Frieden wave-function representations via an Einstein-Podolsky-Rosen-Bohm experiment

J. Syska*

Department of Field Theory and Particle Physics, Institute of Physics, University of Silesia, Uniwersytecka 4, 40-007 Katowice, Poland

(Received 19 July 2013; published 19 September 2013)

The appearance of the spin-$\frac{1}{2}$ and spin-1 representations in the Frieden-Soffer extreme physical information (EPI) statistical approach to the Einstein-Podolsky-Rosen-Bohm (EPR-Bohm) experiment is shown. In order to obtain the EPR-Bohm result, in addition to the observed structural and variational information principles of the EPI method, the condition of the regularity of the probability distribution is used. The observed structural information principle is obtained from the analyticity of the logarithm of the likelihood function. It is suggested that, due to the self-consistent analysis of both information principles, quantum mechanics is covered by the statistical information theory. The estimation of the angle between the analyzers in the EPR-Bohm experiment is discussed.

DOI: 10.1103/PhysRevE.88.032130 PACS number(s): 03.65.Ud, 89.70.Kn

I. INTRODUCTION

In the 1990s the generalization of the maximum likelihood method (MLM) and Fisher information (I_F) analysis in the statistical inference [1] was proposed by Frieden and Soffer [2] (more information is given at the end of this section). It concerns the nonparametric estimation that permits the selection of the equation of motions (or generating equations) of various field theory or statistical physics models. They called it the extreme physical information (EPI) method. Although the EPI method is mainly used to describe physical phenomena [2,3], other applications [2–5] have also been considered. This paper is devoted to the application of EPI for the description of the Einstein-Podolsky-Rosen-Bohm (EPR-Bohm) experiment proposed by Frieden [3]. The basis for this kind of statistical analysis of the phenomena was introduced in the 1920s by Fisher [1]. His statistical model selection method, which is particularly useful for a small sample, was constructed independently from the contemporary development of the physical models, especially quantum mechanics. It appeared that the EPI method leads to the estimation of the equations of physical field theories for which a small size of the sample is also a characteristic feature of the physical models [2,3]. For example, the size of the sample of the EPI model of the EPR-Bohm problem is equal to $N = 1$ [3]. Previously, the EPR-Bohm type problems have mainly been understood as manifestations of the quantum-mechanical reality.

The central quantity in an EPI analysis is the information channel capacity I, which is the trace of the expectation value of the Fisher information matrix. The basic tools of the EPI method of estimation are two information principles.

In [6] the derivations, first of the observed and, second, of the expected *structural information principle* from basic principles, was given. It was based first on the analyticity of the logarithm of the likelihood function, which allows for its Taylor expansion in the neighborhood of the true value of the vector parameter Θ [6], and second on the symmetric and assumed positive definite form of the Fisher information matrix. The Fisher information enters into the EPI formalism as the second order coefficient of this expansion [6]. The output of the solution of the information principles comprises the equations of motion of basic field theories or a distribution generating equation. For example, in [3] the Maxwell equations for $N = 4$ were obtained, while in [4,7] the construction of the information channel capacity for the vector position parameter in the Minkowskian space-time was completed. This has laid the statistical foundations of the kinematical term of the Lagrangian of the physical action for many field theory models that have been derived by the EPI method of Frieden and Soffer [2], where the metricity of the statistical space $\mathcal{S}$ of the system is also used. Additionally, the fact that the formalism of the information principles is used for the derivation of the distribution generating equation signifies that the microcanonical description of the thermodynamic properties of a compound system has to meet the analyticity and metricity assumptions as well (in agreement with the Jaynes principle [8]).

Although the analysis of the EPR-Bohm problem presented below originates in the Frieden approach [3], the paper gives both a mathematical background [4,6] and a modified physical interpretation. First, the EPI analysis follows from a meaning different than in [2,3] of the structural information principle [6], which originates in the above-mentioned analyticity condition of (in this case) the one-dimensional statistical space $\mathcal{S}$ (Sec. II B) [6]. Second, the joint space of events (see Sec. II B) for the construction of the Fisher information for the EPR-Bohm problem is different than in [3]. The information channel capacity that is necessary to solve the EPR-Bohm problem in EPI method is presented in Sec. III. The new meaning of the structural information principle is also connected with a more physical [6,9] and less informational [2,3] oriented interpretation of the physical information K. The variational information principle is connected with the extremization of K [2,3,6,9]. Third, the probability distribution that is characteristic for the EPR-Bohm problem [see Eq. (111) in Sec. V B3] is obtained as the solution of two information principles: the observed structural principle, which is the differential one, and the variational principle (Sec. IV). The condition of the regularity of the probability distribution on the one-dimensional statistical space $\mathcal{S}$, equipped with the Rao-Fisher metric [10–14], is also used (Sec. V B2). This

*Email address: jacek.syska@us.edu.pl

metric appears to be constant on $\mathcal{S}$. Thus, the result is not obtained using the quantum mechanical precondition of the basis wave functions orthogonality, as was done in [3]. Also, a minute aspect of the analysis is that the boundary conditions are put in order [15]. The experimental settings (Sec. II A) for spin-$\frac{1}{2}$ particles (electrons) are the same as in [3]; however, for spin-1 (massless photons), they are different than in [3]. They are arranged in such a way that the joint space of the events of the pair of spin projections in the analyzers (see Sec. II B) is the same in both cases. This makes the EPI results in both cases basically of the same form and, therefore, they are more easily interpreted as unveiling the origin of the rotation group representation to which a particular particle in the EPI-Bohm experiment belongs. Next, it is suggested that the Fermi-Dirac (for electrons) and the Bose-Einstein (for photons) statistics used in the quantum mechanical description of the EPR-Bohm problem [16] seem to be a reminiscence of two consecutive steps. In the case of fermions that are ruled by the Dirac equation, the first step consists of the appearance of the generalized Einstein-Brillouin-Keller quantization conditions [17]. In the case of the free electromagnetic field that fulfills the Maxwell equations, the first step consists of the existence of the conserved generalized helicity [18]. For both cases the relevant equations of motion are obtained in advance by the EPI method [3]. The second step consists of the appearance of the statistical information principles, which, as shown in Secs. III and IV, follow the analyticity of the log-likelihood function of the statistical space of the model. Thus, in the case of fermions, the Pauli exclusion principle has a statistical information theory background.

Next, for the measurement performed by the experimentalist, in Sec. VI the statistical estimation of the angle ϑ between the analyzers in the EPR-Bohm experiment (see Sec. II A) is presented. In this context, both the difference between the inner accuracy of the estimation [3,7] of the angle ϑ in the EPI analysis of the inner $N = 1$-dimensional sample collected by the system alone [2,3] and the accuracy of the estimation of ϑ in the experiment performed by the experimentalist are discussed. In the latter case the asymptotic local unbiasedness of the ϑ estimator is analyzed. Finally, throughout the analysis the priority of the *analytical form* of the observed Fisher information is kept and the *metric form* of the observed Fisher information is absolutely secondary; i.e., on the observed level, it is consequently the full analytical model that is solved (see Sec. IV).

The Frieden-Soffer original form of the physical information and information principles. Frieden and Soffer use two Fisherian information quantities: the intrinsic information J of the source phenomenon and the information channel capacity I, which connects the phenomenon and observer. Using the information channel capacity I and "bound" information J, the physical information $K \equiv I - \mathrm{J}$ was postulated [2,3], which (together with its densities) was used for the construction of the Frieden-Soffer information principles, at both the observed and the expected levels. Although the structural information principle is postulated at the expected level, in the Frieden and Soffer approach it is then reformulated to the observed one, giving, together with the variational principle, two coupled differential equations. Thus, Frieden and Soffer, along with Plastino and Plastino [2], put the solution of the (differential) information principles for various EPI models into practice. The above-mentioned analytical background of the structural information principle was derived in [6] and postulated previously in [9] (where the right direction for the transfer of information during the measurement, i.e., $\mathrm{J} \to I$, for the description of the Frieden interpretation should be used). In our approach [6] to the construction of the information principles (see Sec. IV A) the notion of the (total) *physical information* (TPI) $K = Q + I$ instead of $K = I - \mathrm{J}$ introduced in [2] is used, where Q is the *structural information*. This difference does not affect the derivation of the equation of motion or the generating equation for the problems that have been analyzed until now [2,3], since, identically, $Q = -\mathrm{J}$. Nevertheless, the above-mentioned analyticity condition appears to be fruitful for the EPI modeling (see Sec. IV B).

II. THE PHYSICAL SETTINGS AND BOUNDARY CONDITIONS

A. The physical settings and statistical space $\mathcal{S}$

Let us consider two types of the EPR-Bohm experiment: the first one for the spin-0 charged molecule which decays into bipartite system of two identical spin-$\frac{1}{2}$ particles (electrons e^- in this paper) and the second one for the spin-0 neutral molecule which decays into a bipartite system of two spin-1 (massless) photons. In both cases the total angular momentum along the z axis is zero (see Fig. 1). For the first case, such a bipartite state for the EPR-Bohm experiment can be effectively prepared, e.g., as the final state in the scattering process $e^-e^- \to e^-e^-$, where the spins of the initial electrons of the process are arranged to be one up and the other down along the z axis, whereas their momenta $\vec{p}$ and $-\vec{p}$ are along the y axis [16] [see Fig. 1(a)]. There is the nonzero probability that two scattered particles (here final electrons) move along the x axis with the opposite momenta [19]. In the EPR-Bohm experiment, the measurement of the spin projections of the scattered particles is performed.

The analyzer "a", which is the Stern-Gerlach device, measures the projection S_a of the spin $\vec{S}_1$ of the particle "1" along the direction of the unite vector $\vec{a}$ and similarly the analyzer "b" measures the projection S_b of the spin $\vec{S}_2$ of the particle "2" along the direction of the unite vector $\vec{b}$. The angle between the planes of the vectors $\vec{a}$ and $\vec{b}$ that include the x axis is equal to $\vartheta = \chi_1 - \chi_2$, $0 \leqslant \vartheta < 2\pi$. Similarly, for the second case, the process $e^-e^+ \to \gamma\gamma$ is the relevant one, where electron e^- and positron e^+ are prepared with spins up and down along the z axis and with momenta $\vec{p}$ and $-\vec{p}$ along the y axis, respectively [see Fig. 1(b)]. The electron and positron annihilate into two photons. The created photons 1 and 2 move up and down along the z axis with the opposite momenta, and there is also a nonzero probability that such a process occurs [19]. The polarizations of photons are measured by polarizers with polarization vectors making angles χ_1 and χ_2 with the x axis, respectively [16].

Below, in Sec. II B we derive the boundary conditions for the EPR-Bohm problem in the EPI method, taking care not to appeal to the quantum mechanical vision of reality. However, we take into account the general statement that if $P(AB)$ is the joint probability distribution for two random vaiables A and B

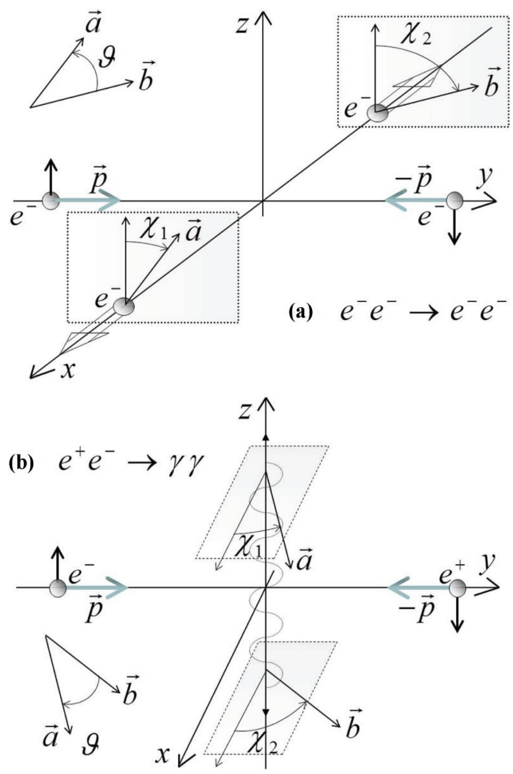

FIG. 1. (Color online) The EPR-Bohm experiment. The possible configurations of processes are presented (see the text). (a) Process $e^-e^- \to e^-e^-$. One of the initial electrons has the spin up and the other down along the z axis. Their momenta along the y axis are $\vec{p}$ and $-\vec{p}$, respectively. The final electrons of the bipartite system move along the x axis with opposite momenta and are registered in the Stern-Gerlach devices. The angle ϑ between the directions $\vec{a}$ and $\vec{b}$ of the Stern-Gerlach devices is signified. χ_1 and χ_2 are the angles of the analyzers with the z axis; $\vartheta = \chi_2 - \chi_1$. (b) Process $e^-e^+ \to \gamma\,\gamma$. The initial electron and positron have their spins up and down along the z axis and their momenta along the y axis are $\vec{p}$ and $-\vec{p}$, respectively. The created photons "1" and "2" of the bipartite system move up and down along the z axis with the opposite momenta, respectively, and their polarizations are measured by two polarizers, which make angles χ_1 and χ_2 with the x axis; $\vartheta = \chi_2 - \chi_1$.

and $P(A)$ and $P(B)$ are their marginal distributions, then [3]

$$I_F[P(AB)] \geqslant I_F[P(A)] + I_F[P(B)] \equiv \tilde{C}, \quad (1)$$

where the equality holds if the variables are independent. Here $\tilde{C}$ is the information channel capacity of the composite system. Relation (1) means that if there is any dependence among the variables, then, if we know the result of the experiment for the first variable, the amount of information needed for the determination of the result of the experiment for the other one diminishes. So, the existence of the dependence in the system increases the Fisher information I_F on the parameters that characterize the probability distribution of the system.

Note on the common notion for electron and photon spin projections. The notation is common for the measurement of the spin projection of final particles, whether they are electrons [Fig. 1(a)] or photons [Fig. 1(b)]. The final electrons [Fig. 1(a)] are recorded in the Stern-Gerlach devices and the final photons [Fig. 1(b)] are recorded in the polarizers. When the spin-$\frac{1}{2}$ particle (e.g., electron 1) is registered then "+" signifies the observed value of its polarization along the direction $\vec{a}$ and the spin projection on $\vec{a}$ that is equal to $S_a = S_+ = +\hbar/2$, whereas "−" signifies the observed value of its polarization along the direction $-\vec{a}$ and the spin projection on $\vec{a}$ equal to $S_a = S_- = -\hbar/2$. When the spin-1 photon (e.g., photon 1) is registered then + signifies the observed value of its polarization along the direction $\vec{a}$ and the spin projection (helicity) *on the z axis* (direction of its propagation) equal to $S_a = S_+ = +\hbar$, whereas − signifies the observed value of its polarization along the perpendicular direction $\vec{a}_\perp$ and the spin projection *on the z axis* equal to $S_a = S_- = -\hbar$. Similar notation is used for the particle 2 (electron or photon) that is registered in the polarizer b.

B. The boundary conditions for the EPR-Bohm problem

In order to determine the problem of the differential equations of the EPI method, it is necessary to indicate the boundary conditions for the probabilities. They follow the phenomenological premises, conservation laws, and special symmetries of the system being examined.

1. Determination of the spin projection

The basic EPI method concept from which the derivation of the Dirac equation follows is the (density of the) information channel capacity I, which for the spinorial field is invariant under the isometry transformations in $N/2$-dimensional complex space $\mathbb{C}^{N/2}$ of fields $\psi \equiv [\psi_n(\mathbf{x})]_{n=1}^{N/2}$ [20]. After decomposing the density of I into the components [3], each of them can be factorized into terms that are the elements of the Clifford group Pin(1,3) [20]. The group Pin(1,3) is a subset of the Clifford algebra $C(1,3)$. As the spinor representations of an orthonormal basis in $C(1,3)$ are the Dirac γ^μ matrices; therefore, in summary, the Dirac equation appears via the factorization of the density of the information channel capacity [3] and due to the previously mentioned observed structural information principle and the variational one, in which the density of the information channel capacity i forms the kinetic term. In this way in [3] for the $N = 8$-dimensional sample the Dirac equation for the spinorial field [i.e., for the complex field $\psi \equiv [\psi_n(\mathbf{x})]_{n=1}^{4}$ of the rank $N = 8$] was obtained. Then, under the assumption of the conservation of momentum, it was supplemented [21] by the EPI method background of the de Broglie-Fourier representation of a particle [22].

Next, in [17] the semiclassical theory for spinning particles (ruled by the Dirac equation) was presented. Here, Keppeler derived a form of the Einstein-Brillouin-Keller quantization conditions generalized for the particles with spin, in which the spin quantum number equal to $\pm 1/2$ appears. In [17] the latitude of the classical spin vector appears as a constant of motion. This type of semiclassical quantization also appears in [23] for the description of the spin of the neutron. The suitable steps for the photon are as follows. For $N = 4$, the EPI method leads to the Maxwell equations [3,7] (with the Lorentz condition additionally imposed). Then, for the free electromagnetic field seen as the composition of the circularly polarized right- and left-handed photons [16,24], the existence of the conserved classical generalized helicity was proven in [18]. Finally, in [25] the stochastic trial to explain the quantization of the spin projection was presented.

[Meanwhile, the classical statistics estimation relevant for stochastic processes is connected with the analysis of the (pure) likelihood function and, hence, it omits the logarithmic structure of the system [4,6,26] (which is present in the EPI method)].

2. *The joint space of events*

Thus, in accordance with the above section, *the boundary condition* connected with the existence of strictly (up to the noise in the measuring devices; see Sec. VI D) two possible spin projections appears, which for a particle with spin $\hbar/2$ are on the arbitrary space direction and for the massless photon with spin $\hbar$ on the direction of the propagation. In agreement with the above *Note on the common notation* (Sec. II A), we let $+$ signify the observed value of the spin projection $S_a = +\hbar/2$ or $S_a = +\hbar$, while $-$ signifies $S_a = -\hbar/2$ or $S_a = -\hbar$ for the particle with spin $\hbar/2$ or a photon with spin $\hbar$, respectively. We introduce the common denotation for *the base space* $\mathbb{S}$ of the random variable S_a for both cases:

$$\mathbb{S} = \{S_-,S_+\} \equiv \{-,+\}. \tag{2}$$

Similar notation is used for the observed value of the spin projection S_b, for which the base space is also $\mathbb{S}$.

The joint space of events Ω_{ab} of the pair of spin projections for particles 1 and 2 is as follows [3]:

$$S_a S_b \equiv S_{ab} \equiv ab \in \Omega_{ab} = \{S_{++},S_{--},S_{+-},S_{-+}\} \equiv \{+\,+\,,-\,-,+\,-,-\,+\}. \tag{3}$$

It can be checked that the form of Ω_{ab} is different for a spin-1 massive particle or a higher than spin-1 particle. The analysis of these cases is not included in this paper.

Let us suppose that four joint conditional probabilities can be defined $P(S_{ab}|\vartheta)$,

$$P(+\,+|\vartheta),\ P(-\,-|\vartheta), P(+\,-|\vartheta),\ P(-\,+|\vartheta), \tag{4}$$

where $\vartheta \in \langle 0,\pi) \equiv V_\vartheta$ and V_ϑ is the parameter space.

Let us remark that in the case of the lack of the joint probability space Ω_{AB} for the random variables, let us say A and B, there is no possibility to define the joint probability distribution $P(A,B)$ for these two variables, even if their marginal distributions $P(A)$ and $P(B)$ exist. This means that they cannot be simultaneously measured. Also, in general, despite the existence of the joint marginal probability distribution $P(A,B)$ for the variables A and B and the existence of the joint marginal probability distribution $P(B,C)$ for the variables B and C, the joint probability distribution $P(A,B,C)$ for variables A, B, and C does not exist. Let us notice that in the proof of the Bell inequality [27,28], it is taken for granted that the joint distribution $P(A,B,C)$ does exist [27,28]. Such a possibility always exists only if the joint events space Ω_{ABC} for these three random variables exists, which is the Cartesian product $\Omega_A \times \Omega_B \times \Omega_C$. On the other hand, Bell-type inequalities are known from Boole's times as the test, which, if failed, comfirms the impossibility of the construction of the joint probability distribution. The above consideration could be extended for an arbitrary number of random variables [28,29]. In the full description of the EPR-Bohm experiment, not only the random variables of the spin projection measured in the analysers a and b but two random angle variables measured for these particles in the moment of their production should be taken into account.

