# Large-amplitude oscillations in the rectangular Fermi accelerator

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Exponential energy growth in a rectangular billiard with an oscillating bar has been clearly demonstrated earlier. Using the log-normal approximation, analytical estimates for the energy growth rate have also been provided. However, these analytical estimates are valid only when the amplitude of oscillation of the bar is small. In this paper, for larger oscillation amplitude, the log-normal approximation is numerically shown to be invalid and analytical estimates are obtained for the true energy growth rate in the case of very large particle velocities or small bar length. It is also shown that when everything else remains constant, the length of the bar which gives rise to maximum energy growth rate decreases as the oscillation amplitude increases and the true value of this maximizing length is also smaller than what is predicted by the log-normal approximation. Thus, the rectangular Fermi accelerator forms a very good example of a contemporary research problem where the limitations of the log-normal approximation can be easily appreciated.

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## I. INTRODUCTION

The Fermi acceleration model was originally proposed by Fermi [1] and later refined by Ulam [2] to explain the acceleration of cosmic rays observed by Hess [3]. This concept has also found immense applications in areas like collisional heating in plasma rf sheaths [4] and models of nuclear fission [5]. The one-dimensional (1D) Fermi-Ulam model [6,7] consists of a particle elastically bouncing between two rigid walls, one of which is fixed and the other of which oscillates periodically. If both the walls are fixed, then the total energy of the particle remains constant, but if one of the walls is moving, the particle gains or loses energy with each collision with this moving wall. It is now well known that for the 1D Fermi-Ulam model, the particle cannot gain energy unboundedly if the wall motion is smooth. However, it has been shown that unbounded energy growth can be achieved when the particle motion takes place in the presence of potentials [8,9] or if one introduces a relativistic factor in the equations of motion [10]. Nonsmooth motion and stochastic fluctuations may also lead to unbounded energy growth [11], which is typically polynomial in time [12].

In the case of two-dimensional (2D) Fermi accelerators with smoothly oscillating walls, it has been shown that unbounded energy growth can be achieved if the frozen billiard [13] is chaotic [14–20] or pseudointegrable [21,22]. The quantum mechanical oscillating chaotic billiards play an important role in the modeling of mesoscopic devices [23]. It has also been shown numerically that there can be a very slow unbounded energy growth even when the static billiard is integrable [24]. In the case of pseudointegrable billiards, it has been predicted that the energy growth is not only unbounded but can also be exponential [22]. In one special case of a rectangular billiard with an oscillating bar in between, it has been shown both analytically and numerically that the energy growth is indeed exponential with a predictable and finely controllable energy growth rate [21]. The static version was previously introduced in Refs. [25,26] and has been used to study weak mixing along filamented surfaces [27]. Robust exponential energy growth has also been demonstrated in a billiard with adiabatic motion of a piston [28].

In Ref. [21], an estimate for the energy growth rate of particles in the rectangle with bar was obtained by using the log-normal distribution, which is a widely used approximation for random multiplicative processes. Though this approximation is valid for small oscillation amplitudes of the bar, it fails to predict the energy growth rate when the oscillations become large. In particular, the true energy growth rate is much lower than the rate predicted by making the log-normal approximation. The goal of this paper is to analyze the energy growth rate for large amplitude oscillations of the bar in more detail and point out important differences from the case of small amplitude oscillations.

In Sec. II, a derivation of the known energy growth rate in a rectangle with bar is briefly described for the sake of completion. A detailed description of the energy growth rate in this Fermi accelerator can be found in Ref. [21]. In Sec. III, the main results of this paper are presented and their implications discussed. The paper is finally concluded with Sec. IV.

## II. RECTANGLE WITH BAR

Consider a rectangular billiard with a bar as shown in Fig. 1. To achieve acceleration, let the bar oscillate slowly: $s = f(\theta) = \overline{f} + \tilde{f}(\theta)$, $\theta \in [0,2\pi]$, $\theta = \omega t$ where $\langle \tilde{f}(\theta) \rangle = 0$. The vertical velocity of the bar is $V(t) = \dot{s} = \omega f'(\theta)$. Let $\theta_n = \omega t_n$ be the phase of the bar at the collision time, $(x_n, y_n)$ be the location of the particle at time $t_n$, and $(u_n, v_n)$ be the velocity vector of the particle immediately after the $n$th collision. The particle undergoes elastic collisions from all the boundaries. Since all wall motions are purely vertical, the horizontal speed $u = |u_n|$ remains a constant of motion, whereas the vertical speed changes when the particle collides with the moving bar:

$$v_{n+1} = 2\omega f_n' - v_n \quad \text{when } \{y_n = f_n, x_n \in [0,\lambda]\} \tag{1}$$

where $f_n = f(\theta_n)$ and $f_n' = f'(\theta_n)$.

