$$

for a finite ν and the fixed λ_0. The first inequality is obtained by considering the case with $\lambda_0/\sigma = 0$ and $a = 1/(1+\nu)$. Since the stiffness of the harmonic potential is unchanged via the feedback control, the difference between the free energies ΔF before and after the measurement vanishes. Equation (78) hence confirms the relative-entropy work relation, Eq. (41). Abreu and Seifert [14] have already derived a relation similar to Eq. (78). Their result corresponds to the special case of Eq. (78) with $\lambda_0 = 0$.

3. Stationary state

Let us proceed to the analysis of the stationary state. Since the propagator of the control parameter, Eq. (2), is given by $G(\lambda,\lambda';x) = p_\mathrm{m}(\lambda/a|x,\lambda')/a$, the reduced propagator, Eq. (38), is calculated as

$$G(\lambda,\lambda') = \frac{1}{a}\sqrt{\frac{\zeta}{\pi}}\exp\left[-\zeta\left(\frac{\lambda}{a}-\lambda'\right)^2\right]. \tag{79}$$

We have used the formula Eq. (A2). The measurement-averaged energy, Eq. (4), is calculated as $\widetilde{E}(x,\lambda) = (k/2)[(1-a)^2x^2 + a^2\sigma^2]$. From Eqs. (37) and (79), the stationary distribution of the control parameter is obtained easily as

$$P_*(\lambda) = \mathcal{N}(\lambda;0,1/\sqrt{2\zeta(a^{-2}-1)}), \tag{80}$$

whose normalization condition requires $a<1$. We obtain the expected work for the stationary distribution, from Eq. (3),

$$\beta\langle W\rangle_* = \frac{a(\nu a - 1)}{1+a}. \tag{81}$$

To see the typical behavior of the variation of the distribution of λ and the expected work $\langle W\rangle$ via each measurement, we plotted those for $a=0.7$ and $\nu=0.5$ in Fig. 8(b) and those for $a=0.7$ and $\nu=1.5$ in Fig. 8(c). For both cases, the initial distributions were set to the δ function, $P_\mathrm{in}(\lambda)=\delta(\lambda)$. As seen in the figures, the perpetual extraction of work is achieved in the former case, whereas not in the latter case.

Here we examine the optimal protocol of the feedback control in detail. $\beta\langle W\rangle_*$ is plotted in Fig. 9(a) as a function of a and ν. For $0<\nu<1$, $\langle W\rangle_* < 0$ regardless of a. For $\nu>1$, $a<\nu^{-1}$ must be satisfied so $\langle W\rangle_*<0$. The optimal protocol for the repeated measurements, which gives the lowest $\langle W\rangle_*$ for a given ν, is calculated from the derivative of Eq. (81),

$$\frac{\partial\beta\langle W\rangle_*}{\partial a} = \frac{\nu a^2 + 2\nu a - 1}{(1+a)^2}. \tag{82}$$

The optimal protocol and its corresponding work are thus given by $a_\mathrm{opt}=1$ and $\beta\langle W\rangle_{*\mathrm{opt}} = (\nu-1)/2$ for $\nu<1/3$, respectively, while $a_\mathrm{opt} = -1+\sqrt{1+\nu^{-1}}$ and $\beta\langle W\rangle_{*\mathrm{opt}} = 2[\sqrt{\nu(\nu+1)}-\nu]-1$ for $\nu>1/3$, respectively. In the limit of completely precise measurements ($\sigma=0$), the optimal expected work is given by $\beta\langle W\rangle_{*\mathrm{opt}} = -1/2$, which means that the extracted work is equal to the energy of the system for the canonical distribution.

The efficiency of the perpetual extraction of work is calculated as

$$\eta_* = \frac{2a(1-\nu a)}{\ln(1+1/\nu)(1+a)}, \tag{83}$$

where we have used Eq. (81) and the relation $\langle I\rangle_* = \overline{I}$ in the present case [see Eq. (77)]. It is plotted in Fig. 9(b) as a function of a and ν giving rise to negative $\langle W\rangle_*$. It is easily confirmed from Eq. (83) that the a giving rise to the highest η_* for a given ν coincides with a_opt, which is for the largest negative $\langle W\rangle_*$ discussed above. This result means that the optimal protocols for the largest extracted work and the highest efficiency are the same in the present case.

The optimal protocol a_opt, the corresponding expected work $\beta\langle W\rangle_{*\mathrm{opt}}$, and the efficiency $\eta_{*\mathrm{opt}}$ in the stationary state as functions of ν are plotted in Fig. 9(c). It is seen that a_opt is a monotonic decreasing function, while $\beta\langle W\rangle_{*\mathrm{opt}}$ and $\eta_{*\mathrm{opt}}$ are monotonic increasing functions. The asymptotic behaviors of these quantities in the limit of $\nu\to\infty$ can be obtained from the explicit expressions derived above. They are $a_\mathrm{opt}\approx 1/(2\nu)$, $\beta\langle W\rangle_{*\mathrm{opt}} \approx -1/(4\nu)$, and $\eta_{*\mathrm{opt}} \approx 1/2 - 1/(96\nu^2)$.