Because of the mutual exclusion of different number $\aleph = 4$ of events S_{ab}, (3), the condition of the probability normalization $P(S_a S_b|\vartheta)$ in the EPR-Bohm problem can be written as follows:

$$\begin{aligned} P\left(\bigcup_{ab} S_{ab}|\vartheta\right) &= \sum_{ab} P(S_{ab}|\vartheta) \\ &= P(S_{++}|\vartheta) + P(S_{--}|\vartheta) \\ &\quad + P(S_{+-}|\vartheta) + P(S_{-+}|\vartheta) \\ &= 1, \quad \forall\, \vartheta \in V_\vartheta. \end{aligned} \tag{5}$$

These four joint probabilities $P(S_{ab}|\vartheta)$ form the statistical (sub)space $\mathcal{S}$,

$$\mathcal{S} = \{P(S_{ab}|\vartheta)|\ \ \vartheta \in \langle 0\,2\,\pi) \equiv V_\vartheta \subset \mathbb{R}^1\}, \tag{6}$$

of the EPR-Bohm problem, which is the one-dimensional submanifold of the $\aleph - 1 = 3$-dimensional probability simplex [30] coordinatized by the parameter ϑ (see the Appendix).

Taking into account the *Note on the common notation* in Sec. II A and keeping the order of the consecutive summands in Eq. (5), we obtain the following. In the case of a spin-$\frac{1}{2}$ particle and using the notation from Fig. 1(a), we see that Eq. (5) has the form [16]

$$\begin{aligned} \sum_{ab} P(S_{ab}|\vartheta) &= P(\vec{a},\vec{b}) + P(-\vec{a},-\vec{b}) \\ &\quad + P(\vec{a},-\vec{b}) + P(-\vec{a},\vec{b}) \\ &= 1, \end{aligned}$$

where for the particle 1 and the polarizer a, the event $\vec{a}$ under the denotation of the probability P signifies the particle polarization in the direction $\vec{a}$, i.e., $S_a = S_+$, and $-\vec{a}$ signifies the event of the particle polarization in the direction $-\vec{a}$, i.e., $S_a = S_-$ (and similarly for the particle 2 and the polarizer b). In the case of spin-1 photon and using the notation from Fig. 1(b), we see that Eq. (5) has the form [16]

$$\begin{aligned} &\sum_{ab} P(S_{ab}|\vartheta) \\ &\quad = P(\vec{a},\vec{b}) + P(\vec{a}_\perp,\vec{b}_\perp) + P(\vec{a},\vec{b}_\perp) + P(\vec{a}_\perp,\vec{b}) = 1, \end{aligned}$$

where for the photon 1 and the polarizer a, the event $\vec{a}$ under the denotation of the probability P means the photon polarization in the direction $\vec{a}$, which signifies the spin projection on the z axis equal to $S_a = S_+$, and $\vec{a}_\perp$ means the event of the photon polarization in the direction perpendicular to $\vec{a}$ and therefore $S_a = S_-$ (and similarly for the photon 2 and the polarizer b).

Note on the method of the estimation of ϑ. In Sec. VI we see that, due to the fact that the frequencies of the events S_{ab}, (3), observed in the analyzers a and b, are locally unbiased estimators of the probabilities (4), the value of the angle ϑ can be robustly estimated by an experimentalist by taking a large enough M-dimensional (outer) sample. However, to make this possible, an analytical model which enables the derivation of the formulas for the probability distribution (4) is needed, and it can be obtained using the EPI method. The dimension of the (inner) sample in the EPI analysis has to be taken as equal to $N = 1$ (Sec. III). Although the result was originally obtained by Frieden [3], we regain it without any reference to the quantum mechanical formalism (i.e., here to the orthogonality of the quantum mechanical wave functions).

C. The formulation of the boundary conditions

For every event S_{ab}, (3), the probability $P(S_{ab})$ of the appearance of S_{ab}, irrespective of the value of ϑ, is equal to

$$P(S_{ab}) = \int_0^{2\pi} P(S_{ab},\vartheta)d\vartheta. \tag{7}$$

Here, using the definition of the conditional probability, the probability $P(S_{ab},\vartheta)$ is defined as

$$P(S_{ab},\vartheta) := P(S_{ab}|\vartheta)r(\vartheta), \tag{8}$$

where $r(\vartheta)$ is the so-called *lack of knowledge function*.

(A) Because the events (3) are mutually exclusive and they span the entire space of events Ω_{ab}, the first boundary condition is the *normalization*

$$\sum_{ab} P(S_{ab}) = P(S_{++}) + P(S_{--}) + P(S_{+-}) + P(S_{-+}) = 1, \tag{9}$$

which is fulfilled irrespective of the value of ϑ.

Because of the normalization conditions (5) and (9) and using Eqs. (7) and (8), it can be noticed that the possible form of $r(\vartheta)$ can be chosen as

$$r(\vartheta) = \frac{1}{2\pi}, \quad \vartheta \in V_\vartheta, \tag{10}$$

which means that in the range $\langle 0,2\pi)$ of the measuring apparatus variability of the angle ϑ (see Fig. 1) and due to the lack our knowledge, every value of $\vartheta \in V_\vartheta$ is equally possible.

Remark on the value of the angle ϑ. The value of ϑ is connected with the arrangement of the measuring analyzers a and b (Stern-Gerlach devices or polarizers), and it is hardly to be (seriously) treated as possessing the distribution. Essentially, ϑ is the parameter which is characteristic for the experiment being carried out. Nonetheless, $r(\vartheta)$ is seen as *a priori* known "probability" by some, which means that quantum mechanics that is deduced in this way should be treated as the Bayesian statistical theory [31].

(B) The consecutive conditions follow from *the symmetry of the system and the law of the total spin conservation* (assuming that *the relative orbital angular momentum of the particles is equal to zero* in the experiment).

Let us consider a simple case $\vartheta = 0$ when both planes in which the measuring devices (Stern-Gerlach or polarizer) are set up are oriented in the same way. From the condition of the total spin conservation it follows that

$$P(+\,+|\vartheta = 0) = P(-\,-|\vartheta = 0) = 0, \tag{11}$$

which means that with this arrangement of the apparatus, we never see both spins simultaneously directed up or down. Also, we never see the spins directed one up and the other down in the following situations:

$$\begin{aligned} &\text{for spin} - \frac{1}{2} \text{ if } \vartheta = \pi \\ &\quad \Rightarrow P(+\,-|\vartheta = \pi) = P(-\,+|\vartheta = \pi) = 0, \\ &\text{and} \\ &\text{for spin} - 1 \text{ if } \vartheta = \frac{\pi}{2} \\ &\quad \Rightarrow P\left(+\,-|\vartheta = \frac{\pi}{2}\right) = P\left(-\,+|\vartheta = \frac{\pi}{2}\right) = 0, \end{aligned} \tag{12}$$

respectively. As a result, from the law of the total spin conservation, we find that if we know that in the case of spin-$\frac{1}{2}$ particle $\vartheta = 0$ or $\vartheta = \pi$ (or in the case of spin-1 photon $\vartheta = 0$ or $\vartheta = \pi/2$), then the observation of one spin projection gives us complete knowledge about the second one. In this case, the spin states are clearly not independent. This conclusion is incorporated in their preparation method intuitively (discussed in Sec. II A).

Next, as $P(S_b|\vartheta)$ is the marginal probability of the appearance of the particular value of the spin projection of the particle 2, it does not depend on S_a, i.e., on the orientation of the spin projection of the particle 1. From the normalization condition for $P(S_b|\vartheta)$, it follows that

$$P(S_b = +|\vartheta) + P(S_b = -|\vartheta) = 1. \tag{13}$$

Next, due to the fact that, because of the symmetry, the definition of the *up* vs *down* spin projection S_b is chosen arbitrarily, we see that Eq. (13) leads to

$$P(S_b = +|\vartheta) = P(S_b = -|\vartheta) = \tfrac{1}{2}. \tag{14}$$

Thus, we have obtained

$$P(S_b|\vartheta) = \tfrac{1}{2} \quad \text{for} \quad S_b = +, -\,, \tag{15}$$

independent of the angle ϑ between the vectors $\vec{a}$ and $\vec{b}$. It follows that

$$\begin{aligned} P(S_b) &= \int_0^{2\pi} P(S_b|\vartheta)r(\vartheta)d\vartheta \\ &= \frac{1}{2}\int_0^{2\pi} r(\vartheta)d\vartheta = \frac{1}{2} \quad \text{for} \quad S_b = +, -\,, \end{aligned} \tag{16}$$

where $r(\vartheta)$ is specified in Eq. (10).

The other important property of the spacial symmetry of the experiment is *the lack of the preference for upward or downward orientation of the the spin*; i.e.,

$$P(S_{+-}|\vartheta) = P\,(S_{-+}|\vartheta) \quad \text{and} \quad P(S_{++}|\vartheta) = P(S_{--}|\vartheta). \tag{17}$$

This means that if we observe the experiment for the system rotated around the x axis by the angle π, then the probability would be exactly the same.

From Eq. (17) and Eqs. (7) and (8), we obtain

$$P(S_{++}) = \int_0^{2\pi} P(S_{++}|\vartheta)r(\vartheta)d\vartheta = \int_0^{2\pi} P(S_{--}|\vartheta)r(\vartheta)d\vartheta = P(S_{--}) \quad (18)$$

and, analogously,

$$P(S_{+-}) = P(S_{-+}). \quad (19)$$

Additionally, we obtain

$$\begin{aligned} P(S_{+-}) &= \int_0^{2\pi} P(S_{+-}|\vartheta)r(\vartheta)d\vartheta \\ &= \left\{ \int_0^{2\pi} P(S_{+-}|\vartheta+\pi)\, r(\vartheta+\pi)\, d\vartheta \right. \\ &= \left. \int_0^{2\pi} P(S_{++}|\vartheta)r(\vartheta+\pi)\, d\vartheta \right\} \quad \text{for spin } \frac{1}{2} \\ &= \left\{ \int_0^{2\pi} P\left(S_{+-}|\vartheta+\frac{\pi}{2}\right) r\left(\vartheta+\frac{\pi}{2}\right) d\vartheta \right. \\ &= \left. \int_0^{2\pi} P(S_{++}|\vartheta)r\left(\vartheta+\frac{\pi}{2}\right) d\vartheta \right\} \quad \text{for spin } 1 \\ &= P(S_{++}), \end{aligned} \quad (20)$$

where in the second and fourth lines the simple change of the (circle) variables $\vartheta \to \vartheta + \pi$ or $\vartheta \to \vartheta + \pi/2$ for spin-$\frac{1}{2}$ or spin-1 particle, respectively, has been used (i.e., the starting point of integration around a closed circle is insignificant). Then, in the third line, for spin-$\frac{1}{2}$ particle we use the equality

$$P(S_{++}|\vartheta) = P(S_{+-}|\vartheta+\pi), \quad \text{for spin } \tfrac{1}{2}, \quad (21)$$

and in the fifth line for spin-1 photon,

$$P(S_{++}|\vartheta) = P\left(S_{+-}|\vartheta+\frac{\pi}{2}\right), \quad \text{for spin } 1. \quad (22)$$

The conditions (21) or (22) follow from the equivalence of the events of obtaining the "+" polarization in the direction $\vec{a}$ of the measuring device and the "−" polarization in the direction $-\vec{a}$ or $\vec{a}_\perp$ in the case of an electron or photon, respectively. Finally, in the last equality in Eq. (20), we use $r(\vartheta) = r(\vartheta+\pi) = r(\vartheta+\frac{\pi}{2}) = 1/(2\pi)$.

As a consequence of Eqs. (18)–(20), we obtain

$$P(S_{-+}) = P(S_{--}). \quad (23)$$

Taking into account Eqs. (18)–(23) and Eq. (9), we obtain

$$P(S_{ab}) = \tfrac{1}{4}, \quad \text{for every } S_{ab} \in \Omega_{ab}. \quad (24)$$

Finally, the Bayes' formula for the *conditional probability* gives

$$P(S_a|S_b) = \frac{P(S_{ab})}{P(S_b)} = \frac{1/4}{1/2} = \frac{1}{2}, \quad \text{for every } S_a, S_b. \quad (25)$$

By reasoning similar to that which gave Eqs. (15), (16), (24), (25) previously, we obtain

$$P(S_a|\vartheta) = \tfrac{1}{2}, \quad P(S_a) = \tfrac{1}{2}, \quad \text{for } S_a = +, -, \quad (26)$$

$$P(S_{ba}) = \tfrac{1}{4}, \quad P(S_b|S_a) = \tfrac{1}{2}, \quad \text{for every } S_a, S_b, \quad (27)$$

respectively.

Final remarks on the boundary conditions. Let us note that, to this point, we have not used the EPI method in the derivation of formulas (11)–(25). This is because these are the boundary conditions for the equations of the EPI method. These conditions follow from (i) the initial observation of the existence of exactly two possible spin projections for the spin-$\frac{1}{2}$ particle (here the electron) or for the spin-1 (massless) photon (see the *Note on the common notation* in Sec. II A), (ii) the normalization of the probability distribution, (iii) the law of the total angular momentum conservation, and (iv) the symmetry of the system. Finally, in what follows, when a solution of the equation generating the distribution is chosen, in Sec. IV B we use the geometrical symmetry of the system under the rotation by the angle 2π once more.

III. THE INFORMATION CHANNEL CAPACITY FOR THE EPR-BOHM PROBLEM

Below we derive the EPI method results for the probabilities (4). While they are consistent with the ones obtained in [3], there are also some noticeable differences (Secs. III) and extensions of the method (e.g., in Secs. V and VI).

The general form of I for independent data. Let an original random variable Y (a discrete or continuous one) take values $\mathbf{y} \in \mathcal{Y}$, where $\mathcal{Y}$ is the base space, and suppose that parameter θ of the regular (in θ) distribution $p(\mathbf{y})$ in which we are interested, is the scalar one. Let the data $y = (\mathbf{y}_n)_{n=1}^N$ be a realization of the N-dimensional sample $\widetilde{Y} = (Y_1, Y_2, \ldots, Y_N) \equiv (Y_n)_{n=1}^N$ and $p_n(\mathbf{y}_n|\theta_n)$ be a point distribution for the nth observation in the N-dimensional sample [4,7]. The set of all possible realizations y of the sample $\widetilde{Y}$ forms the *sample space* $\mathcal{B}$ of the system. We assume that the variables Y_n of the sample $\widetilde{Y}$ are independent. It is also supposed that $p_n(\mathbf{y}_n|\theta_n)$ does not depend on the parameter θ_m for $m \neq n$. Thus, the data are generated in agreement with the point probability distributions, which fulfill the condition $p_n(\mathbf{y}_n|\Theta) = p_n(\mathbf{y}_n|\theta_n)$ $(n = 1, \ldots, N)$, where $\Theta \equiv (\theta_n)_{n=1}^N$, and *the likelihood function* $P(y|\Theta)$ of the sample $y = (\mathbf{y}_n)_{n=1}^N$ is the product

$$P(\Theta) \equiv P(y|\Theta) = \prod_{n=1}^{N} p_n(\mathbf{y}_n|\theta_n). \quad (28)$$

The expected Fisher information matrix on the statistical (sub)space $\mathcal{S}$ at point $P(\Theta)$ [14] is defined as

$$I_F(\Theta) \equiv E_\Theta(\mathrm{i}^{\mathrm{F}}(\Theta)) = \int_{\mathcal{B}} dy\, P(y|\Theta)\, \mathrm{i}^{\mathrm{F}}(\Theta), \quad (29)$$

where $\mathcal{B}$ is the sample space, the differential element is given by $dy \equiv d^N\mathbf{y} = d\mathbf{y}_1 d\mathbf{y}_2 \cdots d\mathbf{y}_N$, and $\mathrm{i}^{\mathrm{F}}(\Theta)$ is the *observed Fisher information matrix* [4,6,7,32]. The subscript Θ in the expected value signifies the true value of the parameter under which the data y are generated. The Fisher information matrix defines on $\mathcal{S}$ the Riemannian Rao-Fisher metric [7,14].

The central quantity of EPI analysis is the information channel capacity I, which is the trace of the (expected) Fisher information matrix. Because under above conditions, the observed Fisher information matrix is diagonal $\mathrm{i}^{\mathrm{F}}(\Theta) = \mathrm{diag}(\mathrm{i}^{\mathrm{F}}{}_{nn}(\Theta))$; hence, the information channel capacity $I(\Theta)$

is equal to

$$I(\Theta) = \sum_{n=1}^{N} \int_{\mathcal{B}} dy\, P(y|\Theta)\, \mathrm{i^F}_{nn}(\Theta) = \int_{\mathcal{B}} dy\, i, \qquad (30)$$

where $i := P(\Theta) \sum_{n=1}^{N} \mathrm{i^F}_{nn}(\Theta)$ is the *information channel density* [4,7].

When expressed by the point probability distributions, the *analytical form* of the information channel capacity $I(\Theta)$ is as follows [2,3]:

$$\begin{aligned} I(\Theta) &= \sum_{n=1}^{N} I_{Fn}(\theta_n) \\ &= -\sum_{n=1}^{N} \int_{\mathcal{Y}} d\mathbf{y}_n\, p_n(\mathbf{y}_n|\theta_n) \frac{\partial^2 \ln p_n(\mathbf{y}_n|\theta_n)}{\partial \theta_n^2} \\ &= -\sum_{n=1}^{N} \int_{\mathcal{Y}} d\mathbf{y}_n \frac{\partial^2 p_n(\mathbf{y}_n|\theta_n)}{\partial \theta_n^2} \\ &+ \sum_{n=1}^{N} \int_{\mathcal{Y}} d\mathbf{y}_n \frac{1}{p_n(\mathbf{y}_n|\theta_n)} \left(\frac{\partial p_n(\mathbf{y}_n|\theta_n)}{\partial \theta_n} \right)^2, \end{aligned} \qquad (31)$$

where in the second line both Eq. (28) and the normalization of the point distributions $p_n(\mathbf{y}_n|\theta_n)$, $n = 1, \ldots, N$ were used [3]. Due to the normalization and the condition of regularity of the probability distribution (in every θ_n) [32], it follows that

$$\begin{aligned} &\int_{\mathcal{Y}} d\mathbf{y}_n \frac{\partial^2 p_n(\mathbf{y}_n|\theta_n)}{\partial \theta_n^2} \\ &= \frac{\partial^2}{\partial \theta_n^2} \int_{\mathcal{Y}} d\mathbf{y}_n\, p_n(\mathbf{y}_n|\theta_n) = \frac{\partial^2}{\partial \theta_n^2} 1 = 0 \end{aligned} \qquad (32)$$

and the information channel capacity (31) for the vector parameter $\Theta \equiv (\theta_n)_{n=1}^N$ can be written in its *metric form* [3,14]:

$$\begin{aligned} I(\Theta) &= \sum_{n=1}^{N} I_{Fn}(\theta_n) \\ &= \sum_{n=1}^{N} \int_{\mathcal{Y}} d\mathbf{y}_n\, p_n(\mathbf{y}_n|\theta_n) \left(\frac{\partial \ln p_n(\mathbf{y}_n|\theta_n)}{\partial \theta_n} \right)^2. \end{aligned} \qquad (33)$$

The Fisher information I_{Fn} is the measure of the precision of the estimation of one scalar parameter θ_n [4,7]. Below the forms of the Fisher information will be adopted for the purpose of the estimation of the angle ϑ (see Fig. 1).

