# Limiting phase trajectories and emergence of autoresonance in nonlinear oscillators

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Existence of stable autoresonance (AR) with continuously growing energy is directly connected with the inherent property of nonlinear systems to remain in resonance when the driving frequency varies in time. However, the physical mechanism underlying the transformation of bounded oscillations into AR remains unclear. As this paper demonstrates, the emergence of AR from stable bounded oscillations is basically analogous to the transition from quasilinear to nonlinear oscillations in the time-invariant oscillator driven by an external harmonic excitation with constant frequency, and AR can occur as a result of the loss of stability of the so-called limiting phase trajectory. We obtain the parametric threshold, which determines the transition from bounded oscillations to AR in the time-dependent system. The accuracy of the obtained approximations is confirmed by numerical simulations.

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## I. INTRODUCTION

The phenomenon of the permanent growth of energy in a classical anharmonic oscillator subject to slow variations of forcing and/or resonant frequencies is referred to as *autoresonance* (AR). After first studies for the purposes of particle acceleration [1,2], AR has become a very active field of research. Theoretical approaches, experimental evidence, and applications of AR in different fields of natural science, from plasmas to planetary dynamics, are reported in numerous papers [2]; additional theoretical and computational results can be found in Refs. [4–6]; recent advances in this field are discussed, e.g., in Refs. [3,7–9].

It was noticed [10] that the physical mechanism behind autoresonance can be interpreted as adiabatic nonlinear phase-locking between the system and the driving signal. However, to the best of the authors' knowledge, the mechanism causing the transition from bounded oscillations to AR has not been reported in the literature.

The results presented in this work clearly indicate that the emergence of AR is similar to the transition from small (quasilinear) to large (nonlinear) oscillations in the system with constant parameters and constant excitation frequency. We demonstrate that AR occurs due to the loss of stability of the so-called limiting phase trajectory (LPT) of small oscillations. It is shown that a critical parameter, which determines a boundary between small and large oscillations in the time-invariant system, may be treated as a lower threshold of autoresonance in the oscillator with slowly varying parameters. Furthermore, it is demonstrated that the threshold parameter numerically obtained, e.g., in Refs. [11,12] is unacceptable in the problem examined in this paper. Conditions of the transition from small to large oscillations in the time-independent Duffing oscillator is used to derive the critical sweeping rate. The obtained analytical estimates are proved to be very close to the results of numerical simulations. Unlike previous investigations [3], systems with both linear and nonlinear-in-time detuning laws are examined.

It is important to note that the Duffing system is chosen for illustrative purposes. The qualitative features of the results hold true for a large class of nonlinear oscillators with slowly time-varying frequencies.

## II. THE MODEL

For brevity, we consider the time-dependent Duffing oscillator with the constant excitation frequency. The equation of motion is given

$$\frac{d^2u}{dt^2} + [1 - \varepsilon\zeta(\tau_1)]u + 8\varepsilon\alpha u^3 = 2\varepsilon F\cos t, \quad (1)$$

where $\varepsilon > 0$ is a small parameter of the system, $\tau_1 = \varepsilon t$, $\zeta(\tau_1) = s + b\tau_1^n$. The initial conditions $u = 0$, $v = du/dt = 0$, at $t = 0$, determine the so-called *limiting phase trajectory* of system Eq. (1), corresponding to the maximum possible energy transfer from the source of energy to the oscillator.

