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Arrow-arrow correlations for the six-vertex model

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The six-vertex model on a square lattice is "exactly solvable" because an exact formula for the free energy can be obtained by the Bethe ansatz. However, exact formulas for the correlations of local bulk observables, such as the orientation of the arrow at a given edge, are, in general, not available. In this Rapid Communication, we consider the isotropic "zero-field" six-vertex model at small Δ. We derive the long-distance asymptotic formula of arrow-arrow correlations, which display power law decays with one anomalous exponent. Our method is based on an interacting fermion representation of the six-vertex model and does not use any information obtained from the exact solution.

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I. INTRODUCTION

The six-vertex model on a square lattice is called "exactly solvable" because of the discovery in 1967 of the exact formula for the free energy [1–7]. Since then, there has been intense research activity on this model and its transfer matrix [8–10]; this also led to important results for systems that are equivalent to the six-vertex model in an exact or approximate form [8,9,11].

Many aspects of the six-vertex model are determined by a parameter Δ. The *critical case* $|\Delta| \leqslant 1$ is of special interest: since the spectrum of the transfer matrix was found to be gapless, correlations of local bulk variables are expected to display a long-distance power law decay. However, a direct study of correlations that validates this prediction has been, in general, very problematic.

In the simpler $\Delta = 0$ case in which the six-vertex model is a *free* fermion system, Sutherland [12] considered the correlation of two parallel arrows and derived an exact formula which displayed staggered prefactors and power law decay. In the general $\Delta \neq 0$ case, except for the case of some nonlocal variables [13] and except for the noncritical regime $\Delta < -1$ [14], exact formulas for correlations are not known. Nonetheless, long-distance properties of the correlation of two parallel arrows have been inferred on the basis of two equivalences [15]: (a) the transfer matrices of the six-vertex model and of the Heisenberg quantum chain commute [16], hence, the correlation of two parallel arrows placed along the same lattice line must coincide with the S^z-S^z equal-time correlation of the Heisenberg model; (b) the Heisenberg quantum chain is in the *Luttinger liquid* universality class [17–20] (or can be studied via conformal field theory [21]), and that fixes the power law decay of correlations.

More recently, a different method has been introduced to describe the critical phases of the six-vertex model, the Coulomb gas picture for the *height variable* [22]. However, it appears that no specific result that can be compared with the analysis of Ref. [15] has been derived along this route.

In this Rapid Communication, we consider the isotropic zero-field six-vertex model and derive long-distance asymptotic formulas for the correlations of two parallel arrows and of two orthogonal arrows at small Δ. Remarkably, our formulas are compatible with the features predicted in Ref. [15], including the occurrence of one *anomalous* critical exponent. Our approach is composed of three steps. First, we recast the model into an *interacting* dimers model following Ref. [23]; then, we transform the dimer problem into an interacting fermions system; finally, we study fermion correlations by the standard Renormalization Group (RG) techniques used in condensed matter theory [24–26]. This way of dealing with the six-vertex model was already suggested in Ref. [27]; here we present the technical details: Since the interaction between dimers turns out to be staggered, regarding that paper a different fermion representation with different symmetries is used. Finally, we stress that our approach does not need any result from the exact solution of the six-vertex model.

II. DEFINITIONS AND RESULTS

Consider a finite square lattice Λ'. A configuration ω of the six-vertex model is obtained by drawing an arrow per each edge of the lattice so that the number of incoming arrows at each vertex is 2—the *ice rule*. At each vertex, there is one of the six possible arrangements of arrows in Fig. 1: Assign them positive weights $p_1, p_2, \ldots, p_6$. If $n_j(\omega)$ is the number of vertices in the configuration ω that displays the arrangement number j, the partition function of the six-vertex model is

$$Z = \sum_{\omega} \prod_{j=1}^{6} p_j^{n_j(\omega)}.$$

We assume a periodic boundary condition so that $n_5(\omega) = n_6(\omega)$ and $p_5 p_6$ counts as one free parameter. Our results—as well as the analysis in Ref. [15]—are for the *zero-field six-vertex model*, i.e., the case of

