Kinetic theory of fully degenerate electrons in the long scale limit

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The kinetic theory of fully degenerate electrons in a weakly coupled plasma is considered. We derive an evolution equation for a generalized Fermi surface that also depends on the electron spin state. The equation allows for the study of weakly nonlinear modifications of the Fermi surface within perturbation theory. We apply the theory to Landau damping of ion-acoustic waves. The transition to the nonlinear stage and the nonlinear modification of the Fermi surface are investigated.

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I. INTRODUCTION

Recently there has been much interest in the properties of quantum plasmas [1–7]. The applications include quantum wells [8], spintronics [9], and plasmonics [10]. Quantum plasma effects can also be of interest in experiments with solid density targets [11]. Important classifications of dense plasmas depend on whether they are strongly or weakly coupled, and whether they are degenerate or nondegenerate [12]. Much attention has been given to the weakly coupled fully degenerate plasma [13–17], which has been studied both within fluid theory and kinetic theory. For a weakly coupled plasma, the effects of the many-body physics (e.g., exchange effects) as well as collisions are corrections that may be omitted as a first approximation. See, e.g., Ref. [18] for the accuracy of this approximation. Furthermore, when the dynamics associated with the spin is neglected, such a plasma can be described by the Wigner equation [1–4]. For long scale lengths the Wigner equation reduces to the Vlasov equation. Although this equation is perfectly classical, it may still incorporate the effect of degeneracy. If the initial conditions correspond to a degenerate state, the property that the phase space density is conserved along particle orbits ensures that the particles are not packed more densely in phase space than allowed by quantum mechanics. Thus the evolution equation assures that $f \leqslant m_e^3/4\pi^3\hbar^3$ if it is fulfilled initially. Here the distribution function is denoted by f, the electron mass by m_e, and Planck's constant by $h = 2\pi\hbar$.

In the present paper we start from kinetic theory valid for weakly coupled plasmas, that is, Eq. (83) of Ref. [19]. In addition to particle dispersion, this equation includes the spin dynamics. For long spatial scales compared to the de Broglie length particle, dispersive effects can be dropped and the equation is simplified. Although this equation in principle captures the long scale physics of fully degenerate particles, for certain types of problems weakly nonlinear perturbation schemes run into problems. The reason is that the weakly nonlinear modifications of the Fermi surface cannot be captured by perturbation theory, as will be discussed in detail in Sec. III. The solution to this problem is to transform the evolution equation in a way that allows for amplitude expansions. Besides allowing perturbation theory to be applied, this has the advantage of transforming one of the independent velocity variables to a dependent variable, thereby reducing the computational difficulties. The transformation applies when the spin dynamics is included, but is equally relevant when it is omitted. In this case the transformation modifies the standard Vlasov equation to an equation describing the nonlinear evolution of the space- and time-dependent Fermi surface of the particles. In the absence of spin effects this description is related to the water-bag model used to study classical nonlinear systems (see, for example, Refs. [20–22]), where the contours of the constant distribution function are studied. In order to demonstrate the usefulness of the formalism, we apply the theory to nonlinear Landau damping of ion-acoustic waves in the case of fully degenerate electrons. It is shown that the transition from the linear to the nonlinear regime occurs when the bounce frequency is comparable to the linear damping rate.

II. BASIC EQUATIONS

The foundation for our treatment is given in Ref. [19]. In particular, we refer to Eq. (63) of Ref. [19], which describes a quantum spin 1/2 plasma in the long scale length limit. Making a semiclassical approximation for the magnetic dipole force term, this equation simplifies somewhat [23] and agrees with Eq. (3) of Ref. [5], which was derived heuristically. This equation constitutes the starting point for our investigation and reads

$$\left\{\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\boldsymbol{\nabla}\right)+\frac{q_e}{m_e}(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\boldsymbol{\nabla}_v+\frac{3\mu}{m_e}\boldsymbol{\nabla}[\boldsymbol{s}\cdot\boldsymbol{B}]\cdot\boldsymbol{\nabla}_v+\frac{2\mu}{\hbar}(\boldsymbol{s}\times\boldsymbol{B})\cdot\boldsymbol{\nabla}_s\right\}f(\boldsymbol{x},\boldsymbol{v},\boldsymbol{s},t)=0. \quad (1)$$

Here $f(\boldsymbol{x},\boldsymbol{v},\boldsymbol{s},t)$ is similar to a classical distribution function (certain quantum peculiarities, i.e., that f can be negative, vanish over scale lengths longer than the characteristic de Broglie wavelength—see Ref. [19] for further details) but lives in a phase space augmented by the two spin variables encoded in $\boldsymbol{s} = (\sin\theta_s\cos\varphi_s, \sin\theta_s\sin\varphi_s, -\cos\theta_s)$, where we use spherical variables. The electron magnetic moment is denoted by $\mu = g\hbar e/4m_e$, the electron charge by $q_e = -e$, and $g \simeq 2.002\,331$ is the electron spin g factor.