Since the horizontal speed stays constant, the time the particle spends oscillating above or below the bar is $T_\lambda = 2\lambda/u$, and the time to return to a vertical section after completing one cycle around the rectangle is $T_L = 2L/u$.

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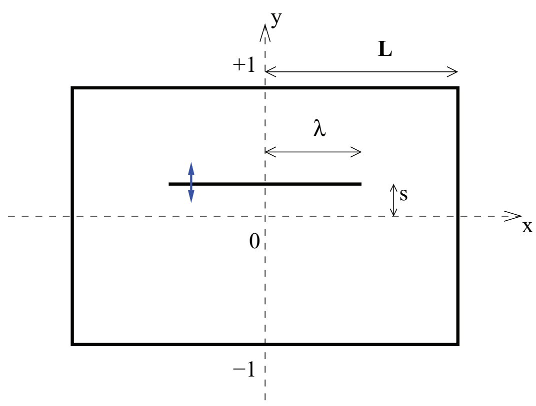


FIG. 1. (Color online) Rectangular billiard, of length $2L$ and height 2 units, with a vertically oscillating horizontal bar of length $2\lambda$ in between. A particle moves within this rectangular region, undergoing elastic reflections from the walls and the horizontal bar. Since the bar is oscillating, collisions with it change the energy of the particle.

Hence, the phase shift of the bar after time $T_\lambda$ is $\theta_\lambda = 2\omega\lambda/u$ and the phase shift of the bar after a full revolution of the particle is $\theta_L = 2\omega L/u$.

As shown in Ref. [21], the motion of particles in this rectangle with bar can be modeled as a random process, with the particle up and down positions being independent random variables. If it is also assumed that the probability to fall into the upper part at the entry phase $\theta$ is proportional to the normalized $y$-interval length, $[1 - f(\theta)]/2$, and is independent of the history of the previous rounds, then $\mu$, the expected value of gain in the kinetic energy in one cycle around the rectangle for a single-particle trajectory that enters the bar with the phase $\theta$ and horizontal velocity $u$ can be estimated as

$$\begin{aligned}\mu_\theta &= \mathbb{E}[\bar{v}^2/v^2]\\ &= \frac{1-f(\theta)}{2}\left(\frac{1-f(\theta)}{1-g(\theta)}\right)^2 + \frac{1+f(\theta)}{2}\left(\frac{1+f(\theta)}{1+g(\theta)}\right)^2\\ &\geqslant 1,\end{aligned}\tag{2}$$

where $g(\theta) = f(\theta + \theta_\lambda)$. The expected value of the energy gain after $N$ cycles is given by

$$\begin{aligned}\mathbb{E}\left[\frac{v^2(NT_L)}{v^2(0)}\right] &= \mu_{\theta_1}\cdot\mu_{\theta_2}\cdots\mu_{\theta_N}\\ &= \exp\left[\sum_{i=1}^{N}\ln\mu_{\theta_i}\right]\\ &= \exp\left[\frac{N}{2\pi}\int_0^{2\pi}\ln\mu_\theta \mathrm{d}\theta\right]\\ &= \exp\left[R_{\theta_\lambda} t\right],\end{aligned}\tag{3}$$

assuming that $\theta_i$ are distributed uniformly over $[0,2\pi]$. Thus, for a finite ensemble of $K$ initial conditions, the observed average energy growth is

$$\left\langle\frac{v^2(t)}{v^2(0)}\right\rangle = \exp\left(R_{\theta_\lambda} t\right) + \mathcal{O}\left(\frac{\sigma}{\sqrt{K}}\right),\tag{4}$$

where $\sigma$ denotes the standard deviation of $v^2(t)/v^2(0)$.