Asymptotic solutions of Eq. (1) for small $\varepsilon$ are derived using the multiple scale method [13]. We introduce the complex-conjugate variables $\varphi$ and $\varphi^*$,

$$\varphi = (v + iu)e^{-it}, \quad \varphi^* = (v - iu)e^{it}, \quad (2)$$

and then construct the function $\varphi$ in the form of the asymptotic series,

$$\varphi(t,\tau_1) = \varphi_0(\tau_1) + \varepsilon\varphi_1(t,\tau_1) + \varepsilon^2 \ldots, \tau_1 = \varepsilon t, \quad (3)$$

with the main slow term $\varphi_0(\tau_1)$ satisfying the equation (see, e.g., Refs. [14,15])

$$\frac{d\varphi_0}{d\tau_1} + i\left[\zeta(\tau_1) - 3\alpha\left|\varphi_0^2\right|\right]\varphi_0 = -iF, \quad \varphi_0(0) = 0. \quad (4)$$

Consider the case of $s > 0$, $\alpha > 0$. The change of variables

$$\tau = s\tau_1, \quad \psi(\tau_1) = \lambda^{-1}\varphi_0(\tau_1), \quad \lambda = (s/3\alpha)^{1/2} \quad (5)$$

and rescaling $\beta = b/s^{n+1}$, $f = F/s\lambda = F\sqrt{3\alpha/s^3}$ reduce Eq. (4) to the two-parameter equation for the complex-valued envelope $\psi$:

$$\frac{d\psi}{d\tau} + i(1 + \beta\tau^n - |\psi|^2)\psi = -if, \quad \psi(0) = 0. \quad (6)$$

 