$$p_1 = p_2 = a, \qquad p_3 = p_4 = b,$$

furthermore, for the sake of simplicity, in this Rapid Communication, we will only consider the *isotropic case* of $a = b$. Without loss of generality, we can fix $p_5 = a$ and can parametrize $p_6 = 2ae^{\lambda}$ for a real λ. The characteristic parameter of the six-vertex model is

$$\Delta = \frac{a^2 + b^2 - p_5 p_6}{2ab} = 1 - e^{\lambda}.$$

The case of $\Delta = 0$ is the free fermion case. Our goal is to show that, at least for small Δ, there are correlations with a power law decay. We consider the *arrow orientation* observable. Let $\mathbf{v}_0$ and $\mathbf{v}_1$ be the two orthogonal vectors that span Λ'; a point

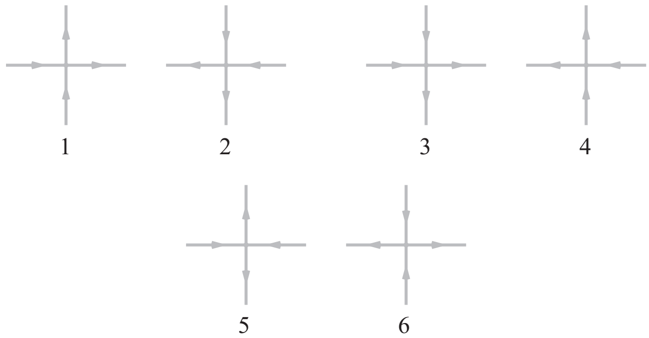

FIG. 1. The six possible configurations of arrows.

of the plane will be parametrized by $\mathbf{x} = x'_0\mathbf{v}_0 + x'_1\mathbf{v}_1$. For d, a horizontal edge centered at a point $\mathbf{x}$, the *horizontal arrow orientation* $\sigma_0(\mathbf{x})$ is equal to 1 if the arrow along d points to the right, otherwise, it is equal to -1; similarly, for d, a vertical edge centered at a point $\mathbf{x}$, the *vertical arrow orientation* $\sigma_1(\mathbf{x})$ is equal to 1 if the arrow along d points up, otherwise, it is equal to -1. In the infinite lattice limit, the arrow correlations have the following large $|\mathbf{x}|$ asymptotic formula: For two vertical arrows:

$$\langle\sigma_1(\mathbf{x}+\mathbf{y})\sigma_1(\mathbf{y})\rangle \sim c_0\frac{x'^2_1 - x'^2_0}{\left(x'^2_0 + x'^2_1\right)^2} + c_-\frac{(-1)^{x'_0+x'_1}}{\left(x'^2_0 + x'^2_1\right)^{\kappa_-}}, \quad (1)$$

for one horizontal and one vertical arrow,

$$\langle\sigma_1(\mathbf{x}+\mathbf{y})\sigma_0(\mathbf{y})\rangle \sim c_0\frac{-2x'_0x'_1}{\left({x'_0}^2 + {x'_1}^2\right)^2} - c_-\frac{(-1)^{x'_0-x'_1}}{\left({x'_0}^2 + {x'_1}^2\right)^{\kappa_-}}. \quad (2)$$

Each of the two above formulas is composed of two terms: The former term has the same power law decay as the $\Delta = 0$ case; the latter term has a power law decay with an *anomalous* (i.e., Δ-dependent) critical exponent $\kappa_- = 1 + O(\Delta)$ and a staggering prefactor $(-1)^{x'_0+x'_1}$ or $(-1)^{x'_0-x'_1}$; besides, c_0 and c_- are $\frac{2}{\pi^2} + O(\Delta)$. Note that $x'_0 + x'_1$ and $x'_0 - x'_1$ are integers. As a by-product of our approach, we can obtain a Feynman graph representation of the expansion of κ_- in powers of λ. For example, at first order,

$$\kappa_- = 1 - \frac{2\lambda}{\pi} + O(\lambda^2), \quad (3)$$

therefore, if λ is positive and small (i.e., Δ is negative and small), then at long distances, the latter terms in (1) and (2) dominate over the former ones.