Similarly, the expectation value of the logarithmic gain can be shown to be positive

$$\begin{aligned}m_\theta &= \mathbb{E}[\ln(\bar{v}^2/v^2)]\\ &= [1-f(\theta)]\ln\frac{1-f(\theta)}{1-g(\theta)} + [1+f(\theta)]\ln\frac{1+f(\theta)}{1+g(\theta)}\\ &\geqslant 0,\end{aligned}\tag{5}$$

which implies

$$\begin{aligned}\mathbb{E}\left[\ln\frac{v^2(NT_L)}{v^2(0)}\right] &= m_{\theta_1} + m_{\theta_2} + \cdots + m_{\theta_N}\\ &= \sum_{i=1}^{N} m_{\theta_i}\\ &= \frac{N}{2\pi}\int_0^{2\pi} m_\theta \mathrm{d}\theta := M_{\theta_\lambda} t.\end{aligned}\tag{6}$$

Equation (2) implies an exponential energy growth for the expected value of the particle's energy (ensemble average), whereas Eq. (5) implies that almost every initial condition also produces an exponentially accelerating trajectory. Assuming that the additive process of Eq. (6) leads to a normal distribution, it can be shown using the properties of the log-normal distribution [29] that

$$R_{LN} = M_{\theta_\lambda} + 0.5 S_{\theta_\lambda}^2,\tag{7}$$

where $S_{\theta_\lambda}^2 t$ is the variance of $\ln\frac{v^2(NT_L)}{v^2(0)}$ and the subscript $LN$ stands for log-normal. If $s = \tilde{f}(\theta) = a\sin\theta$ and $|a| \ll 1$, it can be shown using Eqs. (5), (6), and (7) that

$$R_{\theta_\lambda} \approx 3M_{\theta_\lambda} \approx \frac{6\lambda\omega}{L\theta_\lambda} a^2 \sin^2\frac{\theta_\lambda}{2} \geqslant 0,\tag{8}$$

implying an exponential energy growth for an ensemble of particles. The rate $R_{\theta_\lambda}$ given by Eq. (8) has a global maximum at $\theta_\lambda = \theta_{\max}$ where

$$\theta_{\max} = \tan(0.5\theta_{\max}) \approx 2.35.\tag{9}$$

A phase space interpretation of this exponential energy growth based on a comparison with the 1D Fermi-Ulam model has been presented in Ref. [30]. An important point to note in this context is that though the energy growth rate predicted by the above random walk model is quite accurate, a verification of the basic assumptions of this model requires further investigation. For example, it has been assumed that the probability of the particle going above the bar is proportional to the normalized $y$-interval length, $[1 - f(\theta)]/2$. Though this assumption is reasonable because the system under consideration is ergodic, strictly speaking, ergodicity comes into play only over very long intervals of time or over large ensembles. Another assumption made in the above model is that the probability of the particle going above or below the bar in each cycle is independent of the previous cycles. This assumption also rests on the property of ergodicity and is strictly valid only in the long-time or large ensemble limit. It is for this reason that the above random walk model fails to correctly predict the energy growth rate of each trajectory but correctly captures the energy growth rate of a large ensemble of particles.