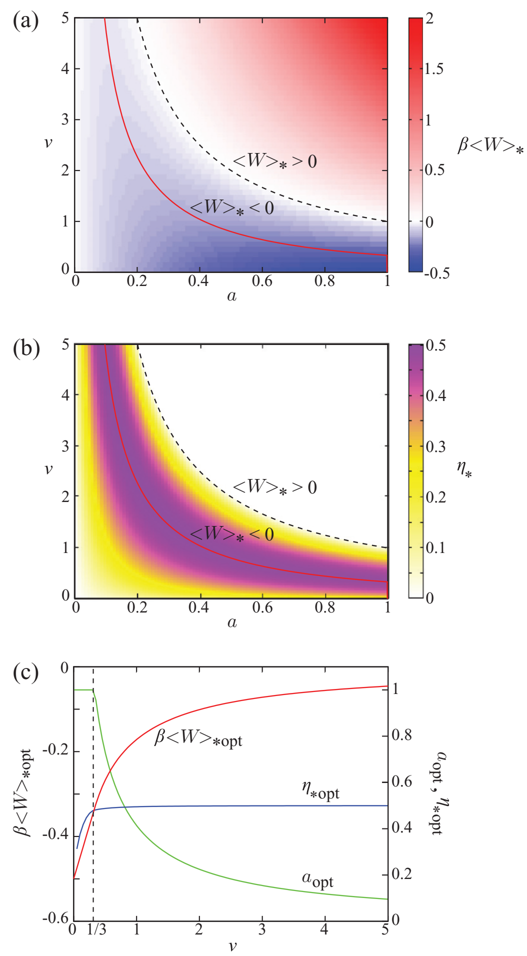

FIG. 9. (Color online) (a) Expected work applied to the system in the stationary state as a function a and ν. Dashed curve separates the regions for positive and negative $\langle W\rangle_*$. Solid curve represents the optimal a giving rise to the largest negative $\langle W\rangle_*$ for a given ν. (b) The efficiency η_* of the perpetual extraction of work in the stationary state as a function a and ν. Solid curve represents the optimal a giving rise to the highest η_* for a given ν, which coincides with the optimal a for $\langle W\rangle_*$ shown above. (c) The optimal protocol $a_{\rm opt}$ and the corresponding expected work $\langle W\rangle_{*\rm opt}$ in the stationary state as functions of ν. The efficiency $\eta_{*\rm opt}$ is also shown.

The stationary distribution of x is obtained by marginalizing $P_*(x,\lambda)$ as

$$P_*(x) = \int_{-\infty}^{\infty} d\lambda\, P_*(x,\lambda) = \int_{-\infty}^{\infty} d\lambda\, P_{\rm eq}(x|\lambda)P_*(\lambda) = \mathcal{N}(x;0,\sqrt{(1+\mu)/(\beta k)}), \tag{84}$$

where $\mu \equiv (1+\nu)/(a^{-2}-1)$ and the formula Eq. (A2) has been used. The width of this Gaussian function is larger than that of $P_{\rm eq}(x|\lambda=0)$. This is clearly due to the influence of the repeated feedback controls, which translate the center of the potential away the origin. In particular, the ratio $r_{\rm opt}(\nu)$ of the width of $P_*(x)$ with $a = a_{\rm opt}$ and $\nu > 1/3$ to that of the canonical distribution function is expressed as

$$r_{\rm opt}(\nu) = \sqrt{1 + \frac{1+\nu}{(-1+\sqrt{1+\nu^{-1}})^{-2}-1}}, \tag{85}$$

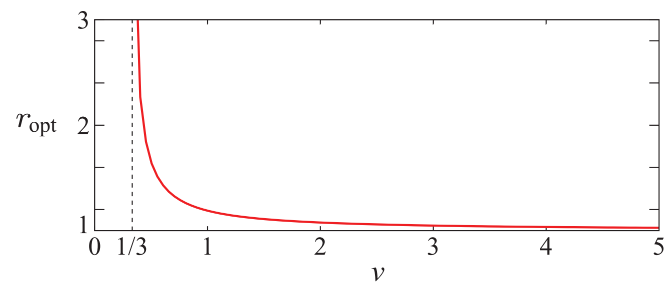

FIG. 10. (Color online) Ratio of the width of the stationary distribution $P_*(x)$ of x with $a = a_{\rm opt}$ to that of the canonical distribution function for $\gamma > 1/3$.

where $r_{\rm opt}$ is a monotonic decreasing function for $\nu > 1/3$, as plotted in Fig. 10. $r_{\rm opt}$ diverges for $\nu < 1/3$ due to $a_{\rm opt} = 1$ as analyzed above. Such a a is in fact not allowed for the achievement of stationary state.