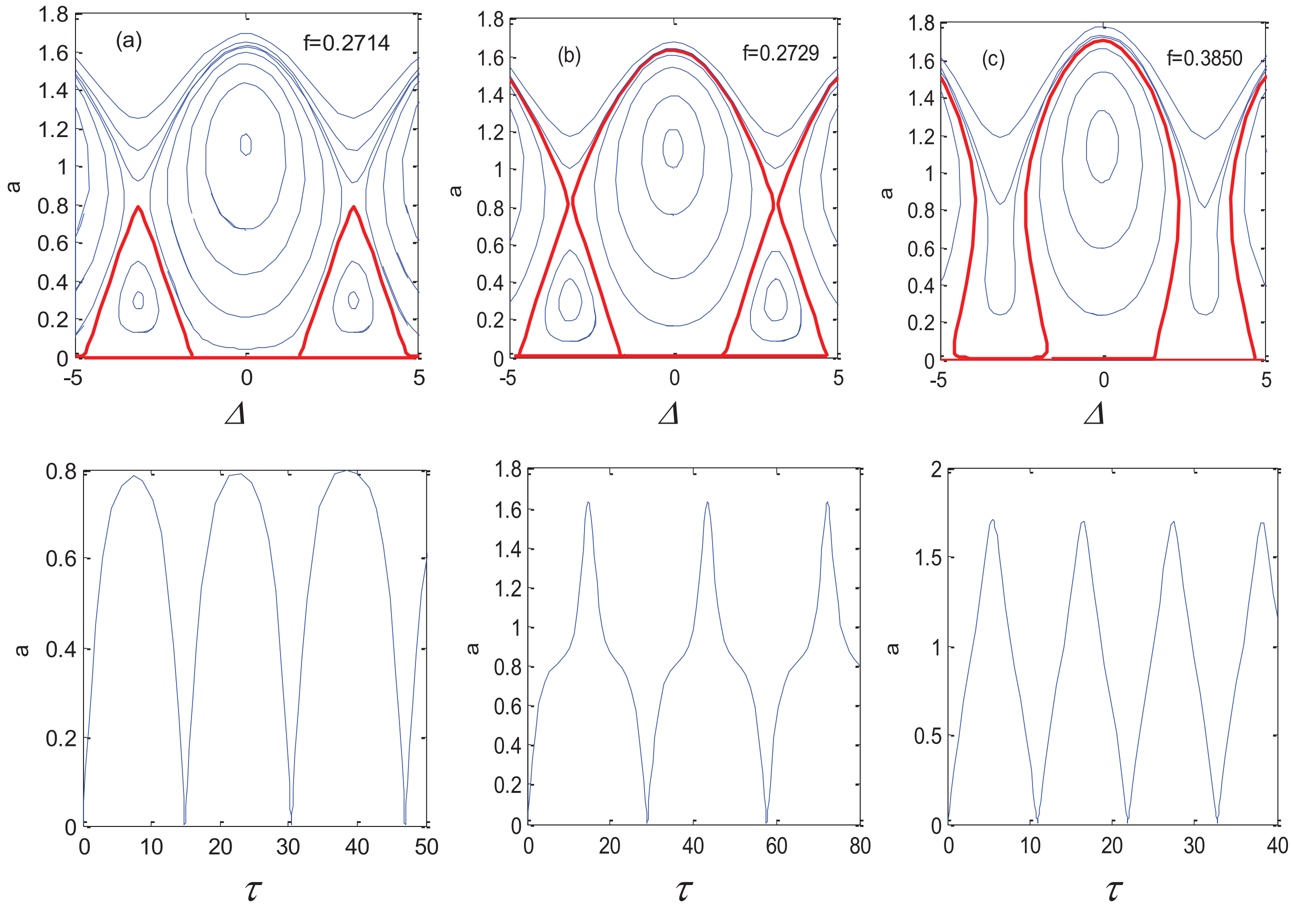


FIG. 1. (Color online) Phase portraits (top row) and envelopes (bottom row) for systems with weak [column (a)], moderate [column (b)], and strong [column (c)] nonlinearity.

The polar representation $\psi = ae^{i\Delta}$ transforms Eq. (6) into the system for the real envelope $a > 0$ and phase $\Delta$

$$\begin{aligned} \frac{da}{d\tau} &= -f \sin\Delta, \\ \frac{d\Delta}{d\tau} &= -(1+\beta\tau^n) + a^2 - a^{-1} f \cos\Delta, \end{aligned} \tag{7}$$

with initial conditions $a(0) = 0$, $\Delta(0) = -\pi/2$. It follows from Eqs. (2)–(5) that the quantities $\lambda a(\tau/s)$ and $\Delta(\tau/s)$ provide the main approximations for the real-valued envelope and the phase of the solution of Eq. (1).

If $s < 0$, $\alpha < 0$, then the change of variables $\tau = |s|\tau_1$, $\psi(\tau_1) = \lambda^{-1}\varphi_0(\tau_1)$, $\lambda = |s/3\alpha|^{1/2}$, and the transformations $\beta = b/|s^{n+1}|$, $f = F/|s\lambda|$ convert Eq. (4) into the complex-valued equation similar to Eq. (6). In what follows, we examine in detail the case of $s > 0$, $\alpha > 0$.

For better understanding of the emergence of the unbounded modes, we first consider the underlying *time-invariant* system, namely,

$$\begin{aligned} \frac{da}{d\tau} &= -f \sin\Delta, \\ \frac{d\Delta}{d\tau} &= -1 + a^2 - a^{-1} f \cos\Delta, \end{aligned} \tag{8}$$

with initial conditions $a(0) = 0$, $\Delta(0) = -\pi/2$ corresponding to the LPT. Figure 1 clearly demonstrates the "limiting" property of the LPTs in the time-invariant system. It is seen that the LPT represents an outer boundary for a set of closed trajectories encircling the stable center in the phase plane $(\Delta, a)$.

It was proved [14] that there exist two critical relationships that define the boundaries between different types of solutions of system Eq. (8). Using our notations, these critical values can be rewritten as

$$f_1 = \sqrt{2/27} \approx 0.2721, \quad f_2 = 2/\sqrt{27} \approx 0.3849. \tag{9}$$

In Fig. 1, it is shown that the threshold $f_1$ corresponds to the boundary between small and large oscillations, namely, at $f = f_1$ the LPT of small oscillations coalesces with the separatrix going through the homoclinic point on the axis $\Delta = -\pi$ [Fig. 1(b)]. This implies that the transition from small to large oscillations occurs due to the loss of stability of the LPT of small oscillations. At $f = f_2$, the stable center on the axis $\Delta = -\pi$ vanishes due to the coalescence with the homoclinic point, and only a single stable center remains on the axis $\Delta = 0$ [Fig. 1(c)].