Note. According to *the main assumption of the EPI method* proposed by Frieden and Soffer, the system alone samples the the space of the positions that is accessible to it using its Fisherian, kinematical degrees of freedom [2,7]. Thus, the bipartite system alone measures [3,7] the values of the spin projections of two particles, i.e., particle 1, which is S_a, and particle 2, which is S_b.

The sample space in the EPR-Bohm problem. The particular form of the information channel capacity takes into account the *measurement channel* [7], i.e., the channel which is indivisible from an experimental point of view. In the EPR-Bohm problem the sample appears to be $N = 1$-dimensional [3], which (as we will see) gives the required form of the estimated probabilities and the measurements of S_a and S_b appear dependent [7]. Thus, the *measurement channel* consists of one joint measurement (performed by the system alone) of the pair of spin projections S_a and S_b, i.e., of the random variable S_{ab}, which takes the value from the joint space of events Ω_{ab} that is given by Eq. (3). Using the general formula (33) the assignments which pertain to the EPR-Bohm problem are

$$\begin{aligned} &\mathbf{y_1} \equiv S_{ab}, \quad \mathcal{Y} \equiv \Omega_{ab}, \quad \theta_1 \equiv \vartheta, \\ &\int_{\mathcal{Y}} d\mathbf{y}_1 \equiv \sum_{S_a=-}^{+} \sum_{S_b=-}^{+} \equiv \sum_{ab}, \quad (N = 1), \end{aligned} \qquad (34)$$

where the summation over ab in $\sum_{ab}$ is performed over the joint space of events Ω_{ab}. As $N = 1$ the sample space $\mathcal{B}$ is equivalent to the base space Ω_{ab}. Because, in accordance with Eq. (34), the dimension of the sample is equal to $N = 1$, the information channel capacity I in Eq. (33) (let us call it the "single-value-ϑ" information channel capacity) reduces to the Fisher information $I_{F\,n=1}(\theta_1) = I_F(\vartheta)$ for $\theta_1 \equiv \vartheta$.

The likelihood function. As a bipartite system (of two particles) performs the measurement of S_{ab} by itself, the likelihood function for the problem stated above is

$$P(S_{ab}|\vartheta) \text{ is the likelihood of the sample for } \vartheta \text{ parameter}, \qquad (35)$$

which means that in Eq. (33) the assignment $p_1 \equiv P(S_{ab}|\vartheta)$ also has to be performed. Now the form of $P(S_{ab}|\vartheta)$ is searched for using the EPI method.

The probability amplitudes q_{ab} are defined in the following way [14,30]:

$$P(S_{ab}|\vartheta) = \tfrac{1}{4} q_{ab}^2(\vartheta) \quad \text{for every} \quad S_{ab} \in \Omega_{ab}. \qquad (36)$$

Because the dimension of the sample is equal to $N = 1$, the rank of the amplitude $q_{ab}(\vartheta)$ of the field is also equal to 1 [3].

A. The expected Fisher information and the information capacity of the channel (ϑ)

As ϑ is the scalar parameter and the dimension of the sample is equal to $N = 1$, thus taking into account the assignments given by Eq. (34), the information channel capacity $I(\vartheta)$, (33), of the *measurement channel* (ϑ) is equal to *the Fisher information* $I_F(\vartheta)$ *of the parameter* ϑ,

$$I(\vartheta) = I_F(\vartheta), \qquad (37)$$

where the *analytical form* (31) of the (expected) Fisher information on parameter ϑ is equal to

$$\begin{aligned} I_F(\vartheta) &= \sum_{a=-}^{+} \sum_{b=-}^{+} P(S_{ab}|\vartheta)\, \mathrm{i^F}_{ab}(\vartheta) \\ &= \sum_{ab} P(S_{ab}|\vartheta) \left(-\frac{\partial^2 \ln P(S_{ab}|\vartheta)}{\partial \vartheta^2} \right) \\ &\equiv \sum_{ab} i_{ab} = \sum_{ab} \left[-\frac{\partial^2 P(S_{ab}|\vartheta)}{\partial \vartheta^2} + (q'_{ab})^2 \right] \\ &= \sum_{ab} \left(-q_{ab} q''_{ab} + \frac{\partial^2 P(S_{ab}|\vartheta)}{\partial \vartheta^2} \right), \end{aligned} \qquad (38)$$

where Eq. (36) and the denotations $\sum_{ab} \equiv \sum_{a=-}^{+} \sum_{b=-}^{+}$, $q'_{ab} \equiv \frac{dq_{ab}(\vartheta)}{d\vartheta}$, and $q''_{ab} \equiv \frac{d^2 q_{ab}(\vartheta)}{d\vartheta^2}$ have been used. In the last equality the relation

$$\frac{\partial^2 P(S_{ab}|\vartheta)}{\partial \vartheta^2} = \frac{1}{2} (q'_{ab})^2 + \frac{1}{2} q_{ab}\, q''_{ab} \tag{39}$$

was also applied. Due to the normalization, (5), and the regularity condition [see Eq. (32)],

$$\sum_{ab} \frac{\partial^2 P(S_{ab}|\vartheta)}{\partial \vartheta^2} = \frac{\partial^2}{\partial \vartheta^2} \sum_{ab} P(S_{ab}|\vartheta) = \frac{\partial^2}{\partial \vartheta^2} 1 = 0, \tag{40}$$

the *analytical form* (38) of the Fisher information transforms into the following *metric form*:

$$\begin{aligned} I_F(\vartheta) &= \sum_{ab} P\,(S_{ab}\,|\vartheta\,)\, \widetilde{\mathrm{i}^{\mathrm{F}}}_{ab}(\vartheta) \\ &= \sum_{ab} \frac{1}{P\,(S_{ab}\,|\vartheta\,)} \left(\frac{\partial P(S_{ab}|\vartheta)}{\partial \vartheta} \right)^2 = \sum_{ab} (q'_{ab})^2. \end{aligned} \tag{41}$$

The *analytical form* of the observed Fisher information $\mathrm{i}^{\mathrm{F}}(\vartheta)$ in Eq. (38) differs from its *metric form* $\widetilde{\mathrm{i}^{\mathrm{F}}}(\vartheta)$ in Eq. (41) by $\frac{-1}{P(S_{ab}|\vartheta)} \frac{\partial^2 P(S_{ab}|\vartheta)}{\partial \vartheta^2}$. However, also due to Eq. (40) and in accordance with Eq. (37), we see that both the *EPI method form* of the (expected) Fisher information for the EPR problem and its single-value-ϑ information channel capacity for the measurement channel (ϑ) are equal to

$$I(\vartheta) = I_F(\vartheta) = -\sum_{ab} q_{ab}(\vartheta) q''_{ab}(\vartheta). \tag{42}$$

Remark. Thus, $I_F(\vartheta)$ is the information about the unknown angle ϑ confined in the $N = 1$-dimensional sample for the random variable S_{ab} (which is the pair of spin projections for particles 1 and 2 of the bipartite system).

The total information capacity I for the parameter ϑ. As the angle ϑ is the parameter whose value can change continuously in the interval $\langle 0, 2\pi)$, in accordance with Eq. (42), there are an infinite number of channels for which the Fisher information on ϑ can be calculated. To handle such a situation, the EPI method uses single, scalar information [2,3] (let us denote it simply I) called the (total) information channel capacity. This quantity is constructed by summing all of the possible single-value-ϑ information channel capacities. Thus, first, the information channel capacity $I\,(\vartheta_k)$ of one channel (ϑ_k), where $\vartheta = \vartheta_k$, is given by Eq. (42), and second, the summation runs over all of the values of ϑ_k. As a result of this summing we obtain the (*total*) information channel capacity I for the parameter $\vartheta \in V_\vartheta = \langle 0, 2\pi)$:

$$\begin{aligned} I &\equiv \sum_k I\,(\vartheta_k) \\ &\rightarrow I = \int_0^{2\pi} I(\vartheta)\, d\vartheta = -\sum_{ab} \int_0^{2\pi} q_{ab}(\vartheta)\, q''_{ab}(\vartheta)\, d\vartheta, \end{aligned} \tag{43}$$

where the integration appears by the reason of the substitution of the summation over the discrete index k by the integration over the continuous set of values of the parameter ϑ. This means that after determining what the single k channel connected with ϑ is, we perform the integration, which runs in accord with Eq. (6) from 0 to 2π in order to obtain the total information channel capacity. The single-value-ϑ information channel capacity (42) has been used in the last equality. *The information channel capacity I is the one which enters into the estimation procedure of the* EPI *method*. The derivation of the form of the amplitude q_{ab} as the solution of the information principles was presented in [3]; however, the sample space for $I_F(\vartheta)$ and the definition of q_{ab} are different in [3] and [33].

IV. THE INFORMATION PRINCIPLES AND GENERATING EQUATION

A. The general form of the information principles

In [9,15] the existence of the (total) *physical information K*,

$$K = I + Q \geqslant 0, \tag{44}$$

was postulated (see also Sec. I on the Frieden-Soffer original form of the physical information and information principles). The choice of the intuitive condition $K \geqslant 0$ is connected with the *expected structural information principle* of the EPI method,

$$I + \kappa\, Q = 0, \tag{45}$$

derived for $\kappa = 1$ in [6], where κ is the so-called efficiency coefficient [3]. The general form of $I(\Theta)$ was given previously by Eq. (30). Here, Q is the *structural information*, whose general form is as [6]

$$Q = \int_{\mathcal{B}} dy\, q = \sum_{n=1}^{N} \int_{\mathcal{Y}} d\mathbf{y}_n\, p_n(\mathbf{y}_n|\theta_n)\, \mathrm{q^F}_n(q_n(\mathbf{y}_n)), \tag{46}$$

where $\mathrm{q^F}_n(q_n(\mathbf{y}_n)) \equiv \mathrm{q^F}_{nn}(q_n(\mathbf{y}_n))$ is the *observed structural information* [6] under the assumption that the variables Y_n of the sample $\widetilde{Y}$ are independent and $p_n(\mathbf{y}_n|\theta_n)$ does not depend on the parameter θ_m for $m \neq n$ (see the text at the beginning of Sec. III). Then the *observed structural information matrix* is equal to $\mathrm{q^F} = \mathrm{diag}(\mathrm{q^F}_n\,(q_n(\mathbf{y}_n)))$. The quantity $q := P(\Theta) \sum_{n=1}^{N} \mathrm{q^F}_n(q_n(\mathbf{y}_n))$ in Eq. (46) is the *structural information density* [4,6,7]. The form of the information principle, which is more fundamental than (45), is the *observed structural information principle* that has the form $\mathrm{q^F} + \mathrm{i^F} = 0$ (derived for the EPR-Bohm problem in the following section). This follows from the analyticity of the logarithm of the likelihood function, which allows for its Taylor expansion in the neighborhood of the true value of the vector parameter $\Theta \equiv (\theta_n)_{n=1}^{N}$ [6] and in information densities it reads [4,6]

$$i + \kappa\, q = 0, \tag{47}$$

where i and q are the information channel density [see Eq. (30)] and structural information density, respectively. It has to be stressed that it is the *(modified) observed structural information principle* (and not the expected one), which is the one that is solved self-consistently together with the *variational information principle* [3,4],

$$\delta(I + Q) = 0, \tag{48}$$

which for the EPR-Bohm problem is introduced in Eq. (63).

[*The modified observed structural information principle*. The observed structural information principle (47) and the modified observed structural information principle $\widetilde{\imath} + C + \kappa\, q = 0$ are connected, and they are equivalent under the integral [in the sense that both of them lead to the expected structural information principle (45)], i.e.,

$$\int_{\mathcal{B}} dy\, (\widetilde{\imath} + C + \kappa\, q) = 0 \Leftrightarrow \int_{\mathcal{B}} dy\, (i + \kappa\, q) = 0 \\ \Rightarrow I + \kappa\, Q = 0, \qquad (49)$$

where the integration is over the entire sample space $\mathcal{B}$ and C is a constant. The transition from the second to the first integral in Eq. (49) is due to Eq. (32) and other transformations, which are equivalent under the integral [3,4]].

B. The information principles for the EPR-Bohm problem

1. The structural information principle

After the assignments (34) and integration over all of the possible values of ϑ [similar to that in Eq. (43) for I], the general form of the structural information Q given by Eq. (46) results in the following form of the (total) structural information in the EPR-Bohm problem for the system described by the set of amplitudes q_{ab}:

$$Q \equiv \frac{1}{4} \sum_{ab} \int_0^{2\pi} q_{ab}^2(\vartheta)\, \mathrm{q^F}_{ab}(q_{ab})\, d\vartheta. \qquad (50)$$

Now the physical information K, (44), in the EPR-Bohm problem is [3,6]

$$K = I + Q = \sum_{ab} \int_0^{2\pi} k_{ab}(\vartheta)\, d\vartheta, \qquad (51)$$

where I is given by Eq. (38). Let us notice that with this general understanding of K, the diversity of the equations of the EPI method is a consequence of the diverse preconditions dictated by physics. These could be, e.g., the continuity equation (which by itself is the result of a statistical estimation [4,6]) and some symmetries that are characteristic for the phenomena and the normalization conditions [3,34].

In Eq. (51), $k_{ab}(\vartheta)$ is the *density of the physical information*, which according to Eqs. (38), (50), and (39) is equal to [35]

$$k_{ab}(\vartheta) = -\tfrac{1}{2}\, q_{ab} q_{ab}'' + \tfrac{1}{2}(q_{ab}')^2 + \tfrac{1}{4}\, q_{ab}^2\, \mathrm{q^F}_{ab}(q_{ab}) \\ = -\tfrac{1}{2}\, q_{ab} q_{ab}'' + \tfrac{1}{4}\, q_{ab}^2\, \widetilde{\mathrm{q}}^{\mathrm{F}}_{ab}(q_{ab}) \\ \text{for every } S_{ab} \in \Omega_{ab}, \qquad (52)$$

where the *modified observed structural information* $\widetilde{\mathrm{q}}^{\mathrm{F}}_{ab}$ used in the EPI method has been introduced:

$$\widetilde{\mathrm{q}}^{\mathrm{F}}_{ab}(q_{ab}) := \frac{2}{q_{ab}^2(\vartheta)}(q_{ab}')^2 + \mathrm{q^F}_{ab}(q_{ab}). \qquad (53)$$

In what follows we see [compare Eq. (68)] that $\widetilde{\mathrm{q}}^{\mathrm{F}}_{ab}$ is free from the first derivative of q_{ab} for the EPR-Bohm problem, which means that $\frac{2}{q_{ab}^2(\vartheta)}(q_{ab}')^2$ cancels a term in $\mathrm{q^F}_{ab}(q_{ab})$. Let us suppose the analyticity of the log-likelihood function $\ln P(S_{ab}|\vartheta)$. After Taylor expanding $\ln P(S_{ab}|\tilde{\vartheta})$ around the true value of ϑ we obtain

$$\mathrm{q^F}_{ab}(P(S_{ab}|\vartheta))\, (\Delta\vartheta)^2 \\ \equiv 2\left[\ln \frac{P(S_{ab}|\tilde{\vartheta})}{P(S_{ab}|\vartheta)} - \frac{\partial \ln P(S_{ab}|\tilde{\vartheta})}{\partial \tilde{\vartheta}}\,\Big|_{\tilde{\vartheta}=\vartheta}\, \Delta\vartheta - R_3\right] \\ = \frac{\partial^2 \ln P(S_{ab}|\tilde{\vartheta})}{\partial \tilde{\vartheta}^2}\Big|_{\tilde{\vartheta}=\vartheta} (\Delta\vartheta)^2 \equiv -\mathrm{i^F}_{ab}(\vartheta)\, (\Delta\vartheta)^2, \\ \text{for every } S_{ab} \in \Omega_{ab}, \qquad (54)$$

where $\Delta\vartheta \equiv (\tilde{\vartheta} - \vartheta)$ and R_3 is the remainder of Taylor series. The observed structural information $\mathrm{q^F}_{ab}$ is defined by the left-hand side (LHS) of Eq. (54) and the right-hand side (RHS) is equal to $-\mathrm{i^F}_{ab}(\vartheta)\, (\Delta\vartheta)^2$, where $\mathrm{i^F}_{ab}(\vartheta)$ is the observed Fisher information on the parameter ϑ [6]. Let us note that after omitting $(\Delta\vartheta)^2$ on both sides, the observed structural information principle $\mathrm{i^F}_{ab} + \mathrm{q^F}_{ab} = 0$ [6] for the EPR-Bohm problem is obtained. This arises purely as a result of the analyticity of the log-likelihood function. After using the denotations of the kind $\frac{\partial^2 \ln P(\vartheta)}{\partial \vartheta^2} \equiv \frac{\partial^2 \ln P(\tilde{\vartheta})}{\partial \tilde{\vartheta}^2}\,|_{\tilde{\vartheta}=\vartheta}$ and $q_{ab}'(\vartheta) \equiv \frac{\partial q_{ab}(\tilde{\vartheta})}{\partial \tilde{\vartheta}}\,|_{\tilde{\vartheta}=\vartheta}$, next passing on the RHS of Eq. (54) from the derivative of $\ln P(S_{ab}|\tilde{\vartheta})$ to the one of $P(S_{ab}|\tilde{\vartheta})$ and using the relation (39) on the RHS of Eq. (54), we can rewrite this equation as follows:

$$\mathrm{q^F}_{ab}\, (\Delta\vartheta)^2 = \frac{1}{q_{ab}^2(\vartheta)}[2 q_{ab}(\vartheta) q_{ab}''(\vartheta) - 2(q_{ab}'(\vartheta))^2]\, (\Delta\vartheta)^2. \qquad (55)$$

Omitting $(\Delta\vartheta)^2$ on both sides, Eq. (55) can be rewritten in the form

$$\widetilde{\mathrm{q}}^{\mathrm{F}}_{ab}(q_{ab}) \equiv \left[\mathrm{q^F}_{ab}(q_{ab}) + \frac{1}{q_{ab}^2(\vartheta)}\, 2\, (q_{ab}'(\vartheta))^2\right] \\ = \frac{1}{q_{ab}^2(\vartheta)} 2\, q_{ab}(\vartheta) q_{ab}''(\vartheta), \qquad (56)$$

where the appearance of q_{ab}, (36), in the argument of $\mathrm{q^F}_{ab}$ means that the probability $P(S_{ab}|\tilde{\vartheta})$ (and its derivatives) present in $\mathrm{q^F}_{ab}$, which is defined by Eq. (54), has been replaced with the amplitude q_{ab} (and its derivatives). Now the forms of the amplitudes q_{ab} that are the solution to the EPR-Bohm problem are sought among the sin and cos trigonometric functions. Thus, because of the form of the RHS of the above equation, terms with the first derivative $q_{ab}'(\vartheta)$ on its LHS also have to cancel each other.

The modified observational structural information principle. Let us rewrite Eq. (56) as

$$-2\, q_{ab}(\vartheta) q_{ab}''(\vartheta) + q_{ab}^2(\vartheta)\, \widetilde{\mathrm{q}}^{\mathrm{F}}_{ab}(q_{ab}) = 0 \\ \text{for every } S_{ab} \in \Omega_{ab}, \qquad (57)$$

which, because of moving the term $\frac{1}{2}\, (q_{ab}')^2$ in Eqs. (55) and (56) [and as a consequence in Eq. (52)] from the Fisher information part to the structural one, will be called the *modified observed structural information principle* of the EPR-Bohm problem. The LHS of Eq. (57) is (up to the factor $\frac{1}{4}$) the density of the physical information $k_{ab}(\vartheta)$ given by Eq. (52). This one is the function of the observed structural information $\mathrm{q^F}_{ab}(q_{ab})$ [which at most can be the function of the amplitudes $q_{ab}(\vartheta)$], of the amplitudes themselves $q_{ab}(\vartheta)$, and of their second derivatives.