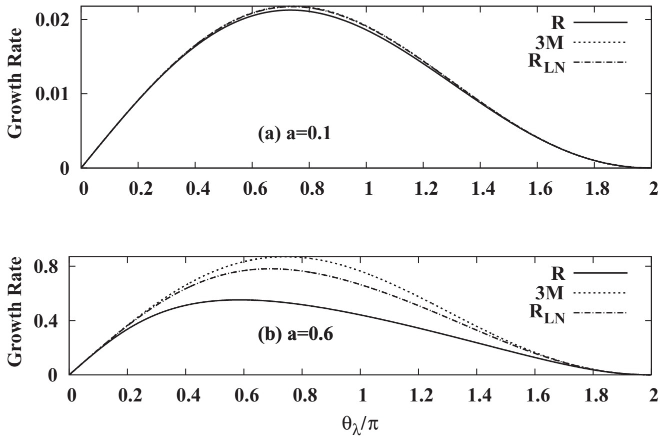


FIG. 2. This figure shows the dependence of $R_{\theta_\lambda}$, $M_{\theta_\lambda}$, and $R_{LN}$ on $\theta_\lambda$. The two subplots (a) $a = 0.1$ and (b) $a = 0.6$ show the remarkable difference between $R_{\theta_\lambda}$ and $R_{LN}$ as $a$ increases. In panel (a), $R_{\theta_\lambda} \approx R_{LN} \approx 3M_{\theta_\lambda}$ for $a = 0.1$, but in panel (b), when $a = 0.6$, we can clearly see that $R_{\theta_\lambda}$ is much smaller than $R_{LN}$, which in turn is lower than $3M_{\theta_\lambda}$.

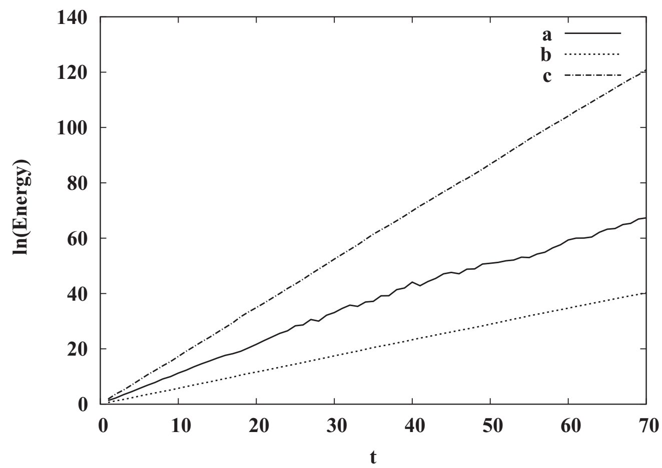


FIG. 3. This figure shows the logarithmic energy growth (with base e) for $a = 0.6$. Curve a is the true energy growth rate, $R_{\theta_\lambda}$, for an ensemble of 1000 particles in the rectangle with bar. Curve b has an energy growth rate of $M_{\theta_\lambda}$ and curve c has an energy growth rate, $R_{LN}$, as predicted by the log-normal approximation. As can be clearly seen, right from $t = 0$, the ensemble energy grows at a much lower rate than predicted by the log-normal approximation.

## III. LARGE AMPLITUDE OSCILLATIONS

In the previous section, a derivation of the exponential energy growth rate of particles in a rectangle with bar has been briefly presented (for details, see Ref. [21]). In this section, we consider the case of large amplitude oscillations of the bar and show that the log-normal approximation does not hold when $a$ is large. For large $a$, we also show that the value of $\theta_\lambda$ which maximizes $R_{\theta_\lambda}$ is substantially lower than the value of $\theta_\lambda$ which maximizes $M_{\theta_\lambda}$ or $R_{LN}$.

### A. True energy growth rate

A method for integrating Eq. (3) exactly is not known to the author and recourse must be taken to a perturbative expansion with the smallness parameter being the oscillation amplitude, $a$. However, this perturbative expansion would be practically meaningful only when $a$ is small since otherwise a large number of terms will be required. Also, when $a$ is large, the perturbative expansion may not converge at all. The result of numerical integration of Eq. (3) to obtain the dependence of $R_{\theta_\lambda}$ on $\theta_\lambda$ is shown in Fig. 2. As can be clearly seen, though for a small value of $a = 0.1$ we have $R_{\theta_\lambda} \approx R_{LN} \approx 3M_{\theta_\lambda}$, for a large value of $a = 0.6$ $R_{\theta_\lambda}$ is much smaller than both $R_{LN}$, which in turn is smaller than $3M_{\theta_\lambda}$. This shows that the log-normal approximation is valid only for small values of $a$, and for larger values of $a$, we need to solve Eq. (3) explicitly.

The fact that for large values of $a$, the true energy growth rate is much smaller than what is predicted by the log-normal approximation is clear from Fig. 3. This figure is not obtained by exactly simulating the Fermi accelerator but by simulating the equivalent random process described in Sec. II. The full simulation for the Fermi accelerator could not be done for such large values of $a$ since the energy growth rate is too large in this case and the computation reaches its accuracy limits for very small values of $t$. In Fig. 3, curve (a) shows the true energy growth rate for the rectangle with bar. Curve (b) has an exponential energy growth rate equal to $M_{\theta_\lambda}$ and curve (c) has an exponential energy growth rate equal to $R_{LN}$. As can be clearly seen, the true energy growth rate, $R_{\theta_\lambda}$, is much lower than that given by $R_{LN}$.