By definition [15], conditions $f < f_1$, $f_1 < f < f_2$ and $f > f_2$ characterize *quasilinear*, *moderately nonlinear*, and *strongly nonlinear* systems, respectively.

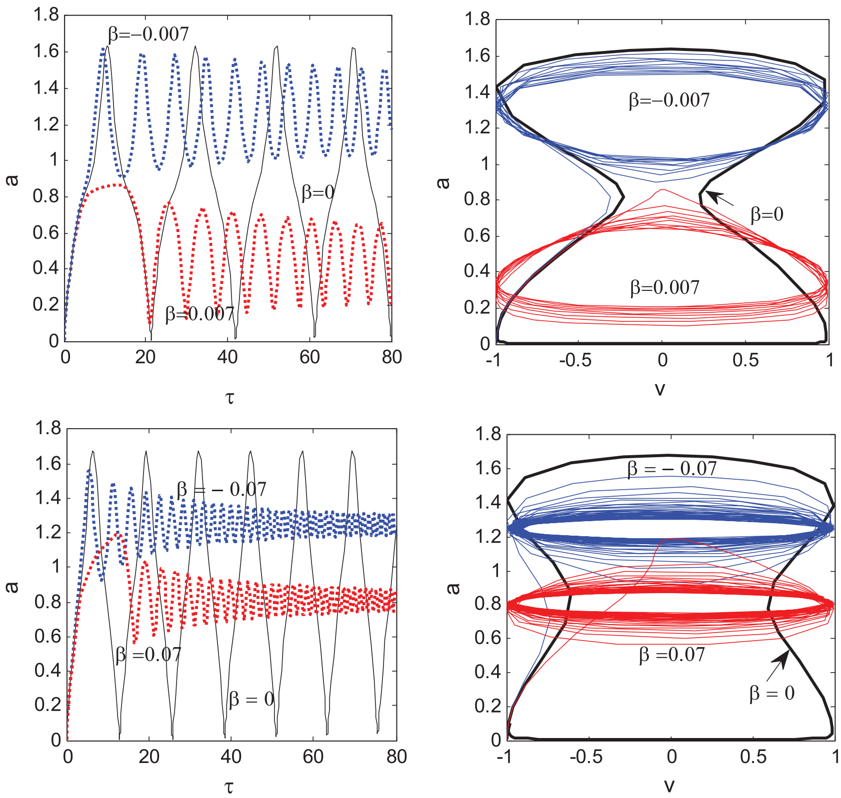


FIG. 2. (Color online) Plots of $a(\tau)$ (left column) and phase portraits (right column) for system Eq. (7); top row: $f = 0.28, \beta = \pm 0.007$; bottom row: $f = 0.34, \beta = \pm 0.07$. Plots of the LPTs in the corresponding time-invarinat systems (solid lines) are shown for comparison.

## III. ANALYSIS OF AUTORESONANCE

In this section, the system with $\alpha > 0$ (hard nonlinearity), $s > 0$, and linear detuning $\beta\tau$ is considered in detail. Numerical results for a quasilinear system with $f_1 < f < f_2$ are presented in Fig. 2. It is seen that in the first half-cycle of oscillations, the envelope $a(\tau)$ is very close to the LPT of the time-independent system; in the case of $\beta > 0$ detuning $\beta\tau$ is increased with an increase of $\tau$, thereby shifting the system to the domain of small oscillations; in the case of $\beta < 0$, detuning is decreased and, thus, shifts the system to the domain of large oscillations. The projection of the trajectory $a(\tau)$ onto the phase plane $(a, v)$ represents the spiral orbit with an attracting focus $(a = a_0, v = 0)$, where $a_0 = \lim a(\tau)$ as $\tau \to \infty$. The calculation of the limiting value $a_0$ is suggested in Refs. [15,16].