The efficiency factor in the EPR-Bohm problem. As we mentioned in Sec. IV A, in general, the efficiency factor κ appears before the density of the structural information q, which in the EPR-Bohm case is equal to $q = \frac{1}{4}\, q_{ab}^2 \widetilde{\mathrm{q}}^{\mathrm{F}}_{ab}(q_{ab})$.

In the EPR-Bohm case $\kappa = 1$ [3]; the value $\kappa = 1$ follows from the fact that except for the information principles, no additional differential constraints are put upon the amplitudes q_{ab}. Thus, the presented EPI model is a pure analytic one [6].

Using Eq. (52) the physical information K, (51), takes the following form:

$$K = I + Q = \sum_{ab} \int_0^{2\pi} \left[-\frac{1}{2} q_{ab} q''_{ab} + \frac{1}{4} q_{ab}^2(\vartheta)\, \widetilde{\mathrm{q}}^{\mathrm{F}}{}_{ab}(q_{ab}) \right] d\vartheta. \quad (58)$$

From Eq. (57) the expected structural information principle [see Eq. (45)], for $\kappa = 1$, follows:

$$I + Q = 0, \quad (59)$$

where $I + Q$ is given by the RHS of Eq. (58).

The differential equation (57) is the first one from the information principles used in the EPI method. The second one is the variational information principle, obtained below [3,6,9].

2. The variational information principle

In order to obtain the variational information principle, we have to transform the physical information K, (58), into the *metric form*, i.e., the one quadratic in q'_{ab}. Therefore, after integration by parts, K can be rewritten as

$$K = I + Q = \sum_{ab} \int_0^{2\pi} \left[k_{ab}^{met}(\vartheta) - \frac{\mathrm{C}_{ab}}{2} \right] d\vartheta, \quad (60)$$

where the constant C_{ab} is equal to

$$\mathrm{C}_{ab} = \frac{1}{2\pi} [q_{ab}(2\pi) q'_{ab}(2\pi) - q_{ab}(0) q'_{ab}(0)] \quad (61)$$

and $k_{ab}^{met}(\vartheta)$ is the *metric form* of the density of the physical information:

$$k_{ab}^{met}(\vartheta) = \tfrac{1}{2}\, q'^2_{ab} + \tfrac{1}{4}\, q_{ab}^2(\vartheta)\, \widetilde{\mathrm{q}}^{\mathrm{F}}{}_{ab}(q_{ab}). \quad (62)$$

The variational information principle [3,6,9] has the form

$$\delta_{(q_{ab})} K \equiv \delta_{(q_{ab})} (I + Q) = \delta_{(q_{ab})} \left\{ \sum_{ab} \int_0^{2\pi} \left[k_{ab}^{met}(\vartheta) - \frac{\mathrm{C}_{ab}}{2} \right] d\vartheta \right\} = 0. \quad (63)$$

The solution of the *variational* problem (63) with respect to q_{ab} is the *Euler-Lagrange equation*:

$$\frac{d}{d\vartheta} \left(\frac{\partial k_{ab}^{met}(\vartheta)}{\partial q'_{ab}(\vartheta)} \right) = \frac{\partial k_{ab}^{met}(\vartheta)}{\partial q_{ab}} \text{ for every } S_{ab} \in \Omega_{ab}. \quad (64)$$

From this equation and for $k_{ab}^{met}(\vartheta)$ as in Eq. (62), the following differential equation is obtained for every amplitude q_{ab}:

$$q''_{ab} = \frac{1}{2} \frac{d\left(\frac{1}{2} q_{ab}^2 \widetilde{\mathrm{q}}^{\mathrm{F}}{}_{ab}(q_{ab})\right)}{dq_{ab}} \quad \text{for every } S_{ab} \in \Omega_{ab}. \quad (65)$$

As $q_{ab}^2(\vartheta) \widetilde{\mathrm{q}}^{\mathrm{F}}{}_{ab}(q_{ab})$ is explicitly the function of q_{ab} only, the total derivative has replaced the partial derivative over q_{ab} present in Eq. (64).

The modified observed structural information principle (57) and the variational information principle (63) [from which the Euler-Lagrange equation (65) follows] serve for the derivation of the equation that generates the distribution.

3. The derivation of the generating equation

Using the relation (65) in Eq. (57), we obtain

$$\frac{1}{2} q_{ab} \frac{d\left(q_{ab}^2 \widetilde{\mathrm{q}}^{\mathrm{F}}{}_{ab}(q_{ab})\right)}{dq_{ab}} = q_{ab}^2 \widetilde{\mathrm{q}}^{\mathrm{F}}{}_{ab}(q_{ab}) \text{ for every } S_{ab} \in \Omega_{ab}. \quad (66)$$

Let us rewrite the above equation in a handier form,

$$\frac{2dq_{ab}}{q_{ab}} = \frac{d\left(\frac{1}{2} q_{ab}^2 \widetilde{\mathrm{q}}^{\mathrm{F}}{}_{ab}(q_{ab})\right)}{\frac{1}{2} q_{ab}^2 \widetilde{\mathrm{q}}^{\mathrm{F}}{}_{ab}(q_{ab})}, \quad (67)$$

from which, after integration on both sides, we obtain

$$\frac{1}{2} q_{ab}^2(\vartheta) \widetilde{\mathrm{q}}^{\mathrm{F}}{}_{ab}(q_{ab}) = \frac{q_{ab}^2(\vartheta)}{A_{ab}^2}, \quad (68)$$

$$\text{hence,} \quad \widetilde{\mathrm{q}}^{\mathrm{F}}{}_{ab}(q_{ab}) = \frac{2}{A_{ab}^2} \quad \text{for every } S_{ab} \in \Omega_{ab},$$

where the constants of integration A_{ab}^2 are in general complex numbers. This result was obtained previously in [3], but the arrival at the structural information principle is different in this paper and the form of both information principles also differs slightly.

The generating equation. By substituting Eq. (68) into Eq. (65), we obtain the searched for differential *generating equation* for the amplitudes q_{ab} [3],

$$q''_{ab}(\vartheta) = \frac{q_{ab}(\vartheta)}{A_{ab}^2} \text{ for every } S_{ab} \in \Omega_{ab} \text{ and } \vartheta \in V_\vartheta, \quad (69)$$

which is the consequence of both information principles, the structural and variational ones.

The solution of the generating equation. As the amplitude q_{ab} is the real one, thus A_{ab}^2 has also to be real and it can be displayed with the aid of the other real constant a_{ab} as $A_{ab} = a_{ab}$ or $A_{ab} = i\, a_{ab}$ [3], where here i is the imaginary unit. Therefore, there are two classes of solutions for Eq. (69). For $A_{ab} = a_{ab}$, the solution of Eq. (69) is purely of an *exponential* character [3]:

$$q_{ab}(\vartheta) = B''_{ab} \exp\left(-\frac{\vartheta}{a_{ab}}\right) + C''_{ab} \exp\left(\frac{\vartheta}{a_{ab}}\right) \quad \text{for every } S_{ab} \in \Omega_{ab}, \quad (70)$$

where the B''_{ab} and C''_{ab} constants are real. For A_{ab}, which is a purely imaginary number,

$$A_{ab} = i\, a_{ab}, \quad (71)$$

we obtain the solution of Eq. (69) that is purely of a *trigonometric* character [3],

$$q_{ab}(\vartheta) = B_{ab} \sin\left(\frac{\vartheta}{a_{ab}}\right) + C_{ab} \cos\left(\frac{\vartheta}{a_{ab}}\right) \quad \text{for every } S_{ab} \in \Omega_{ab}, \quad (72)$$

where a_{ab}, B_{ab}, C_{ab} are the real constants.

The invariance under the rotation. The possible values of the angle ϑ between measuring devices range from $\langle 0, 2\pi)$ (see Fig. 1). The physical periodicity is also inferred from the geometrical symmetry of the measuring system under the rotation of the angle 2π. Thus, the distribution $P(S_{ab}|\vartheta) = \frac{1}{4}\, q_{ab}^2(\vartheta)$, (36), and every amplitude $q_{ab}(\vartheta)$ are also periodic functions of ϑ. Therefore, from the solutions (70) and (72), we choose only the one which has a *trigonometric character* [in fact, we did it below Eq. (56)]. Next, the functions sin and cos in Eq. (72) form the basis for the probability amplitudes $q_{ab}(\vartheta)$. As $q_{ab}(\vartheta)$ is determined on the parameter space $V_\vartheta = \langle 0,2\pi)$, *the orthogonality condition of the base functions*,

$$\int_0^{2\pi} \sin\left(\frac{\vartheta}{a_{ab}}\right)\cos\left(\frac{\vartheta}{a_{ab}}\right) d\vartheta = \frac{a_{ab}}{2}\sin^2\left(\frac{2}{a_{ab}}\pi\right) = 0, \tag{73}$$

gives the form of the constants a_{ab} [3]

$$a_{ab} = \frac{2}{n_{ab}}, \quad \text{where} \quad n_{ab} = \pm 1 \text{ or } \pm 2 \text{ or } \ldots \quad \text{for every} \quad S_{ab} \in \Omega_{ab}, \tag{74}$$

and in Sec. V A we see that only $n_{ab} = \pm 1$ (for every $S_{ab} \in \Omega_{ab}$) or ± 2 (for every $S_{ab} \in \Omega_{ab}$) are permitted where the "plus" or "minus" signs correspond to the right-handed or left-handed polarization, respectively, of two final particles in the bipartite system.

The requirement of the orthogonality of the amplitudes $q_{++}(\vartheta)$ and $q_{+-}(\vartheta)$, (72), is not invoked here. From Sec. V B, it follows that the orthogonality of $q_{++}(\vartheta)$ and $q_{+-}(\vartheta)$ arises afterwards from the condition of the regularity of the probability distribution.

The condition of the minimal capacity I. The orthogonality condition (73) is the one that gives the possible values of a_{ab}, but its solution (74) permits their infinite sequence. Therefore, in [2,3] the condition of the minimal value of the information (kinematical) channel capacity $I \to \min$ is postulated as the one that, first, in accordance with the *additive* form (33) of $I > 0$, fixes the value of N to 1 and, second, also fixes the value of n_{ab} in a unique way [3]. However, in the present paper the second consecutive value of I is also analyzed (see Sec. V).

[Sometimes the nonminimal values of I are also discussed as they lead to the EPI method's models, which are of a physical significance. For example, the information principles of the EPI method analyzed in the realm of classical statistical physics [2,3] led (for the space component of the four-momentum vector) to the Maxwell-Boltzmann velocity law [3], in which case the minimal I appears for $N = 1$, whereas for $N > 1$, nonequilibrium, stationary solutions were obtained (that otherwise follow from the Boltzmann transport equation) [2,3]. For the time component of the four-momentum vector and with $N = 1$ which minimizes I, the Boltzmann probability law of the equipartition of energy in the form (70) was also obtained [2,3]].

Thus, in the present paper $N = 1$ and the consecutive values of I are obtained for increasing values of $|n_{ab}|$ (Sec. V).

Now, the generating equation (69) allows q''_{ab} to be eliminated, thus giving the useful form of the information channel capacity (42):

$$I = -\sum_{ab}\frac{1}{A_{ab}^2}\int_0^{2\pi} q_{ab}^2(\vartheta)\, d\vartheta . \tag{75}$$

The integral in Eq. (75) is calculated as

$$\begin{aligned}\int_0^{2\pi} q_{ab}^2(\vartheta)\, d\vartheta &\equiv 4\int_0^{2\pi} P(S_{ab}\,|\vartheta)\, d\vartheta = 4\int_0^{2\pi}\frac{P(S_{ab},\vartheta)}{r(\vartheta)}\, d\vartheta \\ &= 8\pi\int_0^{2\pi} P(S_{ab},\vartheta)\, d\vartheta = 8\pi P(S_{ab}) \\ &= 8\pi\,\frac{1}{4} = 2\pi,\end{aligned} \tag{76}$$

where Eqs. (36), (8), (10), (7), and (24), respectively, have been used in the successive equalities.

Using Eqs. (75) and (76) and $A_{ab} = i\, a_{ab}$, (71), together with Eq. (74), the information channel capacity I can be expressed via the constants n_{ab}, giving

$$\begin{aligned} I &= -Q = -\sum_{ab}\frac{1}{A_{ab}^2}\int_0^{2\pi} q_{ab}^2 d\vartheta = 2\pi\sum_{ab}\frac{1}{a_{ab}^2} \\ &= \frac{\pi}{2}\sum_{ab} n_{ab}^2 \quad \text{where} \quad n_{ab} = \pm 1 \text{ or } \pm 2 \text{ or } \ldots, \\ &\quad \text{for every} \quad S_{ab} \in \Omega_{ab}. \end{aligned} \tag{77}$$

Here, in the first of the above equalities, the relation $Q = -I$, which was obtained from the expected structural information principle (59), has been used.

V. THE FORMS OF THE AMPLITUDE

A. The first and second minimal *I*

Let us recall only that in the entire EPI-Bohm problem analyzed in this paper, the inner sample taken by the system alone is a $N = 1$-dimensional one (see the paragraph on *The condition of the minimal capacity I* just above).

1. The spin-$\frac{1}{2}$ with $n_{ab} = \pm 1$ case of minimal I

From the above condition, it follows that the minimization condition for I will be fulfilled when [3]

$$n_{ab} = \pm 1 \text{ (for every } S_{ab} \in \Omega_{ab}) \Rightarrow I \to \text{minimal}, \tag{78}$$

for arbitrary S_{ab}. According to Eq. (74), the condition $n_{ab} = \pm 1$ corresponds to the following values of a_{ab}:

$$a_{ab} = \pm 2 \quad \text{for every} \quad S_{ab} \in \Omega_{ab}. \tag{79}$$

The summation in Eq. (77) runs over all pairs S_{ab} of the spin projections for particles 1 and 2, (3); thus, for $n_{ab} = \pm 1$, we obtain the minimal value of I equal to

$$I_{(\min)} = 2\pi \quad \text{for} \quad n_{ab} = \pm 1. \tag{80}$$

Thus, the minimal value of the information channel capacity for the parameter ϑ is obtained. As the amplitude given by Eq. (72) for $a_{ab} = \pm 2$ has the form

$$q_{ab}(\vartheta) = B_{ab}\sin\left(\pm\frac{\vartheta}{2}\right) + C_{ab}\cos\left(\pm\frac{\vartheta}{2}\right), \tag{81}$$

we notice that $q_{ab}(\vartheta + 2\pi) = -q_{ab}(\vartheta)$ and its period is equal to $T_{1/2} = 4\pi$. This means that the solution with $n_{ab} = \pm 1$ is characteristic for the two-dimensional representation of the rotation group operator to which the spin-$\frac{1}{2}$ particles belong. Below we see that the EPR-Bohm problem formulas on the probabilities for the case of the bipartite system of two spin-$\frac{1}{2}$ particles are consistent with this finding [3]. Finally, $n_{ab} = +1$ corresponds to the right-handed polarization of two final electrons and similarly $n_{ab} = -1$ corresponds to the left-handed polarization of two final electrons.

2. The spin-1 with $n_{ab} = \pm 2$ case of second minimal I

The second smallest value of I is obtained for

$$n_{ab} = \pm 2 \;\; (\text{for every} \;\; S_{ab} \in \Omega_{ab}) \Rightarrow I \rightarrow \;\; \text{second minimal}, \tag{82}$$

for arbitrary S_{ab}. According to Eq. (74), the condition $n_{ab} = \pm 2$ corresponds to the following values of a_{ab}:

$$a_{ab} = \pm 1 \quad \text{for every} \;\; S_{ab} \in \Omega_{ab}. \tag{83}$$

The summation in Eq. (77) runs over all pairs S_{ab} of the spin projections for particles 1 and 2, (3), and therefore for $n_{ab} = \pm 2$, we obtain the second minimal ($s.min$) value of the information channel capacity I for the parameter ϑ, which is equal to

$$I_{(s.min)} = 8\pi \quad \text{for} \quad n_{ab} = \pm 2. \tag{84}$$

As the amplitude (72) for $a_{ab} = \pm 1$ has the form

$$q_{ab}(\vartheta) = B_{ab} \sin(\pm\vartheta) + C_{ab} \cos(\pm\vartheta); \tag{85}$$

thus, $q_{ab}(\vartheta + 2\pi) = q_{ab}(\vartheta)$ and the period of the amplitude under the rotation is equal to $T_1 = 2\pi$. Thus, the solution with $n_{ab} = \pm 2$ is characteristic for the bipartite system of two spin-1 particles, which belong to the three-dimensional representation of the rotation group operator. Below, we see that the EPR-Bohm problem formulas on the probabilities for the case of the bipartite system with spin-1 particles are also consistent with this finding. Finally, $n_{ab} = +2$ corresponds to the right-handed polarization of two final photons and, similarly, $n_{ab} = -2$ corresponds to the left-handed polarization of two final photons.

It follows from the above considerations that for $|n_{ab}| > 2$ the basic period of the amplitudes q_{ab} is smaller than 2π, which would be characteristic for bipartite systems of spin-1 massive particles or those higher than spin-1 particles. However, then the base space is different than Ω_{ab} given by Eq. (3) [see also the text below Eq. (3)]. Therefore, we conclude that the EPI method analysis for Ω_{ab} given by Eq. (3) permits $n_{ab} = \pm 1$ or $n_{ab} = \pm 2$ only. We discuss this splitting in Sec. V B5. Finally, from Eqs. (72) and (74), we see that the amplitudes q_{ab} have the form

$$q_{ab}(\vartheta) = B_{ab} \sin\left(n_{ab}\frac{\vartheta}{2}\right) + C_{ab} \cos\left(n_{ab}\frac{\vartheta}{2}\right), \quad \text{where} \;\; n_{ab} = \pm 1 \text{ or } \pm 2 \;\; \text{for every} \;\; S_{ab} \in \Omega_{ab} \;\; \text{and} \;\; \vartheta \in V_{\vartheta}. \tag{86}$$

B. The determination of the constants of amplitudes and the Rao-Fisher metric

In the amplitudes (86) [or particularly in (81) or (85)], there are the constants B_{ab} and C_{ab}, which have to be determined.