For large values of $a$, though it is difficult to analytically integrate Eqs. (3) and (6) for arbitrary values of $\theta_\lambda$, it can be shown that for small values of $\theta_\lambda$

$$R_{\theta_\lambda} = \frac{2\lambda\omega}{L}\left[\frac{3a^2\theta_\lambda^3}{4\sqrt{1-a^2}} + 3\left(\theta_\lambda - \frac{7\theta_\lambda^3}{12}\right)(1-\sqrt{1-a^2}) - \frac{9\theta_\lambda^3}{4}\left(1+\frac{a^4+a^2-2}{2\sqrt{1-a^2}}\right) + \mathcal{O}(\theta_\lambda^5)\right] \quad (10)$$

and

$$M_{\theta_\lambda} = \frac{2\lambda\omega}{L}\left[\frac{a^2\theta_\lambda^3}{4\sqrt{1-a^2}} + \left(\theta_\lambda - \frac{7\theta_\lambda^3}{12}\right)(1-\sqrt{1-a^2}) + \mathcal{O}(\theta_\lambda^5)\right] \quad (11)$$

and

$$S_{\theta_\lambda}^2 = 4M_{\theta_\lambda} + \frac{2\lambda\omega}{L}\left[\theta_\lambda^3\left(8 - \frac{13a^4-8a^2+16}{2\sqrt{1-a^2}}\right) + \mathcal{O}(\theta_\lambda^5)\right]. \quad (12)$$

In the above equations, it can be clearly seen that for large $a$, the expressions for $R_{\theta_\lambda}$ and $3M_{\theta_\lambda}$ are almost the same except for the last term in Eq. (10). It is this term that leads to a lower energy growth rate than what is predicted by the log-normal approximation. On comparing Eqs. (10), (11), and (12), we find that the difference between $R_{\theta_\lambda}$ and $R_{LN} = M_{\theta_\lambda} + 0.5S_{\theta_\lambda}^2$

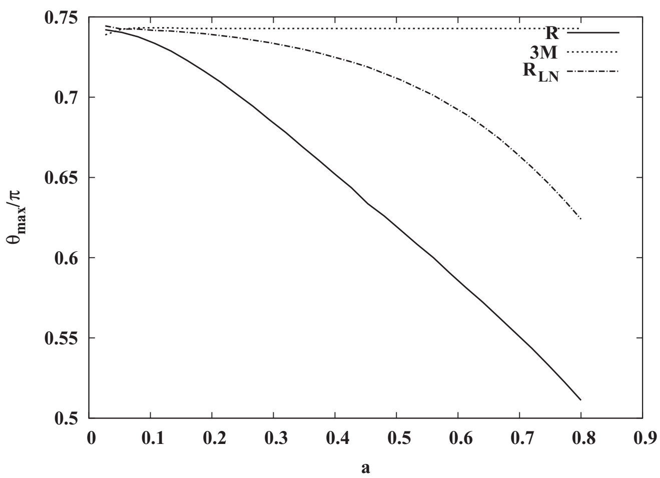


FIG. 4. This figure shows the dependence of $\theta_{\max}$ on the amplitude of oscillation, $a$. As can be seen, $\theta_{\max}$ is almost constant for $M_{\theta_\lambda}$ but decreases with increase in $a$ for both $R_{\theta_\lambda}$ and $R_{LN}$.

is given by

$$\frac{R_{\theta_\lambda} - R_{LN}}{2\lambda\omega/L} = -\frac{9\theta_\lambda^3}{4}\left(1 + \frac{a^4 + a^2 - 2}{2\sqrt{1-a^2}}\right) - \frac{\theta_\lambda^3}{2}\left(8 - \frac{13a^4 - 8a^2 + 16}{2\sqrt{1-a^2}}\right) + \mathcal{O}(\theta_\lambda^5)$$
$$= -\theta_\lambda^3\left(\frac{25}{4} - \frac{17a^4 - 25a^2 + 50}{8\sqrt{1-a^2}}\right) + \mathcal{O}(\theta_\lambda^5), \quad (13)$$

which is always negative and $\mathcal{O}(a^4)$ and thus is not negligible if the oscillation amplitude, $a$, is large. Equation (13) also shows that $R_{\theta_\lambda} < R_{LN}$, implying that the true energy growth rate is much lower than that predicted by the log-normal approximation for large $a$.