Formulas (1)–(3) are the main result of this Rapid Communication. For $\Delta = 0$, (1) and (3) coincide with Sutherland's exact solution, see (6) of Ref. [12]. For $\Delta \neq 0$ as expected from the analysis after formula (15) of Ref. [15], (1) and (3) are in agreement with the asymptotic formulas of the S^z-S^z equal-time correlation in the Heisenberg quantum chain [17–20]. Formula (2) is new.

As an application of (1) and (2), we consider the long-distance behavior of the covariance of the *height variable* [28]. For $\mathbf{u}$, the center of a plaquette of Λ', $h(\mathbf{u})$ is the integer variable such that: If $\mathbf{x}$ is the center of a vertical bond,

$$\mathbf{v}_0 \cdot \nabla h(\mathbf{x}) \equiv h\left(\mathbf{x} + \tfrac{1}{2}\mathbf{v}_0\right) - h\left(\mathbf{x} - \tfrac{1}{2}\mathbf{v}_0\right) = \sigma_1(\mathbf{x}),$$

if $\mathbf{x}$ is the center of a horizontal bond,

$$-\mathbf{v}_1 \cdot \nabla h(\mathbf{x}) \equiv h\left(\mathbf{x} - \tfrac{1}{2}\mathbf{v}_1\right) - h\left(\mathbf{x} + \tfrac{1}{2}\mathbf{v}_1\right) = \sigma_0(\mathbf{x}).$$

Because of the ice rule, $h(\mathbf{u})$ is a scalar potential defined up to a global constant. From (1), we find the large $|\mathbf{x}|$ asymptotic formula,

$$\langle[h(\mathbf{x}+\mathbf{u}) - h(\mathbf{u})]^2\rangle \sim 2c_0 \ln|\mathbf{x}|, \quad (4)$$

hence, $h(\mathbf{u})$ has the same long-distance behavior as a *free* boson field in dimension two. It is remarkable that because of the staggering prefactor, the latter term in (1) does not determine the leading term of (4) regardless of the sign of λ.

In the next sections, we will derive (1)–(3), and we will show how to obtain the application (4).

III. INTERACTING FERMIONS PICTURE

Our point of departure is the equivalence of the six-vertex model on the square lattice Λ' with an interacting dimer model (IDM) on a different square lattice Λ [23]. A dimer configuration ω on Λ is a collection of dimers covering some of the edges of Λ with the constraint that every vertex of Λ is covered by one, and only one, dimer. The general partition function of the IDM is

$$Z_\lambda = \sum_\omega \exp\left\{\lambda \sum_{d,d'\in\omega} v(d,d')\right\}, \quad (5)$$

where the first sum is over all the dimer configurations, the second sum is over any pair of dimers in the configuration ω, finally, λ is the dimer coupling constant, and $v(d,d)$ is an interaction that we have to determine to have the equivalence with the six-vertex model. To do so, first embed Λ' into Λ as shown in Fig. 2, then, use the mapping from the six-vertex configurations to the dimer configurations in Fig. 3. As a consequence, set $v(d,d') = 1$ if d and d' are the two

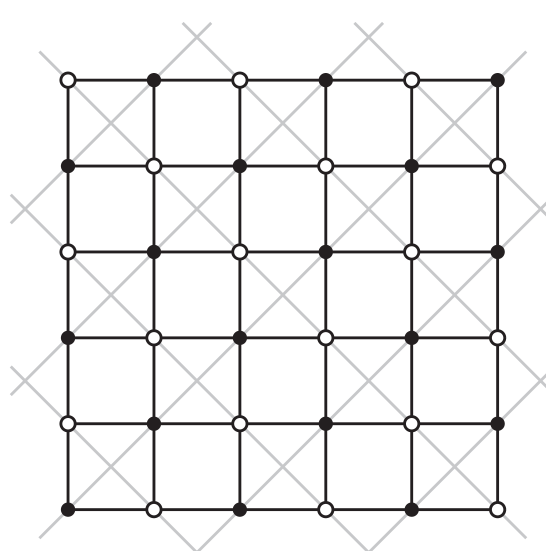

FIG. 2. Superposition of the six-vertex lattice (in gray) and the interacting dimer lattice (in black). Note the role played by the bipartition of the latter.