1. The restrictions from pure boundary conditions

In order to perform the task, we appeal at first to the values of the joint conditional probabilities $P(S_{ab}|\vartheta)$ for $\vartheta = 0$, which was previously determined in Eq. (11) [3]. According to Eq. (36), we know that $P(S_{ab}|\vartheta) = \frac{1}{4} q_{ab}^2$, and thus it follows that

$$q_{ab}(\vartheta) = 0 \quad \text{if only} \quad P(S_{ab}|\vartheta) = 0. \tag{87}$$

Therefore, using Eq. (87) we see that for both the amplitudes given by Eq. (81) for the bipartite system of the spin-$\frac{1}{2}$ particles and for the amplitudes given by (85) for the bipartite system of the spin-1 photons, the boundary condition (11) in $\vartheta = 0$, $P(++|0) = P(--|0) = 0$, leads to [3]

$$C_{++} = C_{--} = 0. \tag{88}$$

Moreover, by appealing to the geometric symmetry of the experiment, $P(S_{+-}|\vartheta) = P(S_{-+}|\vartheta)$ and $P(S_{++}|\vartheta) = P(S_{--}|\vartheta)$, given by Eq. (17), we obtain [3]

$$B_{++} = B_{--} \quad \text{and} \quad B_{+-} = B_{-+}, \tag{89}$$

and

$$C_{+-} = C_{-+}. \tag{90}$$

2. Restrictions from the regularity condition

After using Eqs. (88)–(90) and Eqs. (36) and (86), the normalization condition (5) reads

$$\sum_{ab} P(S_{ab}|\vartheta) = \frac{1}{2}\left[B_{+-} \sin\left(\frac{n_{ab}\vartheta}{2}\right) + C_{+-} \cos\left(\frac{n_{ab}\vartheta}{2}\right)\right]^2 + \frac{1}{2} B_{++}^2 \sin^2\left(\frac{n_{ab}\vartheta}{2}\right) = 1, \quad \text{where} \;\; n_{ab} = \pm 1 \text{ or } \pm 2 \;\; \text{for every} \;\; S_{ab} \in \Omega_{ab}. \tag{91}$$

From the above equation and the regularity condition (40), it follows that

$$\frac{\partial^2}{\partial\vartheta^2} \sum_{ab} P(S_{ab}|\vartheta) = \tfrac{1}{4} n_{ab}^2 [-2\, B_{+-} C_{+-} \sin(n_{ab}\vartheta) + (B_{++}^2 + B_{+-}^2 - C_{+-}^2) \cos(n_{ab}\vartheta)] = 0, \quad \vartheta \in V_{\vartheta}, \quad \text{where} \;\; n_{ab} = \pm 1 \text{ or } \pm 2 \;\; \text{for every} \;\; S_{ab} \in \Omega_{ab}. \tag{92}$$

This condition gives

$$B_{++}^2 = C_{+-}^2 \neq 0 \tag{93}$$

and [together with Eq. (89)]

$$B_{+-} = B_{-+} = 0, \tag{94}$$

as otherwise, i.e., for $C_{+-} = 0$, from the conditions (88)–(90) and Eq. (92), we obtain the zeroing of all coefficients B_{ab}

and C_{ab}, which corresponds to the trivial case of the lack of a solution to the EPR problem in the event of not measuring any of the spin projections (i.e., no EPR-Bohm experiment is occurring). Thus, for the physical *nontrivial solution* for the EPR-Bohm problem, we obtain

$$C \equiv C_{+-} = C_{-+} \neq 0, \tag{95}$$

where the equality of the coefficients follows from Eq. (90). Let us notice that because of Eq. (93), the condition (95), which is the condition of the existence of the nontrivial solution, means that

$$B \equiv B_{++} = B_{--} \neq 0, \tag{96}$$

where again the condition (89) is used in the equality. Finally, in accordance with Eqs. (93), (95), and (96) it follows that

$$B^2 = C^2 \neq 0. \tag{97}$$

Additionally, as was mentioned in Sec. III, the regularity condition that resulted in relation (92) enables the Fisher information to pass from the *analytical form* (38) to the *metric form* (41). Let us notice that the conditions (93) and (94) also saturate the equality: $\frac{\partial}{\partial\vartheta}\sum_{ab} P(S_{ab}|\vartheta) = 0$.

[*Frieden's quantum amplitudes*. In order to obtain condition (94), Frieden used another approach [3]. He introduced the "quantum amplitude" $\psi_{ab}(\vartheta) \propto q_{ab}(\vartheta)$ of the bipartite system, where $q^2_{ab}(\vartheta) = P(S_a|S_b,\vartheta)$ (see [33]). Then, by the Bayes' rule, it follows that $\psi^2_{ab}(\vartheta) \equiv \psi^2_{ab}(\vartheta|S_{ab}) \equiv p(\vartheta|S_{ab}) = \frac{P(S_a|S_b,\vartheta)P(S_b)r(\vartheta)}{P(S_a|S_b)P(S_b)}$ [3]. This equality means that with the appearance of the joint configuration of spins S_{ab}, the probability amplitude $\psi_{ab}(\vartheta)$ which says that the value of the angle is equal to ϑ is proportional to the probability amplitude $q_{ab}(\vartheta) = \sqrt{P(S_a|S_b,\vartheta)}$ [3] of observing the spin projection S_a of the particle 1 under the requirement that the spin projection of the particle 2 amounts to S_b and the angle is equal to ϑ. In quantum mechanics we would say that $\psi_{ab}(\vartheta)$ signifies the probability amplitude of the event that the value of the angle is equal to ϑ under the condition that a joint configuration S_{ab} of spins appears. Then, Frieden required *the orthogonality of quantum amplitudes* $\psi_{++}(\vartheta)$ and $\psi_{+-}(\vartheta)$ on $\langle 0,2\pi)$, from where the orthogonality of the amplitudes q_{++} and q_{+-} follows automatically. From this in [3] the zeroing of B_{+-} and B_{-+} that is seen in Eq. (94) follows].

3. Analysis of the Rao-Fisher metric

Below we convince ourselves that the determination of the constants B_{ab} and C_{ab} that were obtained in Secs. V B1 and V B2 leads to the constancy of the Rao-Fisher metric on the statistical (sub)space $\mathcal{S}$. The probability distribution of the EPR-Bohm problem (that we are looking for) is the discrete one (4). It is determined on the joint space Ω_{ab} of the possible results $S_a S_b \equiv S_{ab} \in \Omega_{ab} = \{+\,+, -\,-, +\,-, -\,+\}$, (3), and normalized to unity in accordance with (5). Let us express the double index ab in a compact form, i.e.,

$$ab = +\,+, -\,-, +\,-, -\,+ \quad \text{corresponds to} \quad j-1 \equiv a_b = 0,1,2,3, \tag{98}$$

respectively. The order of ab can be different than $+\,+$, $-\,-, +\,-, -\,+$.

The probability amplitudes related to the distribution (4) have the following form in accordance with Eq. (36):

$$\tilde{q}_{a_b} \equiv q_{ab}(\vartheta) = \pm\sqrt{4\,P(S_{ab}|\vartheta)}. \tag{99}$$

Note on the Rao-Fisher metric of the statistical (sub)space $\mathcal{S}$. For $\aleph$ results, the general relation that determines the induced Rao-Fisher metric on the submanifold of the $\aleph - 1$-dimensional probability simplex, which is coordinatized by a set of coordinates (θ^α), is as follows [30] [see Sec. VII, Eq. (A5)]:

$$g_{\alpha\beta} = \sum_{j=1}^{\aleph} \frac{\partial q_j}{\partial\theta^\alpha}\frac{\partial q_j}{\partial\theta^\beta}. \tag{100}$$

Using Eq. (86) we see that the amplitudes $q_{ab}(\vartheta)$ have the following derivatives:

$$\frac{\partial q_{ab}}{\partial\vartheta} = \frac{n_{ab}}{2}\left[B_{ab}\cos\left(\frac{n_{ab}\,\vartheta}{2}\right) - C_{ab}\sin\left(\frac{n_{ab}\,\vartheta}{2}\right)\right]. \tag{101}$$

Thus, for the statistical (sub)space $\mathcal{S}$, (6), coordinatized by ϑ and after making use of Eq. (99) and the above EPI result (101), the following form of the induced metric $g^{\vartheta\vartheta}$ on the statistical (sub)space $\mathcal{S}$ is obtained (see $I_F(\vartheta)$, (41), which, in fact, defines $g^{\vartheta\vartheta}$ on $\mathcal{S}$ [14]):

$$\begin{aligned} g^{\vartheta\vartheta}(\vartheta) &= \sum_{a_b=0}^{3}\frac{\partial\tilde{q}_{a_b}}{\partial\vartheta}\frac{\partial\tilde{q}_{a_b}}{\partial\vartheta} = \sum_{ab}\frac{\partial q_{ab}}{\partial\vartheta}\frac{\partial q_{ab}}{\partial\vartheta} \\ &= \frac{1}{4}\sum_{ab} n^2_{ab}\Big[B^2_{ab} - B_{ab}\,C_{ab}\sin(n_{ab}\,\vartheta) \\ &\quad + \left(C^2_{ab} - B^2_{ab}\right)\sin^2\left(\frac{n_{ab}\,\vartheta}{2}\right)\Big], \\ &\quad \text{where}\;\; n_{ab} = \pm 1 \text{ or } \pm 2 \;\;\text{for every}\;\; S_{ab} \in \Omega_{ab}. \end{aligned} \tag{102}$$

After using the conditions (88)–(90), which follow from the boundary conditions (i.e., without using the regularity condition as in Sec. V B2), the above equation reads

$$\begin{aligned} g^{\vartheta\vartheta}(\vartheta) &= n^2_{ab}\Big[\frac{1}{2}(B^2_{++} + B^2_{+-}) - \frac{1}{2}(B_{+-}\,C_{+-})\sin(n_{ab}\,\vartheta) \\ &\quad + \frac{1}{2}(C^2_{+-} - B^2_{++} - B^2_{+-})\sin^2\left(\frac{n_{ab}\,\vartheta}{2}\right)\Big], \\ &\quad \text{where}\;\; n_{ab} = \pm 1 \text{ or } \pm 2 \;\;\text{for every}\;\; S_{ab} \in \Omega_{ab}. \end{aligned} \tag{103}$$

Now, using the relations (93) and (94), which follow from the condition of regularity of the probability distribution, we obtain

$$g^{\vartheta\vartheta}(\vartheta) = \tfrac{1}{2}\,n^2_{ab}\,B^2, \quad \text{where}\;\; n_{ab} = \pm 1 \text{ or } \pm 2 \;\; \text{for every}\;\; S_{ab} \in \Omega_{ab}, \tag{104}$$

where the notation from Eq. (96) is used.

4. The normalization condition to unity

Inserting the results for the coefficients B_{ab} and C_{ab} that were obtained in Secs. V B1 and V B2 into Eq. (86), we obtain

$$q_{++}(\vartheta) = B \sin(n_{++}\,\vartheta/2), \quad q_{--}(\vartheta) = B \sin(n_{--}\,\vartheta/2),$$
$$q_{-+}(\vartheta) = C \cos(n_{-+}\,\vartheta/2), \quad q_{+-}(\vartheta) = C \cos(n_{+-}\,\vartheta/2),$$
$$\text{where} \;\; n_{ab} = \pm 1 \text{ or } \pm 2 \;\; \text{for every} \;\; S_{ab} \in \Omega_{ab}, \qquad (105)$$

with the condition $B^2 = C^2$ given in Eq. (97). From Eq. (105), it can noticed that the equality of the coefficients in relations (95) and (96) is a reflection of the equality of the corresponding amplitudes that follows from the symmetry of the space reflection quantified by Eq. (17) and from using Eq. (36). Let us observe the visible orthogonality of the amplitudes $q_{++}(\vartheta)$ and $q_{+-}(\vartheta)$. It has arisen as the result of the EPI method analysis and from the condition of the regularity of the probability distribution. It can also be noted from the resulting property of the constancy of Rao-Fisher metric on the statistical (sub)space $\mathcal{S}$ of the EPR-Bohm problem discussed in Sec. V B3.

However, we still have to determine the constants B and C. From the condition of the probability $P(S_a S_b|\vartheta)$ normalization to unity, (5), and using Eq. (36), we obtain the equation

$$\tfrac{1}{4}[q_{++}^2(\vartheta) + q_{--}^2(\vartheta) + q_{-+}^2(\vartheta) + q_{+-}^2(\vartheta)] = 1. \qquad (106)$$

Using Eq. (105) in Eq. (106) gives

$$2(C^2 - B^2)\cos^2(n_{ab}\,\vartheta/2) + 2\,B^2 = 4,$$
$$\text{where} \;\; n_{ab} = \pm 1 \text{ or } \pm 2 \;\; \text{for every} \;\; S_{ab} \in \Omega_{ab}. \qquad (107)$$

Comparing the coefficients that stand beside the appropriate functions of ϑ on the left- and right-hand-sides of the above expression, we obtain

$$B^2 = C^2 = 2, \qquad (108)$$

which, after putting it into Eq. (105), gives the final solution of the EPI method for the amplitudes of the bipartite system in the EPR-Bohm problem (compare [3]):

$$q_{++}(\vartheta) = \pm\sqrt{2}\,\sin(n_{++}\,\vartheta/2),$$
$$q_{--}(\vartheta) = \pm\sqrt{2}\,\sin(n_{--}\,\vartheta/2),$$
$$q_{-+}(\vartheta) = \pm\sqrt{2}\,\cos(n_{-+}\,\vartheta/2), \qquad (109)$$
$$q_{+-}(\vartheta) = \pm\sqrt{2}\,\cos(n_{+-}\,\vartheta/2),$$
$$\text{where} \;\; n_{ab} = \pm 1 \text{ or } \pm 2 \;\; \text{for every} \;\; S_{ab} \in \Omega_{ab}.$$

It is well known (see, e.g., [16]) that each of the EPR-Bohm solutions (109) can be uniquely decomposed into the inner products for either the antisymmetric or symmetric tensor product only of two one-particle states (final and initial of the detected particles), which are the electronic ones that are given by spinors or photonic ones given by vectors, respectively. What we have obtained is that the quantization of the spin projection on the particular direction in space, which was introduced as the boundary condition in Sec. II B1, results in the rediscovery of the dimension of the rotation group representation to which the particles belong, which is consistent with the EPI solution obtained for the bipartite system of two spin-$\frac{1}{2}$ particles (Secs. V A1 for $n_{ab} = \pm 1$) or for the bipartite system of two spin-1 photons (Sec. V A2 for $n_{ab} = \pm 2$). Each of the amplitudes of the EPR-Bohm problem is a point on the 3-sphere S^3 of radius 2 [see Eq. (106) and Sec. VII].

After the transformation $q_{ab} \rightarrow q_{ab}^{S^3} \equiv \frac{q_{ab}}{2}$ of the amplitudes, the sphere S^3 becomes the one of the radius 1. The isometry group of the invariant metric on this sphere is SO(4), which is isomorphic to the coset of the group product $SU(2) \times SU(2)/\mathbb{Z}_2$, where $\mathbb{Z}_2 = \{\mathbf{1}, -\mathbf{1}\}$ is the cyclic group of the order 2 [30].

Let us recall that the amplitudes (109) were obtained as the solution of the generating equation Eq. (69), which after using Eqs. (71) and (74) has the following form:

$$q_{ab}''(\vartheta) = -\frac{n_{ab}^2}{4}\,q_{ab}(\vartheta), \quad \text{where} \;\; n_{ab} = \pm 1 \text{ or } \pm 2$$
$$\text{for every} \;\; S_{ab} \in \Omega_{ab} \;\; \text{and} \;\; \vartheta \in V_\vartheta. \qquad (110)$$

Some comments on the possible connection of Eq. (110) with other physical equations are given in the Conclusion.

The result on the probability in the EPR-Bohm experiment. Putting the amplitudes (109) into the relation $P(S_{ab}|\vartheta) = \frac{1}{4}\,q_{ab}^2(\vartheta)$, (36), gives the joint probability of getting a particular combination of spins projections at a fixed value of the angle ϑ:

$$P(+\,+|\vartheta) = \tfrac{1}{2}\sin^2(n_{++}\,\vartheta/2),$$
$$P(-\,-|\vartheta) = \tfrac{1}{2}\sin^2(n_{--}\,\vartheta/2),$$
$$P(-\,+|\vartheta) = \tfrac{1}{2}\cos^2(n_{-+}\,\vartheta/2), \qquad (111)$$
$$P(+\,-|\vartheta) = \tfrac{1}{2}\cos^2(n_{+-}\,\vartheta/2),$$
$$\text{where} \;\; n_{ab} = \pm 1 \text{ or } \pm 2 \;\; \text{for every} \;\; S_{ab} \in \Omega_{ab},$$

which is also the prediction of quantum mechanics [16] both for the system of two spin-$\frac{1}{2}$ particles ($n_{ab} = \pm 1$) and also for the system of two spin-1 photons ($n_{ab} = \pm 2$).

5. The consistence of the Rao-Fisher metric on $\mathcal{S}$ and the one of the EPI method

By inserting Eq. (108) into Eq. (104), the precise form of the Rao-Fisher metric $g^{\vartheta\vartheta}$ is obtained:

$$g^{\vartheta\,\vartheta}(\vartheta) = g^{\vartheta\,\vartheta} = n_{ab}^2 = \text{const.} \quad \text{for} \quad \vartheta \in V_\vartheta,$$
$$\text{where} \;\; n_{ab} = \pm 1 \text{ or } \pm 2 \;\; \text{for every} \;\; S_{ab} \in \Omega_{ab}. \qquad (112)$$

As the statistical (sub)space $\mathcal{S}$ is one-dimensional, the only $g^{\vartheta\,\vartheta}$ component is the Fisher information $I_F(\vartheta)$ on the parameter ϑ [14]. Now, the equality of $I_F(\vartheta)$, (41), and $g^{\vartheta\vartheta}$ given by (the first line in) (102) is by no means trivial. This equality means that the Rao-Fisher metric $g^{\vartheta\vartheta} = I_F(\vartheta)$ on $\mathcal{S}$ was obtained dynamically by the EPI method, which uses the information principles: the modified observed structural one (57) and the variational one (65) described in Sec. IV B [4]. From the above description of the EPI method for the EPR-Bohm problem [3], we know that the Fisher information $I_F(\vartheta)$, (41), is connected with the intrinsic, $N = 1$-dimensional sampling performed by the system alone. It enters into the information channel capacity I [compare Eq. (37)], which (i.e., primarily, its *analytical form*) is then itself or *by means of its density* used in the EPI statistical nonparametric estimation. Thus, let us write the final form of the Fisher information on the parameter ϑ that is inherent for the EPI method and calculated in accordance with Eq. (42) with the amplitudes (109) that are the solution to the EPR-Bohm problem:

$$I_F(\vartheta) = -\sum_{ab} q_{ab}(\vartheta) q''_{ab}(\vartheta) = n_{ab}^2, \quad \text{where } n_{ab} = \pm 1 \text{ or } \pm 2 \text{ for every } S_{ab} \in \Omega_{ab}. \tag{113}$$

The result (113) follows from the EPI analysis of the (inner) $N = 1$-dimensional sampling of the particles spins in the devices a and b *by the bipartite system alone*. The obtained amplitudes q_{ab} given by Eq. (109) describe the whole configuration of the system; i.e., $P(S_{ab}|\vartheta) = \frac{1}{4} q_{ab}^2(\vartheta)$ given by Eq. (111) is the true probability distribution from which, in the further analysis described below, the data are taken by an experimentalist. Thus, Eq. (111) gives the theoretical distribution from which the data are generated during the sampling of the system *by the experimentalist from outside*, i.e., when the (outer) M-dimensional sample is taken, e.g., by two of them from devices a and b. We discuss the outer experiment in Sec. VI. The Rao-Fisher metric (112), which originates in Eq. (102) [and is the particular case of Eq. (100)], is openly related to the $\aleph = 4$ outcomes that can occur in devices a and b that are observed by, this time, the outer observer.

The equivalence of the analytical and metric models. Using Eq. (111), it can be checked that the *analytical form* $I_F(\vartheta)$ calculated from Eq. (38) is equal to the *metric* one calculated from Eq. (41) and to the *EPI method form* (42) [given in Eq. (113)]. Thus, on the expected level, the analytical and metric models are equivalent under the integral due to the regularity condition (40), although it is the full analytical model that is solved.