### B. Maximizing value of $\theta_\lambda$

In Fig. 2, it can also be seen that the value of $\theta_\lambda$ at which $R_{\theta_\lambda}$ and $R_{LN}$ take their maximum values varies with $a$. However, the value of $\theta_\lambda$ at which $M_{\theta_\lambda}$ takes its maximum value is almost constant and does not depend on $a$. A plot of the value of $\theta_\lambda$ at which $R_{\theta_\lambda}$, $M_{\theta_\lambda}$, and $R_{LN}$ take their maximum values for different values of $a$ is shown in Fig. 4. As can be seen in the figure, for larger values of $a$, $R_{\theta_\lambda}$ and $R_{LN}$ take their maximum values for smaller values of $\theta_\lambda$. Also, for a given value of $a$, $\theta_{\max}(R_{\theta_\lambda}) < \theta_{\max}(R_{LN})$. Since $\theta_\lambda = 2\omega\lambda/u$, this implies that when everything else remains same, the bar length for which the true ensemble energy growth is maximum is lower than that predicted by the log-normal approximation.

The dependence of $\theta_{\max}$ on $a$ for $R_{\theta_\lambda}$ has been found to fit well to a fourth-order polynomial with a root mean square error (RMSE) of 0.000285,

$$\left.\frac{\theta_{\max}}{\pi}\right|_{R_{\theta_\lambda}} = -0.7877a^4 + 1.497a^3 - 1.072a^2 + 0.0136a + 0.7425.$$

Fitting the curve to a third-order polynomial gives an RMSE of 0.00167, which is much higher, and fitting to a fifth-order polynomial gives an RMSE of 0.000274, which is very close to the RMSE of a fourth-order fit. Thus, for all practical purposes, a fourth-order fit of $\theta_{\max}$ for $R_{\theta_\lambda}$ is good enough. Similarly, the dependence of $\theta_{\max}$ on $a$ for $R_{LN}$ is also found to fit well to another fourth-order polynomial,

$$\left.\frac{\theta_{\max}}{\pi}\right|_{R_{LN}} = -0.1363a^4 - 0.06412a^3 - 0.02703a^2 - 0.0181a + 0.7442,$$

with an RMSE of 0.0002755. Fitting this curve to a third-order polynomial gives an RMSE of 0.0003896 and a fifth-order fit gives an RMSE of 0.0002802, implying that a fourth-order fit is good enough in this case too. This shows that the functional form for $\theta_{\max}$ is same for both $R_{\theta_\lambda}$ and $R_{LN}$, but the actual values of $\theta_{\max}$ are very different in these two cases for larger values of $a$.

## IV. CONCLUSION

In this paper, it has been shown that the true energy growth rate of an ensemble of particles in the rectangular Fermi accelerator is lower than what is predicted by the log-normal distribution. This can be clearly understood by considering a simple binomial random process. For any given $\theta$ in Eq. (2), let the probability of the particle going up or down the bar be $p$ and $q = 1 - p$ respectively, and let the gain of energy of the particle if it goes up or down be $z_1$ and $z_2$ respectively. If $a$ is small, then $z_1 \approx z_2$ and $p \approx q = 1/2$. Under these conditions, the log-normal distribution is a valid approximation for the random multiplicative process, but when $a$ is large, $p \not\approx q$ and $z_1 \not\approx z_2$. As was shown in Ref. [31], in this case the log-normal approximation can give incorrect estimates for the mean of the product of random variables. It must be noted that there is no critical value of the oscillation amplitude beyond which the corrections become important. The corrections gradually become larger as the oscillation amplitude is increased. The presence of a critical value, if any, will depend on the particular application being considered.

Though the limitations of the log-normal approximation are quite well known [31], this paper provides an example of a contemporary research problem where these limitations can be easily appreciated.

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