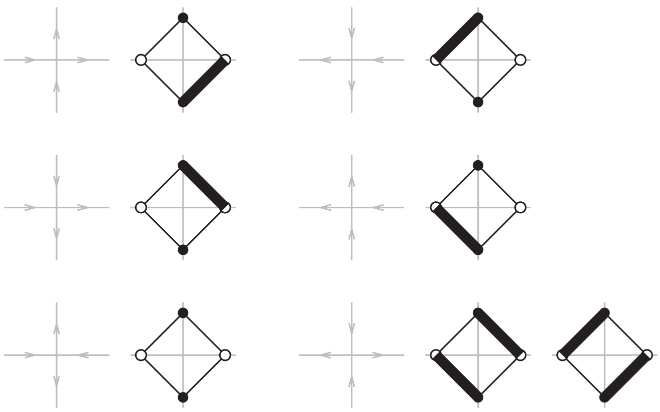

FIG. 3. Map of the six-vertex configurations on Λ' into the dimer model configurations on Λ. The vertex configurations numbers $j = 1,\ldots,5$ correspond to dimer arrangements with weight 1; the vertex configuration number 6 corresponds to two dimers arrangement, each of which has a weight e^{λ}. Note that, in this figure, the black sites are in the vertical positions; when the black sites are in the horizontal positions, all the dimer arrangements have to be counted with a weight 1.

parallel nearest-neighbor dimers in the last two arrangements in Fig. 3, which have the black sites in vertical positions, and $v(d,d') = 0$, otherwise. Let $\mathbf{e}_0$ and $\mathbf{e}_1$ be the two vectors that span Λ, a point in the plane will be $\mathbf{x} = x_0\mathbf{e}_0 + x_1\mathbf{e}_1$, in particular, $\mathbf{v}_0 = \mathbf{e}_0 - \mathbf{e}_1$ and $\mathbf{v}_1 = \mathbf{e}_0 + \mathbf{e}_1$. A natural local bulk observable for the dimer model is the *dimer occupancy*$\nu_j(\mathbf{x})$, which is equal to 1 if the edge $\{\mathbf{x},\mathbf{x}+\mathbf{e}_j\}$ is occupied and is 0 otherwise. The relationships between the dimer occupancy and the arrow orientation of the equivalent six-vertex model are as follows: for $\mathbf{x}$, a white site,

$$\sigma_0(\mathbf{x}) := \nu_0(\mathbf{x}-\mathbf{e}_0) + \nu_1(\mathbf{x}) - \nu_0(\mathbf{x}) - \nu_1(\mathbf{x}-\mathbf{e}_1) \tag{6}$$

for $\mathbf{x}$, a black site,

$$\sigma_1(\mathbf{x}) := \nu_0(\mathbf{x}) + \nu_1(\mathbf{x}) - \nu_0(\mathbf{x}-\mathbf{e}_0) - \nu_1(\mathbf{x}-\mathbf{e}_1). \tag{7}$$

[As a side remark, also for the dimer model, one can introduce an integer height function $H(\mathbf{w})$ for every $\mathbf{w}$ that is the center of a plaquette of the lattice Λ. The standard definition is such that, if $\mathbf{u}$ is the center of a plaquette of Λ' *and* the center of a plaquette of Λ, one has

$$H(\mathbf{u}) = 2h(\mathbf{u}).$$

We will not need $H(\mathbf{w})$ in the rest of the Rapid Communication.] Since black and white lattice sites play a different role in the choice of the dimer potential, we need a fermion representation of the dimer model that takes the bipartition of the square lattice into account. For this reason, the fermion representation that we introduce below is different from the one used in Ref. [27].

Let $\mathcal{L}$ be the Bravais lattice of the black sites of Λ. $\mathcal{L}$ is spanned by the vectors $\mathbf{v}_0$ and $\mathbf{v}_1$; its primitive cell has volume $v_P = 2$, and the reciprocal lattice of $\mathcal{L}$ is spanned by $\hat{\mathbf{v}}_0 = \frac{1}{2}\mathbf{v}_0$ and $\hat{\mathbf{v}}_1 = \frac{1}{2}\mathbf{v}_1$.