6. The generic property of the EPI solutions

Finally, let us notice that in the EPR-Bohm problem condition (112) means the constancy of the Rao-Fisher metric $g^{\vartheta\vartheta}(\vartheta)$, i.e., the independence of $g^{\vartheta\vartheta}$ on the statistical (sub)space $\mathcal{S}$ on the value of the parameter ϑ (see Comment at the end of this section). Its constancy follows, first, from the boundary conditions discussed in Sec. V B1 and, second, from the regularity condition used in Sec. V B2. However, strictly speaking, for the constancy of $g^{\vartheta\vartheta}(\vartheta)$ on $\mathcal{S}$, the normalization *to unity* used in Eq. (106) is not necessary. Thus, using Eq. (105) in Eq. (106), but relaxing the condition of the normalization to unity on the RHS of Eq. (106) (and leaving the normalization to a finite value instead) and using Eq. (97), we obtain

$$2\,B^2 \sin^2(n_{ab}\,\vartheta/2) + 2\,C^2 \cos^2(n_{ab}\,\vartheta/2) = 2\,B^2, \quad \text{where } n_{ab} = \pm 1 \text{ or } \pm 2 \text{ for every } S_{ab} \in \Omega_{ab}, \tag{114}$$

instead of Eq. (107). Thus, the normalization to unity in Eq. (91) would have the form $\sum_{ab} \tilde{P}(S_{ab}|\vartheta) = \frac{1}{2} B^2$ instead and would also produce the results (104) and (105) although without value 2 established in Eq. (108). Nevertheless, the ratio of $g^{\vartheta\vartheta}(\vartheta)$ given by Eq. (104) to $\sum_{ab} \tilde{P}(S_{ab}|\vartheta)$,

$$\frac{g^{\vartheta\vartheta}(\vartheta)}{\sum_{ab} \tilde{P}(S_{ab}|\vartheta)} = \frac{\frac{1}{2} n_{ab}^2 B^2}{\frac{1}{2} B^2} = n_{ab}^2, \quad \text{where } n_{ab} = \pm 1 \text{ or } \pm 2 \text{ for every } S_{ab} \in \Omega_{ab}, \tag{115}$$

is equal to the Fisher information (112), that was obtained with the normalization to unity. Now, the point is that this ratio is a generic property of the solutions [36] of the pair of differential equations (57) and (65), which are two information principles. This means that the ratio in Eq. (115) is conserved for any disturbance of $B^2 > 0$ from the value 2 (with the boundary conditions kept unchanged), where $B^2 = C^2$ (97).

In Secs. V A1 and V A2 we noticed that the EPI method analysis for Ω_{ab} given by Eq. (3) allows $n_{ab} = \pm 1$ or $n_{ab} = \pm 2$ only for the bipartite system, which transform according to the two- or three-dimensional representation of the rotation operator, respectively. This means that the statistical (sub)space $\mathcal{S}$ splits into two stable subspaces and each of these is the solution of the generating equation (110), or more precisely, the self-consistent solution of the pair of information principles, whose particular forms are determined by the observed structural information $\widetilde{\mathrm{q}^{\mathrm{F}}}_{ab}(q_{ab})$, (68), and as a consequence by its expected value given by Eq. (77):

$$Q = -2\pi, \quad n_{ab} = \pm 1 \quad \text{for spin } \tfrac{1}{2}, \tag{116}$$

$$Q = -8\,\pi, \quad n_{ab} = \pm 2 \quad \text{for spin 1}. \tag{117}$$

Comment. It is also easy to check that the statistical model $\mathcal{S}$ is Amari $\alpha = 0$ affine flat, where the $\alpha = 0$ Amari connection [14] is the Riemannian one with respect to the Rao-Fisher metric.

C. Discussion of the EPI result on the probabilities

In quantum mechanics the relations (111) for the bipartite system of two spin-$\frac{1}{2}$ particles (here electrons) are obtained [16] from the conservation of total spin (the orbital term is assumed to be zero) and the indistinguishability of the particles obeying the Fermi-Dirac statistics, which are therefore described by an antisymmetric state under their exchange. In the calculation of the amplitude for the process [which in [16] is the same as the one on Fig. 1(a)], the opposite spins of the initial particles of the bipartite system was taken also into account [16]. In the EPI formalism, relations (111) result from the self-consistent solution of the modified observed structural information principle (57) and the variational one (65). Thus, the Fermi-Dirac statistics seems to be a reminiscence of these information principles, that is, of their solution, which is the generating equation (110), and the precisely defined boundary conditions. Similarly, the relation (111) for two spin-1 photons in the final state of the process [see Fig. 1(b)] in quantum mechanics is obtained [16] due to the indistinguishability of the particles that obey the Bose-Einstein statistics. The opposite momenta of final photons also have to be taken into account. Thus, in the case of the spin-1 photons, the Bose-Einstein statistics (and not the Maxwell-Boltzmann one) also seems to be a reminiscence of the generating equation (110) (and thus of the modified observed structural and variational information principles) and the precise boundary conditions.

The EPI result (111) for the EPR-Bohm problem signifies that the effect of the dependance of the spins projections changes strongly with the value of ϑ. A comparison of Eq. (111) with Eqs. (15) and (26), in general, gives the inequality [3]

$$P(S_{ab}|\vartheta) \neq P(S_a|\vartheta)P(S_b|\vartheta) \;\; \text{for every } S_{ab} \in \Omega_{ab}, \tag{118}$$

which proceeds when the estimation of the probability distribution with the EPI method that uses the differential equations (57) and (65), occurs. That is, the inequality (118) arises in a situation in which the averaging over ϑ is not performed [3] and instead the final forms of the observed (microscopic) information principles are self-consistently solved. From the analysis in this paper, we see that the EPI method coverage of quantum mechanics is due to the *analytical form* (38) of the Fisher information on the parameter ϑ.

When the averaging over ϑ is performed instead, then the factorization appears. Indeed, in Eq. (24) the joint probability $P(S_{ab}) = 1/4$ is determined. On the other hand, from Eq. (16) we see that $P(S_b) = 1/2$ [and analogously from Eq. (26) $P(S_a) = 1/2$] and thus the relation follows:

$$P(S_{ab}) = P(S_a)\,P(S_b) \quad \text{for every} \quad S_{ab} \in \Omega_{ab}. \tag{119}$$

This condition means the independence of the spin projection variables S_a and S_b after the averaging over ϑ is performed [3], i.e., when the "third" (nonrandom) variable, which is ϑ, is under control and therefore its effect is eliminated.

The Ehrenfest analog. The above conclusion is in agreement with the Ehrenfest theorem from which it follows that in the case of averaging over angle ϑ, the entangled EPR-Bohm states again undergo the classical mechanics separation and therefore the relation (119) appears [3].

1. The hidden variables and the EPI result

Factorization (119) with probabilities depending additionally on a set ς of random hidden variables with values in a set Σ is used when the Bell inequalities [16] are derived. The local hidden variables (LHV) theories are put to the test with them.

However, due to the Fisher information properties, the theories fall into two classes because of the value of the dimension N of the sample [9]. For quantum mechanics, and also for classical field theory, N is finite (and the Fisher information is finite), whereas for classical mechanics (for pointlike particles), N is infinite (and the Fisher information is infinite), which means that quantum mechanics cannot be derived from classical mechanics [9]. Therefore, if hidden variables ς have the classical mechanical meaning then the possibility of deriving quantum mechanical probabilities from them is excluded and the factorization rule (119) that is used in the Bell tests could result from classical field theory. However, if classical theory, i.e., its equations of motion follow from the self-consistent (differential) information principles, then the probabilities that are obtained fulfill the relation (118) instead and the Bell-like inequalities cannot be constructed. Thus, it is possible that quantum mechanics can be covered by the EPI method modeling (see also Sec. II B1).

VI. THE UNCERTAINTY OF THE ESTIMATION OF THE ANGLE

Below, the uncertainty of the estimation of the angle ϑ for the bipartite system of two spin-$\frac{1}{2}$ particles [3] or two spin-1 photons is analyzed. The statistical analysis, which led to the EPR-Bohm result (111), was performed in accordance with the EPI postulates [2,3] outlined in the Introduction (unless it takes into account also the influence of the measuring devices [3]).

The basic one is the assumption that the system by itself performs the "inner" sampling of the configuration space and then, in accordance with the information principles, performs the estimation of the generation equation whose solutions, after taking into account the boundary conditions, are the amplitudes of the probability distributions. The dimension of the "inner" sample was chosen as equal to $N = 1$ [see Eq. (34)]. Four pairs (3) of the spin projection $S_a S_b \equiv S_{ab} \in \Omega_{ab}$ are possible and the probability distribution $P(S_{ab}|\vartheta)$ of the random variable S_{ab} was summarized by Eq. (111).

A. The likelihood of the outer sample

As mentioned earlier, there is also the "outer" sample of the dimension M, which is taken by the researcher. The distribution $P(S_{ab}|\vartheta)$ distinguished by $\vartheta \in V_\vartheta$ forms the statistical model $\mathcal{S}$, (6), which is displayed above by the EPI method [3]. For the established ϑ, the probabilities $\lambda_{a_b} \equiv P(S_{ab}|\vartheta)$, $S_{ab} \in \Omega_{ab}$, can be perceived as four parameters of the probability distribution on the space of events Ω_{ab}. [Since $\sum_{ab} P(S_{ab}|\vartheta) = 1$, three of them are independent.] Below, we use the renaming of the double index $ab = ++, --, +-, -+$ to $a_b = 0,1,2,3$, respectively [in accordance with Eqs. (3) and (98) and the text just below Eq. (98)].

Let us denote the values of S_{ab} as s_{a_b} and let the "new" random variable S take the values s_{a_b}. The lower index in s_{a_b} signifies the a_bth value of S. The probability distribution $P(\mathrm{s};\lambda)$ of this random variable is as follows:

$$P(\mathrm{s}_{a_b};\lambda) = \lambda_{a_b} \quad \text{for} \quad S = \mathrm{s} = \mathrm{s}_{a_b}, \quad \text{where} \quad 0 \leqslant a_b \leqslant 3. \tag{120}$$

Now, a statistical space $\mathcal{S}_\Lambda$ can be constructed:

$$\mathcal{S}_\Lambda = \left\{ P(\mathrm{s};\lambda) | \lambda \equiv (\lambda_{a_b})_{a_b=0}^{3} \in \Lambda \subset \mathbb{R}^4, \right. \left. \lambda_{a_b} \geqslant 0 \; (\forall\, a_b), \quad \sum_{a_b=0}^{3} \lambda_{a_b} = 1 \right\}. \tag{121}$$

Note on $g_\Lambda^{\vartheta\vartheta}$. The Rao-Fisher metric $g_\Lambda^{\vartheta\vartheta}$ on the submanifold (coordinatized by ϑ) of the statistical space $\mathcal{S}_\Lambda$ is equal to the Rao-Fisher metric (102) on the submanifold of the simplex of probabilities (111) coordinatized by ϑ, which is the model $\mathcal{S}$ given by Eq. (6) (see the Appendix); that is, it has one component that is equal in accordance with Eq. (112) [and Eq. (A9) in the Appendix] to $g^{\vartheta\vartheta} = n_{ab}^2 = \text{const.}$

The experimental distribution. In an experiment, the researcher obtains the M-dimensional sample with the frequencies $\hat{\lambda}_{a_b}$ of appearances of pairs of the spin projections S_{ab} (3). The frequencies $\hat{\lambda}_{a_b}$ are the *unbiased* and *consistent* estimators of the probabilities $\lambda_{a_b} \equiv P(S_{ab}|\vartheta)$ from Eq. (111), i.e., $E_{\lambda_{a_b}}[\hat{\lambda}_{a_b}] = \lambda_{a_b}$ and (for all $\varepsilon > 0$) $\lim_{M\to\infty} Pr_{\lambda_{a_b}}(|\hat{\lambda}_{a_b} - \lambda_{a_b}| > \varepsilon) = 0$, respectively.

In the above notation, the characteristics of the distributions of the estimators $\hat{\lambda}_{a_b}$ of the parameters λ_{a_b} ($a_b = 0,1,2,3$) for the M-dimensional sample,

$$\widetilde{S}_{(M)} \equiv (S^1, \ldots, S^i, \ldots, S^M), \qquad (122)$$

are calculated with the joint probability distribution,

$$P(\widetilde{S}_{(M)}; \lambda) = \prod_{i=1}^{M} P(S^i; \lambda), \qquad (123)$$

where the random variables S^i, $i = 1,2,\ldots,M$, are independent. The upper index in S^i signifies the ith data point in the sample $\widetilde{S}_{(M)} \equiv (S^1, \ldots, S^i, \ldots, S^M)$. The distribution $P(S^i; \lambda)$ of the variable S^i for each single data point $i = 1,2,\ldots,M$ is the same as the distribution $P(S; \lambda)$ of S that is given by Eq. (120). The asymptotic local unbiasedness of four estimators $\hat{\vartheta}$ of the parameter ϑ in the limit $M \to \infty$ is proven below.

B. The estimator $\hat{\vartheta}$

Usually in an experiment the angle ϑ is fixed. The frequencies $\hat{\lambda}_{a_b}$ are measured and only then are relations (111) tested (as, e.g., the signature of their supposed quantum mechanical origin). We saw that EPR-Bohm relations (111) appear as the result of the self-consistent solution of two, in their origin classical, statistical information principles with the conditions of regularity, (92), and normalization to unity, (106) (see Secs. V B2–V B4).

The statistical approach to the EPR-Bohm problem enables the implementation of all statistical techniques of the investigation of the properties of the estimators. Thus, let us discuss the inverse problem, i.e., the quality of the ϑ estimation via the frequencies $\hat{\lambda}_{a_b}$ of events $S_{ab} \in \Omega_{ab}$. The inverse of every function of the four given by Eq. (111) allows the angle ϑ to be expressed as depending on one probability λ_{a_b}, i.e., $\vartheta = \vartheta(\lambda_{a_b})$. We see that four estimators $\hat{\vartheta}$ of $\vartheta \in V_\vartheta$ can be built, one for each event $S_{ab} \in \Omega_{ab}$:

$$\begin{aligned} \hat{\vartheta}(\hat{\lambda}_{a_b}) &= \frac{2}{n_{ab}} \arcsin\left(\sqrt{2\,\hat{\lambda}_{a_b}}\,\right), \quad ab = +\,+ \text{ or } -\,-, \\ \hat{\vartheta}(\hat{\lambda}_{a_b}) &= \frac{2}{n_{ab}} \arccos\left(\sqrt{2\,\hat{\lambda}_{a_b}}\,\right), \quad ab = +\,- \text{ or } -\,+, \\ &\text{where } n_{ab} = \pm 1 \text{ or } \pm 2 \quad \text{for every } S_{ab} \in \Omega_{ab}. \end{aligned} \qquad (124)$$

It will be proven below that each of these four estimators $\hat{\vartheta}$ of the angle ϑ is asymptotically locally unbiased [14]; i.e.,

$$E_{\vartheta+\Delta\vartheta}(\hat{\vartheta}) = \vartheta + \Delta\vartheta + \tilde{o}(\Delta\vartheta) \qquad (125)$$

at every ϑ, where $\tilde{o}\,(\Delta\vartheta)$ is a small number of higher order in $\Delta\vartheta$. (The proof is the same for every estimator.)

To determine whether relation (125) really holds, let us Taylor expand the estimator $\hat{\vartheta}$, which is treated as a function of the frequency $\hat{\lambda}_{a_b}$ for each fixed $S_{ab} \in \Omega_{ab}$ around the corresponding point λ_{a_b}, respectively,

$$\begin{aligned} \hat{\vartheta}(\hat{\lambda}_{a_b}) &= \hat{\vartheta}(\lambda_{a_b}) + \left.\frac{\partial\hat{\vartheta}(\hat{\lambda}_{a_b})}{\partial\hat{\lambda}_{a_b}}\right|_{\lambda_{a_b}} (\hat{\lambda}_{a_b} - \lambda_{a_b}) + o(\Delta\hat{\lambda}_{a_b}) \\ &= \vartheta - \left.\frac{\partial\hat{\vartheta}(\hat{\lambda}_{a_b})}{\partial\hat{\lambda}_{a_b}}\right|_{\lambda_{a_b}} \lambda_{a_b} + \left.\frac{\partial\hat{\vartheta}(\hat{\lambda}_{a_b})}{\partial\hat{\lambda}_{a_b}}\right|_{\lambda_{a_b}} \hat{\lambda}_{a_b} + o(\Delta\hat{\lambda}_{a_b}), \end{aligned} \qquad (126)$$

where $\hat{\vartheta}(\lambda_{a_b}) = \vartheta$ and $o(\Delta\hat{\lambda}_{a_b})$ consists of the higher terms of the expansion, where $\Delta\hat{\lambda}_{a_b} \equiv (\hat{\lambda}_{a_b} - \lambda_{a_b})$. Up to the first order, only the statistic $\hat{\lambda}_{a_b}$ is present. Because $\hat{\lambda}_{a_b}$ is an unbiased estimator of λ_{a_b}, its expectation value at the parameter $\widetilde{\vartheta} = \vartheta + \Delta\vartheta$ is equal to $\lambda_{a_b|\vartheta+\Delta\vartheta} \equiv P(S_{ab}|\vartheta + \Delta\vartheta)$ and the Taylor expansion of $E_{\vartheta+\Delta\vartheta}(\hat{\lambda}_{a_b})$ around ϑ gives

$$E_{\vartheta+\Delta\vartheta}(\hat{\lambda}_{a_b}) = \lambda_{a_b|\vartheta+\Delta\vartheta} = \lambda_{a_b} + \left.\frac{\partial\lambda_{a_b|\widetilde{\vartheta}}}{\partial\widetilde{\vartheta}}\right|_{\vartheta} \Delta\vartheta + o(\Delta\vartheta), \qquad (127)$$

where $o\,(\Delta\vartheta)$ are the higher terms of the expansion.

Now, the lowest order term of the expectation value $E_{\vartheta+\Delta\vartheta}(o(\Delta\hat{\lambda}_{a_b}))$ of the last term $o(\Delta\hat{\lambda}_{a_b})$ in Eq. (126) is equal to

$$\begin{aligned} &E_{\vartheta+\Delta\vartheta}\left[\frac{1}{2}\left.\frac{\partial^2\hat{\vartheta}(\hat{\lambda}_{a_b})}{\partial\hat{\lambda}^2_{a_b}}\right|_{\lambda_{a_b}} (\Delta\hat{\lambda}_{a_b})^2\right] \\ &\quad = \frac{1}{2}\left.\frac{\partial^2\hat{\vartheta}(\hat{\lambda}_{a_b})}{\partial\hat{\lambda}^2_{a_b}}\right|_{\lambda_{a_b}} \sigma^2_{\vartheta+\Delta\vartheta}(\hat{\lambda}_{a_b}) \overset{M\to\infty}{\longrightarrow} 0, \end{aligned} \qquad (128)$$

where in the last line the asymptotic property of the variance of the frequency $\hat{\lambda}_{a_b}$ (in the four-nomial distribution) is used.

Let us take the expectations on both sides of Eq. (126):

$$\begin{aligned} E_{\vartheta+\Delta\vartheta}(\hat{\vartheta}(\hat{\lambda}_{a_b})) &= \vartheta - \left.\frac{\partial\hat{\vartheta}(\hat{\lambda}_{a_b})}{\partial\hat{\lambda}_{a_b}}\right|_{\lambda_{a_b}} \lambda_{a_b} + \left.\frac{\partial\hat{\vartheta}(\hat{\lambda}_{a_b})}{\partial\hat{\lambda}_{a_b}}\right|_{\lambda_{a_b}} \\ &\quad \times E_{\vartheta+\Delta\vartheta}(\hat{\lambda}_{a_b}) + E_{\vartheta+\Delta\vartheta}(o(\Delta\hat{\lambda}_{a_b})). \end{aligned} \qquad (129)$$

After using Eqs. (127) and (128) in Eq. (129) and noticing that $\frac{\partial\lambda_{a_b|\widetilde{\vartheta}}}{\partial\widetilde{\vartheta}}\Big|_{\vartheta} = \frac{\partial\lambda_{a_b}}{\partial\vartheta}$ and $\frac{\partial\hat{\vartheta}(\hat{\lambda}_{a_b})}{\partial\hat{\lambda}_{a_b}}\Big|_{\lambda_{a_b}} = \frac{\partial\vartheta}{\partial\lambda_{a_b}}$, we obtain that, asymptotically,

$$\begin{aligned} &E_{\vartheta+\Delta\vartheta}(\hat{\vartheta}(\hat{\lambda}_{a_b})) \\ &\quad = \vartheta + \left.\frac{\partial\hat{\vartheta}(\hat{\lambda}_{a_b})}{\partial\hat{\lambda}_{a_b}}\right|_{\lambda_{a_b}} \frac{\partial\lambda_{a_b}}{\partial\vartheta} \Delta\vartheta + \left.\frac{\partial\hat{\vartheta}(\hat{\lambda}_{a_b})}{\partial\hat{\lambda}_{a_b}}\right|_{\lambda_{a_b}} o(\Delta\vartheta) \\ &\quad = \vartheta + \Delta\vartheta + \tilde{o}(\Delta\vartheta), \end{aligned} \qquad (130)$$

where $\tilde{o}\,(\Delta\vartheta) = \frac{\partial\hat{\vartheta}(\hat{\lambda}_{a_b})}{\partial\hat{\lambda}_{a_b}}|_{\lambda_{a_b}}\, o\,(\Delta\vartheta)$. Thus, the asymptotic local unbiasedness (125) of each estimator $\hat{\vartheta}$, (124), of the angle $\vartheta \in V_\vartheta$ is proven. Therefore, the Rao-Cramér inequality for each $\hat{\vartheta} \in V_\vartheta = \langle 0, 2\pi)$ can be asymptotically used.