Figure 3 depicts the emergence of AR from stable bounded oscillations under changes of the rate $\beta > 0$. It is seen that initially the shape of small oscillations is close to the LPT of quasilinear oscillations, while the shape of AR is similar to the LPT of the system with moderate nonlinearity. It follows then that the transition from bounded to unbounded oscillations in the system with slowly varying frequency is of the same nature as the transition from small to large oscillations in the system with constant parameters; namely, it occurs due to the destruction of the LPT of quasilinear oscillations.

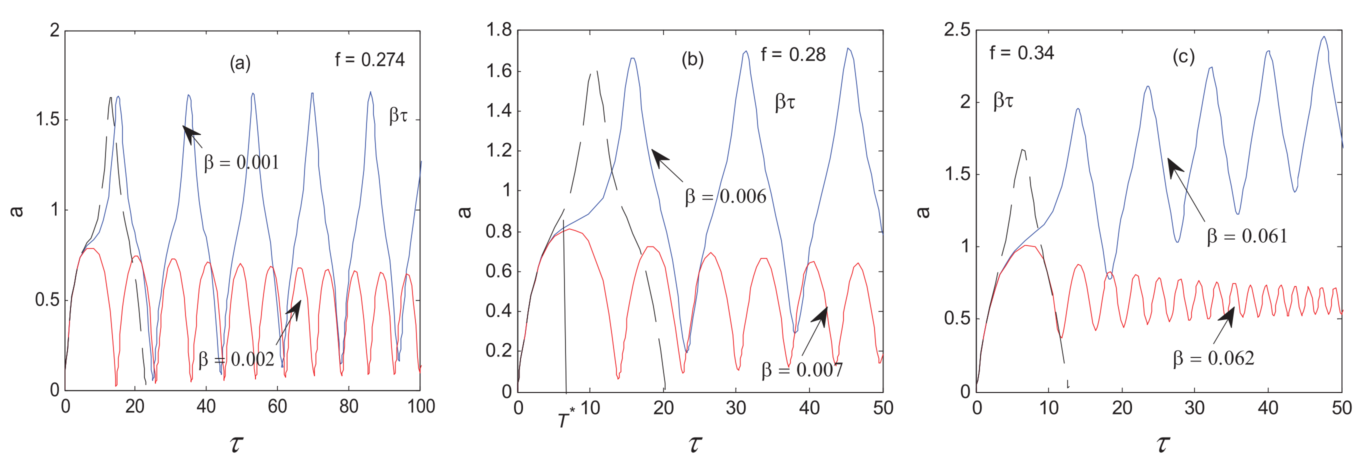


FIG. 3. (Color online) Transitions to AR in system Eq. (7) with different values of $f$ and detuning $\beta\tau$; the cycle of oscillations in the time-independent system (dashed line) is demonstrated for comparison.

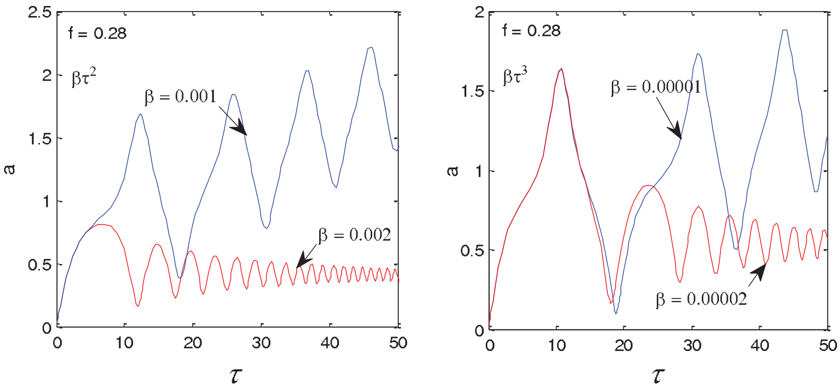


FIG. 4. (Color online) Transition from bounded oscillations to AR in system Eq. (7) with $f = 0.28$ and the quadratic-in-time detuning laws $\beta\tau^2$ (left) and $\beta\tau^3$ (right).