When $\lambda = 0$, the dimer model is equivalent to a lattice fermion field without interaction. Namely,

$$Z_0 = \int D\psi\ \exp\left\{-\sum_{\mathbf{x},\mathbf{y}\in\mathcal{L}} K_{\mathbf{x},\mathbf{y}}\psi^+_{\mathbf{x}}\psi^-_{\mathbf{y}+\mathbf{e}_0}\right\}, \tag{8}$$

where $\{\psi^+_{\mathbf{x}},\psi^-_{\mathbf{x}+\mathbf{e}_0}\colon \mathbf{x}\in\mathcal{L}\}$ are Grassmann variables and $D\psi$ indicates the integration with respect to all of them, $K_{\mathbf{x},\mathbf{y}}$ is one of the possible *Kasteleyn matrices* for the square lattice dimer model,

$$K_{\mathbf{x},\mathbf{y}} = \delta_{\mathbf{x},\mathbf{y}} - \delta_{\mathbf{x}-2\mathbf{e}_0,\mathbf{y}} - \delta_{\mathbf{x}-\mathbf{e}_0+\mathbf{e}_1,\mathbf{y}} - \delta_{\mathbf{x}-\mathbf{e}_0-\mathbf{e}_1,\mathbf{y}}.$$

Equation (8) is the partition function of a free *fermion field*, i.e., a Grassmann-valued Gaussian field with moment generator,

$$\langle e^{i\sum_{\mathbf{x}}(\psi^+_{\mathbf{x}}\eta^-_{\mathbf{x}}+\eta^+_{\mathbf{x}}\psi^-_{\mathbf{x}+\mathbf{e}_0})}\rangle_0 = e^{\sum_{\mathbf{x},\mathbf{y}} S(\mathbf{x}-\mathbf{y})\eta^+_{\mathbf{x}}\eta^-_{\mathbf{y}}},$$

where the $\eta_{\mathbf{x}}$'s are external Grassmann variables, S is the inverse Kasteleyn matrix,

$$S(\mathbf{x}) = \frac{v_P}{2}\int_{1\mathrm{BZ}}\frac{d\mathbf{k}}{(2\pi)^2}\frac{e^{i\mathbf{k}\cdot(\mathbf{x}+\mathbf{e}_0)}}{i\ \sin(\mathbf{k}\cdot\mathbf{e}_0) - \cos(\mathbf{k}\cdot\mathbf{e}_1)},$$

and 1BZ is the first Brillouin zone. The Fourier transform of S has two poles at the *Fermi momenta* $\mathbf{k} = \omega\mathbf{p}_F$ for $\omega = \pm 1$ and $\mathbf{p}_F = \left(0,\frac{\pi}{2}\right)$. Therefore, in view of the study of the scaling limit, it is convenient to decompose

$$S(\mathbf{x}) = \sum_{\omega=\pm} e^{i\omega\mathbf{p}_F\cdot\mathbf{x}} S_\omega(\mathbf{x})$$

for

$$S_\omega(\mathbf{x}) = \frac{v_P}{2}\int_{1\mathrm{BZ}}\frac{d\mathbf{k}}{(2\pi)^2}\frac{e^{i\mathbf{k}\cdot(\mathbf{x}+\mathbf{e}_0)}\chi(\mathbf{k})}{i\ \sin(\mathbf{k}\cdot\mathbf{e}_0) - \omega\ \sin(\mathbf{k}\cdot\mathbf{e}_1)},$$

where $\chi(\mathbf{k})$ is 1 in a neighborhood of $\mathbf{k} = 0$ and such that $\chi(\mathbf{k}-\mathbf{p}_F) + \chi(\mathbf{k}+\mathbf{p}_F) = 1$. Correspondingly, for $\varepsilon = \pm$,

$$\psi^\varepsilon_{\mathbf{x}} = \sum_\omega e^{i\varepsilon\omega\mathbf{p}_F\cdot\mathbf{x}}\psi^\varepsilon_{\mathbf{x},\omega}, \tag{9}$$

where $(\psi^+_{\mathbf{x},+},\psi^+_{\mathbf{x},-})$ and $(\psi^-_{\mathbf{x},+},\psi^-_{\mathbf{x},-})^T$ are Dirac spinors with covariances,

$$\begin{aligned}\langle\psi^+_{\mathbf{x},\omega}\psi^+_{\mathbf{y},\omega'}\rangle_0 &= \langle\psi^-_{\mathbf{x}+\mathbf{e}_0,\omega}\psi^-_{\mathbf{y}+\mathbf{e}_0,\omega'}\rangle_0 = 0,\\ \langle\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{y}+\mathbf{e}_0,\omega'}\rangle_0 &= \delta_{\omega,\omega'}S_\omega(\mathbf{x}-\mathbf{y}).\end{aligned} \tag{10}$$