C. The intrinsic error of the estimation of the parameter ϑ in the EPI method

The frequency $\hat{\lambda}_{a_b}$ has the normal distribution asymptotically for some unknown (but arbitrary) value of ϑ. It follows that using one of the functions given by Eq. (124), a form of the asymptotic distribution of $\hat{\vartheta}$ can be obtained. Therefore, the confidence interval for the parameter ϑ can also be numerically calculated. The question is: What is the minimal error of ϑ estimation?

From the point of view of the experimentalist, the value s_{a_b} of the random variable S (Sec. VI A) of pair of spin projections of the particles 1 and 2, respectively, is observed in a single measurement. The Rao-Cramér inequality, which gives the bound on the accuracy of the estimation of the parameters, which in our case is the angle ϑ, uses the Fisher information on the parameters confined in the M-dimensional sample (122) that is taken by the experimentalist. It is equal to $\mathrm{M}\, g^{\vartheta\vartheta}(\vartheta)$ [14], where $g^{\vartheta\vartheta}(\vartheta) = g^{\vartheta\vartheta} = n_{ab}^2$, (112), is the Fisher information (see Note in Sec. VI A) for the $\mathrm{M} = 1$-dimensional sample. This situation has the following consequences.

The Rao-Cramér inequality for the variance of each of the four estimators $\hat{\vartheta}$ of the angle ϑ given by Eq. (124) has the form [32]

$$\sigma^2(\hat{\vartheta}) \geqslant \frac{1}{\mathrm{M}\, g^{\vartheta\vartheta}(\vartheta)}. \tag{131}$$

Even if we do not know the form of the estimator $\hat{\vartheta}$, the Rao-Cramér inequality (131) gives a lower Rao-Cramér bound (LRCB) for its variance, only if the estimator is unbiased. In our case, it is asymptotically unbiased. Inserting the value of $g^{\vartheta\vartheta}(\vartheta)$ into the inequality (131), we obtain

$$\sigma^2(\hat{\vartheta}) \geqslant \frac{1}{\mathrm{M}\, n_{ab}^2}\ \mathrm{rad}^2, \quad n_{ab} = \pm 1, \pm 2. \tag{132}$$

In the case of the bipartite system of two spin-$\frac{1}{2}$ particles, $n_{ab} = \pm 1$, and from the relation (132), it follows that

$$\sigma^2(\hat{\vartheta}) \geqslant \frac{1}{\mathrm{M}}\ \mathrm{rad}^2, \tag{133}$$

whereas for the bipartite system of two spin-1 photons, $n_{ab} = \pm 2$ and the relation (132) gives

$$\sigma^2(\hat{\vartheta}) \geqslant \frac{1}{4\,\mathrm{M}}\ \mathrm{rad}^2. \tag{134}$$

Conclusion. The inequality (132) states that the observation of one pair of spin projections s_{a_b} (*in the light of the complete ignorance about the angle* ϑ) [see Eq. (10)] gives finite information on ϑ. The lowest Rao-Cramér bound on the error of the angle ϑ estimation $1/(\sqrt{\mathrm{M}}\,|n_{ab}|)$ rad, $n_{ab} = \pm 1, \pm 2$ is quite large for $\mathrm{M} = 1$, which is connected with the flat "ignorance" function $r(\vartheta)$ given in Eq. (10) [3]. We can also notice that with an increase of $|n_{ab}|$ from 1 to 2 and, thus, with an increase of the spin of each particle in the bipartite system, this estimation error decreases. This means that if, e.g., the photons are observed, then the EPI method estimates ϑ more precisely than in the case of the detection of electrons. As usual, the error decreases with an increase of $\sqrt{\mathrm{M}}$.

D. The Rao-Cramér-Frieden inequality

In the measurement of the state of a system by an outer observer, we obtain the data that are influenced by the measuring apparatus. The general EPI analysis in the presence of the measurement and its noisy influence on the internal technical data (here ϑ) could be the topic of the separate study [3].

In the analysis of the EPR-Bohm experiment [15], the measurement data of the spin projection (let us, e.g., denote them by $\bar{S}_a$ for the particle 1) are generated by the true values of the quantities S_a, S_b, ϑ with the noisy presence of the measuring apparatus. In fact, this noise arises in the Stern-Gerlach devices (or polarizers) a and b, but for other reasons than the (assumed as equal to zero) fluctuations of the spin projection (see Sec. II B1).

The Fisher information that takes into account the noise. The Fisher information I_{noise} on the parameter ϑ obtained from the data, which takes into account the *noise* of the measurement, fulfills the relation

$$g^{\vartheta\vartheta}(\vartheta) \geqslant I_{\mathrm{noise}}(\vartheta). \tag{135}$$

Because $I_{\mathrm{noise}}(\vartheta)$ enters into (131) in place of $I_F(\vartheta)$, this condition means the deterioration of the quality of the estimation in comparison with the one that follows from the Rao-Cramér inequality (132).

The information inequality. The information channel capacity I is the sum (43) over the channels with the corresponding Fisher information $I_F(\vartheta)$, (113) [equal also to $g^{\vartheta\vartheta}(\vartheta)$ in Eq. (112)]; therefore,

$$\begin{aligned} I_{(\mathrm{min})} &= 2\pi > I_F(\vartheta) = g^{\vartheta\vartheta}(\vartheta) = 1 \quad \text{for } n_{ab} = \pm 1, \\ I_{(s.\mathrm{min})} &= 8\pi > I_F(\vartheta) = g^{\vartheta\vartheta}(\vartheta) = 4 \quad \text{for } n_{ab} = \pm 2, \end{aligned} \tag{136}$$

where $I_{(\mathrm{min})}$ and $I_{(s.\,\mathrm{min})}$ are the first and second minimal values of I given by Eq. (80) for $n_{ab} = \pm 1$ and by Eq. (84) for $n_{ab} = \pm 2$, respectively. From Eqs. (136), (131), and (135), we asymptotically obtain for an M-dimensional sample

$$\begin{aligned} \sigma^2(\hat{\vartheta}) &\geqslant \frac{1}{\mathrm{M}\, I_{\mathrm{noise}}(\vartheta)} \geqslant \frac{1}{\mathrm{M}\, g^{\vartheta\vartheta}(\vartheta)} \\ &= \frac{1}{\mathrm{M}\, I_F(\vartheta)} > \frac{1}{\mathrm{M}\, I}, \end{aligned} \tag{137}$$

where I is equal to $I_{(\mathrm{min})}$ or $I_{(s.\,\mathrm{min})}$ for $n_{ab} = \pm 1$ or $n_{ab} = \pm 2$, respectively.

[In the lack of noise, it is $1/[\mathrm{M} g^{\vartheta\vartheta}(\vartheta)]$ and not $1/(\mathrm{M} I)$, which is the proper lower bound on the variance of the estimator $\hat{\vartheta}$ of ϑ].

Without the anticipating contribution of the system [described here by $I_F(\vartheta)$ of the EPI method], the measurement would be absolutely impossible. In this sense the information $I_{\mathrm{noise}}(\vartheta)$ is generated by the Fisher information $I_F(\vartheta)$, (38), which as it is obtained by *the EPI estimation procedure is the part of the information that manifests itself in the measurement*. Thus, (with a lack of noise) this composition of the Rao-Cramér inequality with the result of the EPI method asymptotically leads to the inequality

$$\sigma^2(\hat{\vartheta})\, I_F(\vartheta) \geqslant \frac{1}{\mathrm{M}}. \tag{138}$$

E. The Frieden approach to ϑ

In the original Frieden analysis, the parameter ϑ in four joint conditional probabilities $P(S_{ab}|\vartheta)$, (4), is, in fact, the expected one, i.e., $E_{\vartheta}(\hat{\vartheta}_{N=1}) = \vartheta$. Here, the estimator $\hat{\vartheta}_{N=1}$ is an additional random variable that characterizes the inner property of the system on the same rights as the projections

S_a and S_b of its component particles 1 and 2, and the index $N = 1$ signifies that the analysis concerns the inner sample that is taken by the system alone. Only under this condition can the Rao-Cramér inequality be applied directly to the result of the EPI method on $I_F(\vartheta)$, giving $\sigma^2(\hat{\vartheta}_{N=1}) \geqslant 1/I_F(\vartheta)$. It can be rewritten as follows [3]:

$$\sigma^2(\hat{\vartheta}_{N=1})\, I_F(\vartheta) \geqslant 1. \tag{139}$$

Frieden also analyzed this kind of inequality for the position-momentum representations and obtained a kind of Heisenberg-like inequality [3]; therefore, it can be called the Rao-Cramér-Frieden inequality. The difference between Eqs. (139) and (138) is such that Eq. (139) characterizes the inner accuracy of estimating the angle ϑ as a result of the EPI estimation and by the system alone, whereas Eq. (138) characterizes the accuracy of estimating ϑ in the outer experiment in which data are generated by the theoretical distribution (111) obtained as a result of the EPI estimation.

VII. CONCLUSION

One of the main intellectual indicators, which seemed to support the indispensability of quantum mechanics, is the well-known EPR-Bohm relations [16] that are obtained for both spin-$\frac{1}{2}$ particles (electrons in this paper) and for massless spin-1 photons. In the quantum mechanical approach, they are obtained under the indistinguishability property of particles and Fermi-Dirac (for electrons) and the Bose-Einstein (for photons) statistics (for other conditions, see [16], which were also mentioned in Sec. V C). In the EPI method, the statistical formalism invented by Frieden and Soffer, the EPR-Bohm result (111) for the probability distribution for the bipartite system of particles, was originally derived in [3]; however, the one presented in this paper [4] differs in a few points. First, the boundary conditions (Sec. II B) were formulated in a way that is easier to understand [15]. Second, the observed physical information used directly in the structural information principle was consistently obtained from the analyticity condition of the log-likelihood function without any jump from its *analytic* to its *metric form*. The generating equation (69), which is the central output of the EPI method, was obtained from the observed structural and variational information principles (57) and (63), respectively. Third, in [3] the appeal to the orthogonality property of quantum mechanical amplitude (introduced there) was used, whose condition as the introductory one we succeeded in avoiding. Instead, the regularity condition of the probability distribution (40) (or as a consequence, the condition of the independence of the Rao-Fisher metric (112) on the statistical (sub)space $\mathcal{S}$ on the value of the parameter ϑ) was used. This (in addition to the boundary conditions) enabled the integration constants of the general solution of the generating equation (69) to be determined, which led to the final form (109) of EPR-Bohm amplitudes. Then, the well-known solution (111) for the probability distribution in the EPR-Bohm problem [3] was obtained.

Frieden's coverage of quantum mechanics. In the Frieden approach, instead of using Eq. (92), the quantum mechanical amplitudes $\psi_{ab}(\vartheta) \propto q_{ab}(\vartheta)$ are constructed [3] (see the paragraph *Frieden's quantum amplitudes* in Sec. V B2 and [33]). Together with the approach used in this paper (which does not use the quantum mechanics notions in the EPI method estimation), the Frieden's construction of the quantum mechanical amplitudes means that when (inversely) defining quantum amplitudes via the classical ones, the classical statistics coverage of the quantum mechanical approach (even without any mention of the hidden variables) is obtained. We call this the Frieden's "coverage of quantum mechanics."

Fourth, the analysis was made more homogeneous in the sense that the amplitude of the bipartite system exhibits either the periodicity that is characteristic for spin-$\frac{1}{2}$ particles or for the spin-1 massless photons [possessing the same mathematical form (111)]. Thus, it is slightly easier to notice that the EPI method of the EPR-Bohm problem determines the wave function representations than in [3]. Fifth, this approach inevitably connects the EPR-Bohm result (111) with the frequencies of the events that are registered by the experimentalist (Sec. VI). Finally (in agreement with the previous point), the estimation of the angle ϑ is pinned to the outer M-dimensional sample taken by the experimentalist. In addition to these, the differences between the Frieden approach and the one presented in this paper are pointed out in the text.

Let us note that the generating equation [written, e.g., in the form given by Eq. (110)] is the stationary one, indifferently whether two measuring devices are close to each other or infinitely far apart. Interestingly, Eq. (110) can be postulated, e.g., from the stationary telegraphic equation [37] for the field that is rotating with an infinite velocity in the ϑ direction (which is formally equivalent to the Klein-Gordon equation [37] for the tachyon propagating in the ϑ direction). Thus, the generating equation describes the instantaneous twist around the propagation axis of the field of a bipartite system at the moment of detection. The spatial part of the field of a bipartite system is one of the amplitudes (109) of the EPR-Bohm problem. The lack of a finite time component is seen in the boundary conditions and is in agreement with present-day experiments [38]. However, this could "gravitate" to the conclusion that the information principles (57) and (65), which result in the generating equation (110), describe the physics of the collapse of the wave function of a bipartite system in the EPR-Bohm problem. That would provide us with a richer theoretical structure than the quantum mechanical one, which is based merely on the definition of the transition amplitude for the problem [16]. When this richer structure is disclosed, it might appear that the bipartite system is an extended, composed object, which, when detected, knows about this event in the whole medium of its entity by the kind of interaction that propagates inside it. Could it be a gravitational one whose speed of propagation is experimentally unknown until the present day [39] or an electromagnetic one whose velocity is experimentally measured only locally and obviously outside of such compact systems as discussed in the EPR-Bohm problem? If the medium is not known, the velocity could, in fact, even be infinite. This is in agreement with the EPR-Bohm-type experiments [38] (unless one believes in entanglement without any interaction).

Next, the EPI method provides the general formalism for the description of entangled states [2,3,7]. This is particularly true in the case of the EPR-Bohm problem for which pairs S_{ab} of spin projections of particles 1 and 2 of the bipartite system are detected in the experiment with observed frequencies and the entanglement concerns these frequencies. Indeed, the outer data, which form the M-dimensional sample, are generated from the (supposed) probability distribution (111), which this time is obtained theoretically. They are reflected in the Rao-Fisher metric $g^{\vartheta\vartheta}$ of the statistical (sub)space $\mathcal{S}_\Lambda$ and the statistical model $\mathcal{S}$ (see the Appendix) and therefore in the Fisher information $I_F(\vartheta)$ and the information channel capacity I of the EPI method (see Sec. V B5).

[The Rao-Fisher metric $g^{\vartheta\vartheta}$ is equal to the Fisher information $I_F(\vartheta)$, (41), which enters into the information channel capacity I, (43), via Eq. (37) (see Sec. V B5)].

Therefore (in the case of the inseparability of I into the sum of the proper subsystems' information channel capacities and Q into the corresponding structural information terms), the possibility of describing the entangled states follows from the fact that the structural information principle $I = -Q$, (45), describes the relation of the outer data of the M-dimensional sample (connected with I) *with the unobserved configuration of the system*, which is described by structural information Q.

In this paper, the unobserved configuration of the system depends on the angle ϑ between the particles spin projections in two measuring devices. Therefore, in spite of the lack of any direct insight into the *inner* properties of the system, we can (in accordance with Sec. VI) make an inference about the (*inner*) angle ϑ by simply observing the frequencies of the pairs of the spin projections. This can be done with the finite accuracy that is also discussed in Sec. VI. The mere fact that the possibility of such an inference exists determines the situation that is called the *entanglement of the states* of the particles of a bipartite system that is perceived in the dependence of the frequencies of both particle spin projections on the inner angle ϑ of the system [40]. This is the reason why the experimentalist is able to estimate the value of ϑ. However, in a more complete approach to the EPI-method estimation, its maximal accuracy also has to be discussed [7].

Recently, increasing problems with the experimental validation of the uncertainty relation (UR) [41,42] in its quantum mechanical Heisenberg form have been reported. [Unless one forgets that the intrinsic quantum mechanical uncertainties of the complementary variables on the quantum state are, for the purpose of the verification of any Heisenberg uncertainty principle (as it takes place for any form of UR), always estimated experimentally.] One of them is connected with an experiment of the successive projective measurements of two noncommuting neutron spin components [43]. The other group is connected with the diffraction-interferometric experiments for a photon, where both UR and the meaning of the half-widths of a pair of functions (time and frequency), which are related by the Fourier transform, are examined [24]. Thus, the quantum mechanical state-dependent formulation of the simultaneous measurability of the observables (that also uses the notion of their closer unknown noise operators) was proposed, which led to a deep theoretical reformulation of UR in the form of the measurement disturbance relationship (MDR) [44–46]. However, the proper implementation of the Rao-Cramér inequality can also be considered [7,14] and the call to abandon UR on behalf of the more information oriented inequality has already been sounded. It was discussed in Secs. VI C–VI E for the EPR-Bohm problem.

Next, the irrelevance of the hidden variable idea for the construction of Bell-like inequalities in the EPI method was discussed in Sec. V C1 and, therefore, we conclude that the Bell-like tests do not put quantum theory on a pedestal at all. Conversely, the EPI method suggests that although quantum mechanics is the experimentally reliable one, the foundation of its underlying method could be of the statistical information theory background. The introductory steps for the construction of both the Maxwell and Dirac equations using the EPI method and the quantization of the helicity of the free electromagnetic field and spin for the Dirac field were mentioned in the Introduction and Sec. II B1. In this paper, using the EPI method, which follows the previous findings, it was shown (Secs. V A1–V A2) that as a solution of the generating equation with the constants of integration n_{ab} equal to ± 1 or ± 2 for every $S_{ab} \in \Omega_{ab}$, the bipartite EPR-Bohm amplitudes for the rotation group representation of the spin-$\frac{1}{2}$ or spin-1 particles, respectively, appear. That is, the Fermi-Dirac statistics used in the case of electrons for a quantum mechanical description of the EPR-Bohm problem seems to be a reminiscence of the statistical information principles. Thus, the Pauli exclusion principle may also have a statistical information theory background. The same statistical information background is also suggested for the Bose-Einstein statistics that is used for a description of the EPR-Bohm-like problem for photons.

Finally, the general mathematical thought that is behind the results of this paper is the self-consistency of the solution of a proper set of partial differential equations. After they are solved self-consistently, all degrees of freedom are removed and what remains is one particular state of the system. In this paper, this is a particular solution of the bipartite amplitude of the EPR-Bohm problem that is obtained by the self-consistent solution of the pair of (differential) information principles. In the case of an electronic field fluctuation coupled to its electromagnetic self-field, the self-consistent treatment of the Dirac equation and classical Maxwell equations resulted in the (obtained iteratively) solution for the Lamb shift [47]. In the case of the self-consistent model of classical field interactions of the electroweak model that is solved in the presence of nonzero extended fermionic charge density fluctuations, the solution obtained in [48] was a spin zero electrically uncharged droplet, interpreted as the state of mass equal to $\sim$126.5 GeV, which was observed recently in an LHC experiment. In [23,49,50] the dynamical compactification of a six-dimensional model of the space-time to the four-dimensional, locally Minkowskian space-time, which resulted from the self-consistent solution of the coupled Einstein and Klein-Gordon equations, was presented.