As seen in Fig. 3, under very slow sweep the transition from bounded oscillations to AR takes place if the parameter $f$ is close to the critical value $f_1$. It is shown that at $f = 0.274$, the transition occurs at $0.001 < \beta < 0.002$; the difference between $f$ and $f_1 = 0.2721$ is less than 1%; at $f \approx 0.28$ the transition takes place at $0.006 < \beta < 0.007$; the difference between $f$ and $f_1$ is less than 2.8%. On the other hand, for $f = 0.34$, the critical rate $0.061 < \beta < 0.062$; the difference between $f$ and $f_1$ is about 20%. This implies that the inequality

$$f > f_1 \tag{10}$$

can be interpreted as *the necessary condition* of the emergence of AR.

Figure 4 indicates that the transition from bounded to unbounded oscillations in systems with nonlinearly time-dependent detuning follows the same scenario as in the previous example; that is, AR occurs after the destruction of the LPT of small oscillations.

The obtained numerical results motivate the derivation of an analytical threshold between bounded and unbounded oscillations. First, we recall the reported results. The change of variables $\Psi = |\beta|^{-1/4}\psi$, $\tilde{\tau} = |\beta|^{1/2}(\tau + 1/\beta)$, $\mu = |\beta|^{-3/4} f$ reduces Eq. (6) to the Schrödinger-type equation

$$\frac{d\Psi}{d\tilde{\tau}} + i(\tilde{\tau} - |\Psi|^2)\Psi = -i\mu, \quad \Psi(\tilde{\tau}_0) = 0, \tag{11}$$

where $\tilde{\tau}_0 = |\beta^{-1/2}|\mathrm{sign}\beta$. It was found numerically [11,12] that in this system autoresonance occurs for $\mu > \mu_{\mathrm{th}} = 0.41$. The threshold $\mu_{\mathrm{th}} = 0.41$ was first treated as independent of $\tilde{\tau}_0$ [11] but a more thorough study [12] demonstrated that for $\tilde{\tau}_0 > 0$ or, by definition, $\beta > 0$ the threshold value $\mu_{\mathrm{th}}$ grows significantly when $\tilde{\tau}_0$ increases or $\beta$ decreases. Thus, even if we omit a discussion of the applicability of a numerically found threshold to a large class of physical problems, we should underline that this threshold is unusable in the problem under consideration, wherein the effect of small values of $\beta > 0$ is examined.

## IV. CRITICAL RATE

In this section, we evaluate the rate $\beta$, at which the transition from bounded to unbounded oscillations occurs. We show that admissible values of $\beta$ obtained from the above defined threshold $\mu_{\mathrm{th}} = 0.41$ significantly exceed the rate found by numerical simulations.

In order to evaluate the critical rate, we employ the fact that for sufficiently small $\tau$ the solution $a(\tau)$ of system Eq. (7) is very close to the LPT of the time-independent system Eq. (8). We recall that the LPT of the moderately nonlinear ($f_1 < f < f_2$) time-invariant systems has a distinctive inflection at $\tau = T^*$ [Fig. 3(b)]. We introduce the time-dependent parameter $\tilde{f}(\tau) = f/(1 + \beta\tau^n)^{3/2}$ such that $\tilde{f}(0) = f > f_1$. The analysis numerical results presented in Figs. 2–4 allows one to conclude that an adiabatically varying system in which $\tilde{f}(0) > f_1$ gets captured into the domain of small oscillations if $\tilde{f}(T^*) < f_1$. Under this assumption, the critical rate is given by

$$\beta^* = (T^*)^{-n}[(f/f_1)^{2/3} - 1]. \tag{12}$$

If $\beta < \beta^*$, the system admits the persistence of AR. In order to check the correctness of Eq. (12), we calculate the critical rate $\beta^*$ in the system with linear-in-time detuning ($n = 1$). First, the instant $T^*$ is defined from the obtained numerical results. In the next step, the analytical estimate of $T^*$ and the respective value $\beta^*$ will be derived.