If we now let $\lambda \neq 0$, one can verify that (5) becomes

$$Z_\lambda = Z_0\langle e^{(1-e^\lambda)V(\psi)}\rangle_0, \tag{11}$$

where $V(\psi)$ is quartic in the Grassmann variables with a simple explicit formula which is given in the next section. Besides, it is not difficult to find the leading term for long distances of the dimer correlations: If T indicates a truncated correlation and $\mathbf{z} = \mathbf{x}-\mathbf{y}$, for two horizontal dimers,

$$\begin{aligned}&\langle\nu_0(\mathbf{x})\nu_0(\mathbf{y})\rangle - \langle\nu_0(\mathbf{x})\rangle\langle\nu_0(\mathbf{y})\rangle\\ &\quad\sim(-1)^{z_0+z_1}\sum_\omega\langle\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega};\psi^+_{\mathbf{y},\omega}\psi^-_{\mathbf{y},\omega}\rangle^T\\ &\qquad+(-1)^{z_0}\sum_\omega\langle\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},-\omega};\psi^+_{\mathbf{y},-\omega}\psi^-_{\mathbf{y},\omega}\rangle^T,\end{aligned} \tag{12}$$

whereas, for one horizontal and one vertical dimer,

$$\begin{aligned}&\langle \nu_1(\mathbf{x})\nu_0(\mathbf{y})\rangle - \langle \nu_1(\mathbf{x})\rangle\langle \nu_0(\mathbf{y})\rangle \\ &\sim(-1)^{z_0+z_1}\sum_\omega i\omega\langle \psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega};\psi^+_{\mathbf{y},\omega}\psi^-_{\mathbf{y},\omega}\rangle^T \\ &\quad + (-1)^{z_0}\sum_\omega i\omega\langle \psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},-\omega};\psi^+_{\mathbf{y},-\omega}\psi^-_{\mathbf{y},\omega}\rangle^T. \quad (13)\end{aligned}$$

(The above formulas hold regardless of the colors of the sites $\mathbf{x}$ and $\mathbf{y}$.) The renormalization group argument that we will provide in the next section will indicate that, to compute the correlations up to subleading terms, we can just replace the fields $\psi^+_{\mathbf{x},\omega}$, $\psi^-_{\mathbf{x},\omega}$ with the Thirring model fields $\psi^\dagger_\omega(\mathbf{x})$, $\psi_\omega(\mathbf{x})$ times a prefactor of $2^{-(1/4)}e^{i\omega(\pi/8)}$ per each of them; from the exact solution of the Thirring model correlations [29–34] (see also Ref. [35]),

$$\begin{aligned}\langle \psi^\dagger_\omega(0)\psi_\omega(0);\psi^\dagger_\omega(\mathbf{x})\psi_\omega(\mathbf{x})\rangle^T &= c_{T,0}\frac{x_0'^2 - x_1'^2 - 2i\omega x_0'x_1'}{\left(x_0'^2+x_1'^2\right)^2}, \\ \langle \psi^\dagger_\omega(0)\psi_{-\omega}(0);\psi^\dagger_{-\omega}(\mathbf{x})\psi_\omega(\mathbf{x})\rangle^T &= \frac{c_{T,-}}{\left(x_0'^2+x_1'^2\right)^{\kappa_-}},\end{aligned} \quad (14)$$

where the critical exponent is $\kappa_- = 1 - \frac{\lambda_T}{2\pi} + O(\lambda_T^2)$ and λ_T is a parameter of the Thirring model: At first order, $\lambda_T = 4\lambda + O(\lambda^2)$ (see the next section).