Thus, in this paper it can also be seen that the necessity to introduce the quantum (mechanical or field theory) approach is the result of the previous neglect of the self-consistency of a proper set of partial differential equations, which seems to be the primary property of physical structures. Finally, each of the solutions (109) of the EPR-Bohm problem describes the compound, extended in space, state of a bipartite system.

ACKNOWLEDGMENTS

This work has been supported by the Modelling Research Institute, 40-059 Katowice, Drzymały 7/5, Poland, and by the Department of Field Theory and Particle Physics, Institute of Physics, University of Silesia.

APPENDIX: THE RAO-FISHER METRICS ON THE PROBABILITY SIMPLEX AND ITS SUBMANIFOLDS

Let in the result of an experiment the finite number $\aleph$ of possible outcomes s_j, $j = 1,2,\ldots,\aleph$ of a random variable S be obtained. They span a base space of events Ω. The probabilities of the outcomes s_j are $P_j \equiv P(s_j)$ and

$$P_j \equiv P(s_j) \geqslant 0 \quad \text{and} \quad \sum_{j=1}^{\aleph} P_j = 1. \tag{A1}$$

Let us choose $\lambda_j := P_j$ as the jth component of the set $\lambda \equiv (\lambda_j)_{j=1}^{\aleph}$ of parameters. Thus, the probability distribution P is parametrized by λ, i.e., $P_j \equiv P(s_j;\lambda)$. Now, let us construct the statistical space:

$$\mathcal{S}_\Lambda = \Bigg\{ P(s;\lambda) | \lambda \equiv (\lambda_j)_{j=1}^{\aleph} \in \Lambda \subset \mathbb{R}^{\aleph}, \quad \lambda_j \geqslant 0 \ \ (\forall\, j) \quad \text{and} \quad \sum_{j=1}^{\aleph} \lambda_j = 1 \Bigg\}.$$

In the case of the EPR-Bohm problem the number of events is equal to $\aleph = 4$ and the probability distribution $P(s_j;\lambda)$, (120), is determined by Eq. (111), where in Eq. (98) the following denotation of the index j was introduced: The value $j - 1 \equiv a_b = 0,1,2,3$ corresponds to $ab = + +, - -, + -, - +$, respectively.

The $(\aleph - 1)$-*simplex of the probability distributions* P, is defined as follows:

$$\Delta^{\aleph-1} = \Bigg\{ (P_1, P_2, \ldots, P_\aleph) \in \mathbb{R}^{\aleph}; \quad P_j \geqslant 0,\ j = 1,\ldots,\aleph, \quad \text{for} \quad \sum_{j=1}^{\aleph} P_j = 1 \Bigg\}. \tag{A2}$$

It is the convex set, i.e., each probability vector $\vec{P} = (P_j)_{j=1}^{\aleph} \in \Delta^{\aleph-1}$ can be expressed as $\vec{P} = \sum_{j=1}^{\aleph} \varepsilon_j P_j$, where $\sum_{j=1}^{\aleph} \varepsilon_j = 1$ [30].

The squared infinitesimal distance on the probability simplex $\Delta^{\aleph-1}$ is equal to

$$ds^2 = \sum_{j=1}^{\aleph} g^{jj} dP_j\, dP_j = \sum_{j=1}^{\aleph} \frac{1}{P_j} dP_j\, dP_j, \tag{A3}$$

where the diagonality of the Rao-Fisher metric,

$$(g^{ij}) = \text{diag}\left(\frac{\delta^{ij}}{P_j} \right), \tag{A4}$$

on the $\aleph - 1$ simplex $\Delta^{\aleph-1}$ [30] was used.

Let us perform the transformation $P_j \to q_j$ given by $4\, P_j = q_j^2$, $j = 1,\ldots,\aleph$, from probabilities to amplitudes [compare Eq. (99)]. Thus, on the coordinatized by a set of d parameters $\Theta = (\theta^\alpha)_{\alpha=1}^{d}$ statistical (sub)space of the $(\aleph - 1)$-dimensional probability simplex $\Delta^{\aleph-1}$, the metric g^{ij} induces the Rao-Fisher metric $g_{\alpha\beta}$, which has the form [30]

$$\begin{aligned} g_{\alpha\beta} &= \sum_{j=1}^{\aleph} P_j \frac{\partial \ln P_j}{\partial \theta^\alpha} \frac{\partial \ln P_j}{\partial \theta^\beta} = \sum_{j=1}^{\aleph} \frac{1}{P_j} \frac{\partial P_j}{\partial \theta^\alpha} \frac{\partial P_j}{\partial \theta^\beta} \\ &\equiv \sum_{i,j=1}^{\aleph} g^{ij} \frac{\partial P_i}{\partial \theta^\alpha} \frac{\partial P_j}{\partial \theta^\beta} = \sum_{j=1}^{\aleph} \frac{\partial q_j}{\partial \theta^\alpha} \frac{\partial q_j}{\partial \theta^\beta}, \end{aligned} \tag{A5}$$

where in the second line the diagonality, (A4), of the Rao-Fisher metric g^{ij} on the $\aleph - 1$ simplex, was emphasized.

Due to the normalization $\sum_{j=1}^{\aleph} P_j = 1$, it follows that [compare Eqs. (5) and (106)]

$$\sum_{j=1}^{\aleph} q_j q_j = 4. \tag{A6}$$

Thus, we notice that the amplitudes' sphere is of the radius 2.

For example, for the EPR-Bohm problem, in this paper the statistical (sub)space $\mathcal{S}$, (6), with $d = 1$ parameter ϑ is investigated:

$$\mathcal{S} = \{ P(S_{ab}|\vartheta) |\ \ \vartheta \in \langle 0\, 2\pi) \equiv V_\vartheta \subset \mathbb{R}^1 \}.$$

The model $\mathcal{S}$ is coordinatized by parameter ϑ, one-dimensional submanifold of $\aleph - 1 = 3$-dimensional probability simplex Δ^3. Thus, the Rao-Fisher metric (A5) on $\mathcal{S}$ of the EPR-Bohm problem, is equal to

$$g^{\vartheta\,\vartheta}(\vartheta) = \sum_{j=1}^{\aleph=4} \frac{\partial \tilde{q}_j}{\partial \vartheta} \frac{\partial \tilde{q}_j}{\partial \vartheta},$$

where $\tilde{q}_j \equiv q_{ab}(\vartheta)$ are defined in (99). After the calculations presented in Sec. V B, the Rao-Fisher metric (112) on $\mathcal{S}$ appeared to be constant, i.e., $g^{\vartheta\,\vartheta} = n_{ab}^2 = \text{const.}$, where $n_{ab} = \pm 1$ or ± 2 for the bipartite state of electrons or photons, respectively.

Now, let us determine the Rao-Fisher metric g_Λ^{kl} on the statistical (sub)space $\mathcal{S}_\Lambda$, (A2), of the $(\aleph - 1)$-dimensional probability simplex. Knowing, in accordance with Eq. (A5), that it is coordinatized by the set of parameters $\lambda \equiv (\lambda_k)_{k=1}^{d=\aleph}$, we obtain

$$\begin{aligned} g_\Lambda^{kl} &= \sum_{i,j=1}^{\aleph} g^{ij} \frac{\partial P_i}{\partial \lambda_k} \frac{\partial P_j}{\partial \lambda_l} = \sum_{i,j=1}^{\aleph} g^{ij} \frac{\partial P_i}{\partial P_k} \frac{\partial P_j}{\partial P_l} \\ &= \sum_{i,j=1}^{\aleph} g^{ij} \delta_i^k \delta_j^l = g^{kl}. \end{aligned} \tag{A7}$$

We see that the components g_Λ^{kl}, $k,l = 1,2,\ldots,\aleph$, of the searched for metric are equal to the corresponding components g^{kl} of the Rao-Fisher metric on the probability simplex $\Delta^{\aleph-1}$, i.e., in accord with Eq. (A4), $g_\Lambda^{ij} = \frac{\delta^{ij}}{P_j}$. Thus, we can finally compute the Rao-Fisher metric $g_\Lambda^{\vartheta\vartheta}$ on the submanifold coordinatized by one parameter ϑ, which is induced by the metric g_Λ^{kl} on the statistical space $\mathcal{S}_\Lambda$:

$$\begin{aligned} g_\Lambda^{\vartheta\vartheta} &= \sum_{k,l=1}^{\aleph} g_\Lambda^{kl} \frac{\partial P_k}{\partial \vartheta} \frac{\partial P_l}{\partial \vartheta} = \sum_{l=1}^{\aleph} \frac{1}{P_l} \frac{\partial P_l}{\partial \vartheta} \frac{\partial P_l}{\partial \vartheta} \\ &= \sum_{l=1}^{\aleph} \frac{\partial q_l}{\partial \vartheta} \frac{\partial q_l}{\partial \vartheta} = g^{\vartheta\vartheta}. \end{aligned} \tag{A8}$$

In conclusion, the Rao-Fisher metric $g_{\Lambda}^{\vartheta\vartheta}$ on the submanifold (coordinatized by ϑ) of the statistical space $\mathcal{S}_{\Lambda}$ is equal to the Rao-Fisher metric $g^{\vartheta\,\vartheta}$ on the submanifold (coordinatized by ϑ) of the simplex of probabilities $\Delta^{\aleph-1}$. Thus, due to the Rao-Fisher metric $g^{\vartheta\vartheta}$ derived on $\mathcal{S}$ for the EPR-Bohm problem, (112), we finally obtain:

$$g_{\Lambda}^{\vartheta\,\vartheta} = g^{\vartheta\,\vartheta} = n_{ab}^2 = \text{const.},$$
$$\text{where} \quad n_{ab} = \pm 1 \text{ or } \pm 2 \quad \text{for every} \quad S_{ab} \in \Omega_{ab}. \tag{A9}$$

[1] R. A. Fisher, Philos. Trans. R. Soc., London A **222**, 309 (1922); *Statistical Methods and Scientific Inference*, 2nd ed. (Oliver and Boyd, Edinburgh, 1959).
[2] B. R. Frieden and B. H. Soffer, Phys. Rev. E **52**, 2274 (1995); B. R. Frieden, Found. Phys. **16**, 883 (1986); Phys. Rev. A **41**, 4265 (1990); **66**, 022107 (2002); B. R. Frieden, A. Plastino, A. R. Plastino, and B. H. Soffer, Phys. Rev. E **66**, 046128 (2002); Phys. Lett. A **304**, 73 (2002).
[3] B. R. Frieden, *Science from Fisher Information: A Unification* (Cambridge University Press, Cambridge, UK, 2004).
[4] J. Syska, arXiv:1211.3674, http://el.us.edu.pl/ekonofizyka/images/f/f2/Fisher.pdf.
[5] I. Bednarek, M. Makowski, E. W. Piotrowski, J. Sładkowski, and J. Syska, Generalization of the Aoki-Yoshikawa Sectoral Productivity Model (unpublished); J. Syska, in [4]; J. Sładkowski, Int. Conf. Stat. Phys. Sigma Phi (2011).
[6] E. W. Piotrowski, J. Sładkowski, J. Syska, S. Zając, Phys. Status Solidi B **246**, 1033 (2009).
[7] J. Sładkowski and J. Syska, Phys. Lett. A **377**, 18 (2012).
[8] E. Jaynes, Phys. Rev. **106**, 620 (1957); **108**, 171 (1957).
[9] J. Syska, Phys. Status Solidi B **244**, 2531 (2007).
[10] C. R. Rao, Bull. Calcutta Math. Soc. **37**, 81 (1945).
[11] B. Efron, Ann. Stat. **3**, 1189 (1975).
[12] A. P. Dawid, Ann. Stat. **3**, 1231 (1975); **5**, 1249 (1977).
[13] S. Amari, POST RAAG Report 106, 1980; Ann. Stat. **10**, 357 (1982).
[14] S. Amari and H. Nagaoka, *Methods of Information Geometry, Translations of Mathematical Monographs* (Oxford University Press, Oxford, 2000), Vol. 191; A. Fujiwara and H. Nagaoka, Phys. Lett. A **201**, 119 (1995); J. Math. Phys. **40**, 4227 (1999); K. Matsumoto, Ph.D. thesis, University of Tokyo, 1998.
[15] D. Mroziakiewicz, Master's thesis, Institute of Physics, University of Silesia, 2008 [in Polish].
[16] E. B. Manoukian, *Quantum Theory: A Wide Spectrum* (Springer, Dordrecht, 2006).
[17] S. Keppeler, Phys. Rev. Lett. **89**, 210405 (2002).
[18] G. N. Afanasiev and Yu. P. Stepanovsky, Nuovo Cimento A **109**, 271 (1996); JINR-E2-95-413, jdsweb.jinr.ru/record/5958 (1995).
[19] E. B. Manoukian and N. Yongram, Eur. Phys. J. D **31**, 137 (2004).
[20] J. Syska and R. Szafron, Informational Criteria for the Field Theory Models Building (unpublished).
[21] B. R. Frieden and B. H. Soffer, Physica A **388**, 1315 (2009).
[22] L. de Broglie, Ph.D. thesis, Université de Paris, 1924; *An Introduction to the Study of Wave Mechanics* (Dutton, New York, 1928) [reprinted (Methuen, London, 2007)]; P. Roman, in *Theory of Elementary Particles* (North-Holland, Amsterdam, 1961), p. 103.
[23] J. Syska, Int. J. Theor. Phys. **49**, 2131 (2010).
[24] Ch. Roychoudhuri, in *The Nature of Light: What Is a Photon?* edited by Ch. Roychoudhuri, A. F. Kracklauer, and K. Creath (CRC Press, Boca Raton, 2008), pp. 363–377.
[25] G. Grössing, J. M. Pascasio, and H. Schwabl, Found. Phys. **41**, 1437 (2011).
[26] F. D. M. Haldane, Phys. Rev. Lett. **60**, 635 (1988); J. Syska, in *Frontiers in Field Theory*, edited by O. Kovras (Nova Science, New York, 2005), Chap. 6, pp. 125–154.
[27] J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, Cambridge, UK, 1987).
[28] A. Khrennikov, Entropy **10**, 19 (2008).
[29] L. Accardi, in *The Wave-Particle Dualism: A Tribute to Louis de Broglie on his 90th Birthday*, edited by S. Diner, D. Fargue, G. Lochak, and F. Selleri (D. Reidel Publishing Company, Dordrecht, 1984), pp. 297–330; A. Fine, Phys. Rev. Lett. **48**, 291 (1982); I. Pitowsky, *ibid.* **48**, 1299 (1982); Phys. Rev. D **27**, 2316 (1983); P. Rastall, Found. Phys. **13**, 555 (1983); K. Hess and W. Philipp, Proc. Natl. Acad. Sci. USA **98**, 14224 (2001); Europhys. Lett. **57**, 775 (2002); A. Khrennikov, Nuovo Cimento B **115**, 179 (2000); J. Math. Phys. **41**, 1768 (2000); **41**, 5934 (2000).
[30] I. Bengtsson and K. Życzkowski, *Geometry of Quantum States* (Cambridge University Press, Cambridge, UK, 2006).
[31] J-P. Gazeau, *Coherent States in Quantum Physics* (Wiley-VCH, Berlin, 2009).
[32] Y. Pawitan, *In All Likelihood: Statistical Modeling and Inference Using Likelihood* (Oxford University Press, Oxford, 2001).
[33] In [3] the *metric form* of the Fisher information is [instead of (41)] equal to $I_F(\vartheta) = 4\sum_{a=-}^{+} q_{ab}'^2(\vartheta)$, where in [3] $q_{ab}^2(\vartheta) = P(S_a|S_b,\vartheta)$ [compare Eq. (36)]. Frieden's q_{ab} and the one in this paper differ numerically by $1/\sqrt{2}$.
[34] F. Pennini, A. Plastino, B. H. Soffer, and C. Vignat, Phys. Lett. A **373**, 817 (2009).
[35] Beginning with the *metric form* (41) of the Fisher information, the *metric form* of the density of the physical information can be written $k_{ab}(\vartheta) = q_{ab}'^2 + \frac{1}{4}q_{ab}^2(\vartheta)\mathrm{q}_{ab}^{\mathrm{F}}(q_{ab})$, with the *metric form* of the information channel capacity $I = \sum_{ab}\int_0^{2\pi} q_{ab}'^2 d\vartheta = \sum_{ab}\int_0^{2\pi}(-q_{ab}q_{ab}'' + \tilde{C}_{ab})d\vartheta$, where in the last equality the integration by parts has been performed and $\tilde{C}_{ab} = \frac{1}{2\pi}[q_{ab}(2\pi)q_{ab}'(2\pi) - q_{ab}(0)q_{ab}'(0)]$. Then a calculation similar to the one presented in the text of this paper shows that $\tilde{C}_{ab} = 0$ [3,4].
[36] C. Bonatti, in *Encyclopedia of Mathematical Physics*, edited by J. P. Francoise, G. L. Naber, and T. S. Tsun, Vol. 2 (Elsevier, Amsterdam, 2006), pp. 494–502; A. L. Pereira and M. C. Pereira, J. Juliusz Schauder Cent. **20**, 283 (2002).
[37] K. Ohtaka, Green's Functions, in *Mathematical Tools for Physicists*, edited by G. L. Trigg (Wiley-VCH Verlag, Weinheim, 2005), p. 167.

[38] E. Megidish, A. Halevy, T. Shacham, T. Dvir, L. Dovrat, and H. S. Eisenberg, Phys. Rev. Lett. **110**, 210403 (2013); J-D. Bancal, S. Pironio, A. Acín, Y-C. Liang, V. Scarani, and N. Gisin, Nat. Phys. **8**, 867 (2012); J. Yin, Y. Cao, H-L. Yong, J-G. Ren, H. Liang, S-K. Liao, F. Zhou, Ch. Liu, Y-P. Wu, G-S. Pan, L. Li, N-L. Liu, Q. Zhang, Ch-Z. Peng, and J-W. Pan, Phys. Rev. Lett. **110**, 260407 (2013).

[39] Zhu Yin, Chin. Phys. Lett. **28**, 070401 (2011).

[40] Let us remark that the treatment of the physical information, K, which is different than in [3] (see Sec. I), implies that the conclusions in [3] have a different shade, i.e., citing after [3] "The relay of information J to the *observed particle* gives rise to a sampled datum from the output law, the spin value S_a. This implies that the information relay causes the particle to *obey* the output spin law (111). The cause is some unknown interactive mechanism, perhaps a" 'rich and complex inner structure which can respond to information and direct its self-motion accordingly.' In this way, the relay of information J ties together the spin states of the unobserved and observed particles. Thus, J "'gives form' (as required) to the spin properties of the *un*observed particle."

[41] J. Y. Bang and M. S. Berger, Phys. Rev. D **74**, 125012 (2006).

[42] Mu Ben-Rong, Wu Hou-Wen, and Yang Hai-Tang, Chin. Phys. Lett. **28**, 091101 (2011).

[43] J. Erhart, S. Sponar, G. Sulyok, G. Badurek, M. Ozawa, and Y. Hasegawa, Nat. Phys. **8**, 185 (2012).

[44] M. Ozawa, AIP Conf. Proc. **1363**, 53 (2011); Ann. Phys. **311**, 350 (2004).

[45] K. Fujikawa, Phys. Rev. A **85**, 062117 (2012); K. Fujikawa and K. Umetsu, Prog. Theor. Exp. Phys. **2013**, 013A03 (2013).

[46] C. Branciard, Proc. Natl. Acad. Sci. USA **110**, 6742 (2013).

[47] A. O. Barut, J. Kraus, Y. Salamin, and N. Ünal, Phys. Rev. A **45**, 7740 (1992).

[48] J. Syska, arXiv:1305.1237.

[49] M. Biesiada, R. Mańka, and J. Syska, Int. J. Mod. Phys. D **09**, 71 (2000); R. Mańka and J. Syska, UŚL-TP-95/03, University of Silesia, 1995.

[50] J. Syska, Redshift in a six-Dimensional Classical Kaluza-Klein Type Model (unpublished).