It is interesting to compare our results with those for a single measurement. The optimal protocol for a single measurement provided by Abreu and Seifert [14] is $a_{\rm single} = 1/(1+\nu)$ for their Gaussian initial distribution function. Although $a_{\rm single}$ gives a negative $\langle W\rangle_*$ for an arbitrary ν when used in our repeated measurements, it differs from $a_{\rm opt}$. This result is reasonable since our protocol, Eq. (70), moves the center of potential according to how distant the measured position of the particle is from the origin, not from the present center of potential. It means that the next center of potential depends on the present one. To remove such an effect, we could adopt the following protocol:

$$\tilde{\lambda}(x_{\rm m},\lambda) = \lambda + a(x_{\rm m}-\lambda), \tag{86}$$

which moves the potential according to the distance between the measured position of the particle and the present center of potential. The propagator of the control parameter, Eq. (2), for this protocol is $G(\lambda,\lambda';x) = p_{\rm m}((\lambda-\lambda')/a + \lambda'|x,\lambda')/a$, the reduced propagator, Eq. (38), is calculated as

$$G(\lambda,\lambda') = \frac{1}{a}\sqrt{\frac{\zeta}{\pi}}\exp\left[-\frac{\zeta}{a^2}(\lambda-\lambda')^2\right], \tag{87}$$

to be compared with Eq. (79). It is clear from Eq. (A2) that the successive operation of G's on the PDF of λ forces it to spread unlimitedly regardless of a finite a, which means that the stationary solution does not exist for this protocol. The measurement-averaged energy is calculated as $\widetilde{E}(x,\lambda) = (k/2)[(1-a)^2(x-\lambda)^2 + a^2\sigma^2]$. The expected work for a given center λ_0 of the potential is thus given by

$$\beta\overline{W}_{\rm ind} = \tfrac{1}{2}a[a(1+\nu)-2], \tag{88}$$

independent of λ_0, as expected. This means that the measurements under the protocol Eq. (86) are truly independent of each other. The expected work $\beta\overline{W}_{\rm ind}$ for this protocol is clearly lower than that for the protocol for the stationary state, Eq. (75), with any λ_0. $\overline{W}_{\rm ind}$ as a function of a takes the lowest value at $a = a_{\rm single}$. This result is consistent with those obtained by Abreu and Seifert [14], as mentioned above.

IV. CONCLUSIONS

We developed a succinct theory of the perpetual extraction of work from a generic classical nonequilibrium system subject to a heat bath via repeated measurements under a Markovian feedback control. The condition for the realization of the perpetual extraction of work was formulated by treating both the control parameter and the dynamical variable as probabilistic variables. It was demonstrated that a problem for perpetual extraction of work in a nonequilibrium system is reduced to a problem of Markov chain in the higher-dimensional phase space. We derived a version of the detailed fluctuation theorem in a form suitable for the analyses of perpetual extraction of work. As simple applications of the theory, two exactly solvable models were analyzed. The one is a nonequilibrium two-state system and the other is a particle confined to a one-dimensional harmonic potential in thermal equilibrium. For the former example, it was demonstrated that the observer on the way to the stationary distribution can lose energy and that he or she can extract work larger than that achieved in the stationary distribution. For the latter example, it was demonstrated that the optimal protocol for the extraction of work via repeated measurements can differ from that via a single measurement. The validity of our version of the detailed fluctuation theorem, which determines the upper bound of the expected work in the stationary state, was also confirmed for both examples. The results obtained in the present work provide valuable insights into the implication of thermodynamics and information theory for perpetual extraction of work. Our framework will be useful for exploration of realistic modeling of a machine that extracts work from its environment. The examination of the dependence of the stationary distribution on the initial PDF and the incorporation of memory effects into the theory will be interesting in the future.