IV. RENORMALIZATION GROUP ANALYSIS

The fermion interaction $V(\psi)$ has the explicit formula,

$$\begin{aligned}&\sum_{\substack{\omega_1,\omega_2\\ \omega_1',\omega_2'}}\frac{v_P^4}{(2\pi)^7}\int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{q}_1 d\mathbf{q}_2 \widehat{\psi}^+_{\mathbf{k}_1,\omega_1}\widehat{\psi}^+_{\mathbf{k}_2,\omega_2}\widehat{\psi}^-_{\mathbf{q}_1,\omega_1'}\widehat{\psi}^-_{\mathbf{q}_2,\omega_2'} \\ &\times\delta\left(\sum_{j=1}^2(\mathbf{k}_j+\omega_j\mathbf{p}_F) - \sum_{j=1}^2(\mathbf{q}_j+\omega_j'\mathbf{p}_F)\right) v_{\underline{\omega};\underline{\omega}'}(\underline{\mathbf{k}};\underline{\mathbf{q}}),\end{aligned} \quad (15)$$

where $v_{\underline{\omega};\underline{\omega}'}(\underline{\mathbf{k}};\underline{\mathbf{q}}) \equiv v_{\omega_1,\omega_2,\omega_1',\omega_2'}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}_1,\mathbf{q}_2)$ is

$$\begin{aligned}&2\,\sin\left[(\mathbf{k}_1-\mathbf{k}_2)\frac{\mathbf{v}_1}{2}+(\omega_1-\omega_2)\frac{\pi}{4}\right] \\ &\quad\times\sin\left[(\mathbf{q}_1-\mathbf{q}_2)\frac{\mathbf{v}_0}{2}-(\omega_1'-\omega_2')\frac{\pi}{4}\right]. \quad (16)\end{aligned}$$

We follow the RG method in Ref. [36]. Integrating out the large momentum scales, we obtain an effective interaction,

$$\begin{aligned}&\sum_{n\geqslant 1}\sum_{\substack{\omega_1,\ldots,\omega_n\\ \omega_1',\ldots,\omega_n'}}\int\frac{d\mathbf{p}_1\cdots d\mathbf{q}_n}{(2\pi)^{4n-1}}\widehat{\psi}^+_{\mathbf{p}_1,\omega_1}\cdots\widehat{\psi}^+_{\mathbf{p}_n,\omega_n}\widehat{\psi}^-_{\mathbf{q}_1,\omega_1'}\cdots\widehat{\psi}^-_{\mathbf{q}_n,\omega_n'} \\ &\times\delta\left(\sum_{j=1}^n(\mathbf{p}_j+\omega_j\mathbf{p}_F)-\sum_{j=1}^n(\mathbf{q}_j+\omega_j'\mathbf{p}_F)\right)\widehat{w}_{n;\underline{\omega};\underline{\omega}'}(\underline{\mathbf{p}};\underline{\mathbf{q}}),\end{aligned} \quad (17)$$

where $\widehat{w}_{n;\underline{\omega};\underline{\omega}'}$'s are series of Feynman graphs. Some symmetries are of crucial importance. For $R(k_0,k_1) = (k_1,-k_0)$ and $\vartheta(k_0,k_1) = (k_1,k_0)$ because of the explicit formulas of $v_{\underline{\omega};\underline{\omega}'}(\underline{\mathbf{k}};\underline{\mathbf{q}})$ and of the Fourier transform of $S_\omega(\mathbf{x})$, we have

$$\begin{aligned}\widehat{w}_{n;\underline{\omega};\underline{\omega}'}(R\underline{\mathbf{p}};R\underline{\mathbf{q}}) &= e^{i(\pi/2)\sum_j\omega_j}\widehat{w}_{n;\underline{\omega}';\underline{\omega}}(\underline{\mathbf{q}};\underline{\mathbf{p}}), \\ \widehat{w}_{n;\underline{\omega};\underline{\omega}'}(\vartheta\underline{\mathbf{p}};\vartheta\underline{\mathbf{q}}) &= e^{i(\pi/2)\sum_j\omega_j}\widehat{w}_{n;-\underline{\omega};-\underline{\omega}'}(\underline{\mathbf{p}};\underline{\mathbf{q}}).\end{aligned} \quad (18)$$