We recall that the point of inflection is determined by the conditions $da/d\tau \neq 0$, $d^2a/d\tau^2 = 0$. It follows from Eq. (8) that the latter condition corresponds to $d\Delta/dt = 0$; i.e., the envelope $a(\tau)$ achieves the point of inflection when the phase $\Delta$ achieves its minimum (Fig. 5).

We find from Fig. 5 that $T^* \approx 6.5$ for $f = 0.274$; this yields $\beta^* \approx 0.00075$, though the computational result gives $0.001 < \beta < 0.002$. Then, $T^* \approx 5$ and $\beta^* \approx 0.004$ for $f = 0.28$, while the numerical simulation gives $0.006 < \beta < 0.007$. Note that for $f = 0.28$ the threshold parameter $\mu_{\mathrm{th}} = |\beta_{\mathrm{th}}|^{-3/4} f = 0.41$ yields $\beta_{\mathrm{th}} = (f/\mu_{\mathrm{th}})^{4/3} = 0.577$, which is vastly larger than the real threshold rate. In a similar way, we find that for $f = 0.34$ the critical rate $\beta^* = 0.053$, while the numerical simulation gives $0.061 < \beta < 0.062$. Note that at $f = 0.34$ the inflection of the curve $a(\tau)$ is practically indistinguishable, but the phase has the pronounced minimum at $T^* \approx 3$ (Fig. 5).

It is important to note that, in contrast to the results reported in Refs. [11,12], Eq. (12) defines the critical rate for systems with both linear and nonlinear-in-time detuning laws. For example, in the case of quadratic detuning $\beta\tau^2$ and $f = 0.28$, we find $\beta^* = 0.0008$; at the same time, the numerical simulation gives $0.001 < \beta < 0.002$ (Fig. 4).

The analytical derivation of the inflection time $T^*$ and the point of inflection $a^*$, $\Delta^*$ is built upon the results obtained in

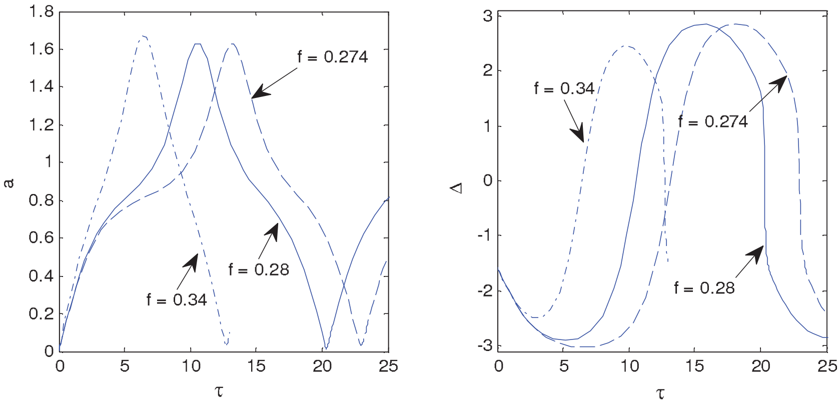


FIG. 5. (Color online) Envelopes $a(\tau)$ and phases $\Delta(\tau)$ for different values of the parameter $f$.