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APPENDIX: PRODUCT OF TWO GAUSSIAN FUNCTIONS

Let $\mathcal{N}(x;b,c)$ be a normalized one-dimensional Gaussian function centered at b of width c:

$$\mathcal{N}(x;b,c) \equiv \frac{1}{\sqrt{2\pi c^2}} \exp\left[-\frac{(x-b)^2}{2c^2}\right]. \tag{A1}$$

It is easily confirmed that the product of two Gaussian functions is another Gaussian function expressed as

$$\mathcal{N}(x;b_1,c_1)\mathcal{N}(x;b_2,c_2) = \frac{C}{\sqrt{2\pi}c_1c_2} \exp\left(\frac{B^2-D}{2C^2}\right)\mathcal{N}(x;B,C), \tag{A2}$$

where

$$B \equiv \frac{b_1c_2^2 + b_2c_1^2}{c_1^2 + c_2^2}, \tag{A3}$$

$$C \equiv \frac{c_1c_2}{\sqrt{c_1^2+c_2^2}}, \tag{A4}$$

$$D \equiv \frac{b_1^2c_2^2 + b_2^2c_1^2}{c_1^2 + c_2^2}. \tag{A5}$$

An integral formula

$$\int dx\,(x-a)^2\mathcal{N}(x;b,c) = (a-b)^2 + c^2 \tag{A6}$$

for an arbitrary a is used in the present study.

[1] V. Serreli, C. Lee, E. R. Kaym, and D. A. Leigh, Nature **445**, 523 (2007).
[2] B. J. Lopez, N. J. Kuwada, E. M. Craig, B. R. Long, and H. Linke, Phys. Rev. Lett. **101**, 220601 (2008).
[3] S. Toyabe, T. Sagawa, M. Ueda, E. Muneyuki, and M. Sano, Nat. Phys. **6**, 988 (2010).
[4] A. Bérut, A. Arakelyan, A. Petrosyan, S. Ciliberto, R. Dillenschneider, and E. Lutz, Nature **483**, 187 (2012).
[5] L. Szilárd, Z. Phys. **53**, 840 (1929).
[6] E. T. Jaynes, Phys. Rev. **106**, 620 (1957).
[7] R. Landauer, IBM J. Res. Dev. **5**, 183 (1961).
[8] C. H. Bennett, Int. J. Theor. Phys. **21**, 905 (1982).
[9] K. Maruyama, F. Nori, and V. Vedral, Rev. Mod. Phys. **81**, 1 (2009).
[10] P. Hänggi and F. Marchesoni, Rev. Mod. Phys. **81**, 387 (2009).
[11] U. Seifert, Rep. Prog. Phys. **75**, 126001 (2012).
[12] A. E. Allahverdyan and D. B. Saakian, Europhys. Lett. **81**, 30003 (2008).
[13] R. Marathe and J. M. R. Parrondo, Phys. Rev. Lett. **104**, 245704 (2010).
[14] D. Abreu and U. Seifert, Europhys. Lett. **94**, 10001 (2011).
[15] D. Abreu and U. Seifert, Phys. Rev. Lett. **108**, 030601 (2012).
[16] F. J. Cao and M. Feito, Phys. Rev. E **79**, 041118 (2009).
[17] M. Esposito and G. Schaller, Europhys. Lett. **99**, 30003 (2012).
[18] D. Mandala and C. Jarzynski, Proc. Natl. Acad. Sci. USA **109**, 11641 (2012).
[19] R. Kawai, J. M. R. Parrondo, and C. Van den Broeck, Phys. Rev. Lett. **98**, 080602 (2007).
[20] T. Sagawa and M. Ueda, Phys. Rev. Lett. **102**, 250602 (2009).
[21] R. Dillenschneider and E. Lutz, Phys. Rev. Lett. **102**, 210601 (2009).
[22] T. Sagawa, J. Phys.: Conf. Ser. **297**, 012015 (2011).
[23] J. M. Horowitz and S. Vaikuntanathan, Phys. Rev. E **82**, 061120 (2010).
[24] Y. Fujitani, T. Kojoh, and K. Inoué, J. Phys. Soc. Jpn. **72**, 1300 (2003).
[25] T. Schmiedl and U. Seifert, Phys. Rev. Lett. **98**, 108301 (2007).
[26] H. Suzuki and Y. Fujitani, J. Phys. Soc. Jpn. **78**, 074007 (2009).
[27] See, e.g., C. Meyer, *Matrix Analysis and Applied Linear Algebra* (SIAM, Philadelphia, 2000).
[28] L. Granger and H. Kantz, Phys. Rev. E **84**, 061110 (2011).
[29] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, 2nd ed. (Wiley-Interscience, Hoboken, NJ, 2006).

[30] T. Sagawa and M. Ueda, Phys. Rev. Lett. **104**, 090602 (2010).
[31] M. Ponmurugan, Phys. Rev. E **82**, 031129 (2010).
[32] S. Lahiri, S. Rana, and A. M. Jayannavar, J. Phys. A: Math. Theor. **45**, 065002 (2012).
[33] C. Jarzynski, Phys. Rev. E **73**, 046105 (2006).
[34] J. M. R. Parrondo, C. Van den Broeck, and R. Kawai, New J. Phys. **11**, 073008 (2009).
[35] J. M. Horowitz and Juan M. R. Parrondo, Europhys. Lett. **95**, 10005 (2011).