From power counting, there are two kinds of terms that are not irrelevant: the quartic and the quadratic ones. Using (18), the quartic terms give a local contribution,

$$\lambda'\sum_{\mathbf{x},\omega}\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega}\psi^+_{\mathbf{x},-\omega}\psi^-_{\mathbf{x},-\omega} \quad (19)$$

for $\lambda' = \widehat{w}_{2;+,-;-,+}(0,0;0,0) - \widehat{w}_{2;+,-;+,-}(0,0;0,0) = -4\lambda + O(\lambda'^2)$, the effective coupling constant. The quadratic terms cannot generate any local contribution of the form $\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},-\omega}$ because of the δ function in (17) and the fact that the momenta are small, therefore, using (18), their only local contribution is

$$z\sum_{\mathbf{x},\omega}\psi^+_{\mathbf{x},\omega}\partial_\omega\psi^-_{\mathbf{x},\omega}, \quad (20)$$

for $z = \frac{1}{2}[-i\,\partial_{p_0}\widehat{w}_{1;+;+}(\mathbf{p};\mathbf{p}) - \partial_{p_1}\widehat{w}_{1;+;+}(\mathbf{p};\mathbf{p})]_{\mathbf{p}=0} = O(\lambda^2)$, the field renormalization counterterm and where ∂_ω is the Fourier transform of $ik_0 - \omega k_1$. An important fact to note is that, in (20), we have not included any mass term: The localization of the quadratic terms would give $m\sum_{\mathbf{x},\omega}\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega}$, for $m = \widehat{w}_{1;\omega;\omega}(0;0)$, however, because of (18), $\widehat{w}_{1;\omega;\omega}(\mathbf{p};\mathbf{q}) = i\omega\widehat{w}_{1;\omega;\omega}(R\mathbf{q};R\mathbf{p})$ and hence,

$$m = 0.$$

At infrared scales, the β function of this fermion model asymptotically coincides with the β function of the Thirring model, which is vanishing [25]. Hence, scaling each $\psi^\varepsilon_{\mathbf{x},\omega}$ of a factor of $2^{-(1/4)}e^{i\omega(\pi/8)}$ to match the standard normalization of the free part of the Thirring model, by comparison with Ref. [33], we obtain $\lambda_T = -\lambda' + O(\lambda^2)$ where the higher orders are determined by the irrelevant terms.

V. HEIGHT COVARIANCE

For simplicity of notation, we derive (4) in the case of $\mathbf{u} = (-N-\frac{1}{2})\mathbf{v}_0 + \frac{1}{2}\mathbf{v}_1$ and $\mathbf{x} = (2N+1)\mathbf{v}_0$ only, but the formula for any $\mathbf{u}$ and $\mathbf{x}$ follows from the same argument. The height covariance is then given by

$$\sum_{i,j=-N}^{N}\langle \mathbf{v}_0\cdot\nabla h(\mathbf{x}_i)\mathbf{v}_0\cdot\nabla h(\mathbf{x}_j)\rangle,$$

where $\mathbf{x}_j = j\mathbf{v}_0 + \frac{1}{2}\mathbf{v}_1$. In the double summation, the $O(N)$ term cancels [37]: Indeed, since the summation of the height difference over any closed contour is vanishing by definition and since the arrow correlations have a decay faster than the inverse distance, one can replace $j = -N,\ldots,N$ with $|j| \geqslant N+1$. Hence, the height covariance becomes

$$-\sum_{|i|\leqslant N}\sum_{|j|\geqslant N+1}\langle\sigma_1(\mathbf{x}_i)\sigma_1(\mathbf{x}_j)\rangle.$$

Now plug (1) into this formula: The staggered term gives a contribution that is bounded in N, whereas, the unstaggered one gives $2c_0\ln N$ plus terms that are bounded in N.

VI. CONCLUSION

We have shown that the interacting fermions representation of the six-vertex model, in combination with a Renormalization Group approach, provides precise formulas for the long-distance decay of the arrow-arrow correlations for small Δ. More in general, using the ideas in Refs. [32,33], one could show that the scaling limit of the n-point arrow correlations, apart from staggering prefactors, are linear combinations of Thirring model correlations.

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