Ref. [14]. We recall that system Eq. (8) is integrable; the first integral of motion takes the form

$$K = a\left(\frac{a^3}{4} - \frac{a}{2} - f\cos\Delta\right) = 0. \tag{13}$$

Using Eq. (13) to exclude $\Delta$, we obtain the equation for the variable $a(\tau)$,

$$\frac{d^2a}{d\tau_1^2} + \psi(a) = 0, \tag{14}$$

with initial conditions $a = 0$, $da/d\tau_1 = f$, at $\tau_1 = 0$. The function $\psi(a)$ is given by

$$\psi(a) = \frac{a}{4}\left(\frac{a^2}{2} - 1\right)\left(\frac{3a^2}{2} - 1\right) = \frac{d\Psi}{da},$$
$$\Psi(a) = \int_0^a \psi(x)dx = \frac{a^2}{8}\left(\frac{a^2}{2} - 1\right)^2. \tag{15}$$

It follows from Eq. (14) that the point of inflection $a^*$ is defined by the condition $\psi(a^*) = 0$; i.e.,

$$a^* = \sqrt{2/3} = 0.8165. \tag{16}$$

The respective value of the phase $\Delta^*$ can be found from the condition $d\Delta/d\tau = 0$. It is easy to derive from Eqs. (8) and (16) that $\cos\Delta^* = a^*[(a^*)^2 - 1]/f$, where $a^*[(a^*)^2 - 1] = -0.2721 = -f_1$; therefore,

$$\cos\Delta^* = -f_1/f. \tag{17}$$

Figure 6 depicts the potential $\Psi(a)$. Since the maximum of $\Psi(a)$ is defined by the condition $d\Psi/da = \psi(a) = 0$, the potential barrier (the dashed line) goes through the point of inflection $a = a^*$. It follows then that the sought time $T^*$ equals the time $T_1^*$ needed to reach the potential barrier. The latter parameter was approximately calculated in Ref. [14]. Using our notations, we obtain from Ref. [14] that

$$T_1^* \approx 3\ln\frac{(f^2 - f_1^2)^{1/2}}{f - f_1} = \frac{3}{2}\ln\frac{f + f_1}{f - f_1}. \tag{18}$$

For example, in the cases $f = 0.274$ and $f = 0.28$, we obtain the approximations $T_1^* \approx 7.48$ and $T_1^* \approx 5.9$, which exceed the corresponding numerical values $T^*$ (Fig. 5) for 15%. It follows then that the substitution of $T_1^*$ for $T^*$ into Eq. (12) gives the rate $\beta_1^* < \beta^*$. Therefore, detuning with rate $\beta < \beta_1^* < \beta^*$ allows the occurrence of autoresonance.

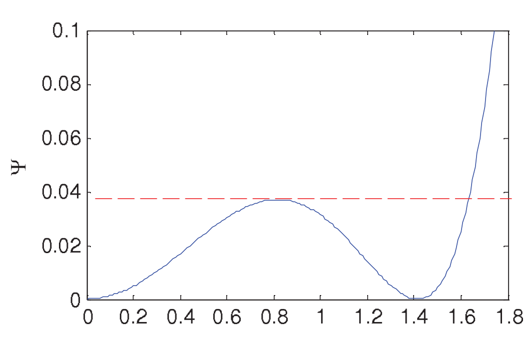


FIG. 6. (Color online) Plot of the potential $\Psi(a)$; the dashed line depicts the potential barrier.

## V. CONCLUSIONS

The emergence of autoresonance in the slowly time-dependent Duffing oscillator was investigated using the concept of LPT. It was shown that the emergence of AR from stable bounded oscillations is similar to the transition from small to large oscillations in the time-invariant oscillator driven by an external harmonic excitation with constant frequency, and AR results from the loss of stability of the so-called limiting phase trajectory separating the domains of small and large oscillations. The LPT concept allows finding the critical parameters, which determine the change of bounded oscillations to AR with continuous growth of energy.

Note that the considered Duffing model was chosen only for illustrative purposes. The obtained results can be extended to a more general case of the arrays of nonlinear oscillators.

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