For fully degenerate electrons in thermodynamic equilibrium the phase space density is constant and equal to its maximum value $f_{\max} = 3n/(16\pi^2 v_F^3)$ inside the Fermi sphere and $f = 0$ outside its boundary. Here n is the number density and $v_F = (\hbar/m_e)(3\pi^2)^{1/3}n^{1/3}$ is the Fermi velocity, which means $f_{\max} = m_e^3/16\pi^4\hbar^3$ independently of the density. Classically we know that when the equilibrium is perturbed, f is constant along particle orbits, and we will find that a

similar property holds for Eq. (1). To show this we make the ansatz

$$f(\boldsymbol{x},\boldsymbol{v},\boldsymbol{s},t) = f_{\max} H(v - \tilde{v}(\boldsymbol{x},\phi_v,\theta_v,\boldsymbol{s},t)), \tag{2}$$

where H is a step function that is unity when the argument is smaller than zero and vanishes otherwise, v is the magnitude of the velocity, $\tilde{v}$ describes the shape of the (dynamically deformed) Fermi surface, and ϕ_v and θ_v are spherical angles in velocity space. This ansatz turns out to solve Eq. (1) provided $\tilde{v}$ fulfills

$$\frac{qE_r}{m_e} = \left\{ \left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla \right) + \left(\frac{q_e}{m_e}(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}) + \frac{3\mu}{m_e}\nabla[\boldsymbol{s}\cdot\boldsymbol{B}] \right)\cdot\nabla_{v\perp} + \frac{2\mu}{\hbar}(\boldsymbol{s}\times\boldsymbol{B})\cdot\nabla_s \right\}\tilde{v}, \tag{3}$$

where $\boldsymbol{v} = \tilde{v}\hat{\mathbf{r}}_v$ with $\hat{\mathbf{r}}_v$ a unit vector in the direction of the velocity, the index r means the component along $\hat{\mathbf{r}}_v$, and $\nabla_{v\perp}$ represents a velocity gradient perpendicular to $\hat{\mathbf{r}}_v$. Using the ansatz (2) we find that the electron charge density is

$$\rho = f_{\max} q_e \int d\Omega \frac{\tilde{v}^3}{3}, \tag{4}$$

and the electron current density is

$$\boldsymbol{J} = f_{\max}\left(q_e \int d\Omega \frac{\tilde{v}^4}{4}\hat{\mathbf{r}}_v + \mu\nabla\times\int d\Omega \boldsymbol{s}\frac{\tilde{v}^3}{3}\right), \tag{5}$$

where $d\Omega = d^2s d^2v$, and the integration runs over the Bloch sphere in spin space and over the Fermi surface in velocity space. We thus have a description where the particle distribution is represented in terms of $\tilde{v}$, which can be viewed as a generalized Fermi surface (cf. Ref. [24]) depending also on the spin state. Equation (3) is a coordinate-free representation for the Fermi surface dynamics (with a spin-dependent Fermi surface), but it is practical to use spherical variables for the velocity coordinates. In this case we write $\hat{\mathbf{r}}_v = (\sin\theta_v\cos\varphi_v, \sin\theta_v\sin\varphi_v, \cos\theta_v)$, $\nabla_{v\perp} = (1/\tilde{v})\hat{\theta}_v\partial/\partial\theta_v + (1/\tilde{v}\sin\theta_v)\hat{\varphi}_v\partial/\partial\varphi_v$, where the spherical unit vectors are $\hat{\theta}_v = (\cos\theta_v\cos\varphi_v, \cos\theta_v\sin\varphi_v, -\sin\theta_v)$ and $\hat{\varphi}_v = (-\sin\varphi_v, \cos\varphi_v, 0)$ in Cartesian unit vectors. Similarly the spin gradient is written $\nabla_s = \hat{\theta}_s\partial/\partial\theta_s + (1/\sin\theta_s)\hat{\varphi}_s\partial/\partial\varphi_s$ and the integration element reads $d\Omega = \sin\theta_v\sin\theta_s d\theta_v d\varphi_v d\theta_s d\varphi_s$. We note that the number of independent variables is reduced by one compared to Eq. (1). More importantly, since we do not need to take velocity derivatives in the radial direction, we can avoid the appearance of delta functions in the calculations, which for certain classes of problems leads to insurmountable difficulties. There is a number of dimensionless parameters (for example, $\hbar k^2/m_e\omega$, $\mu B/m_e v_F^2$, where k and ω are characteristic wave number and frequencies, respectively; see Ref. [6] for a more detailed discussion) that determine the relative importance of spin effects. Generally when there there are no magnetic fields involved, the spin dynamics can be omitted. In this case we can drop all terms proportional to μ , and drop the spin dependence in $\tilde{v}$. We then get a system that only accounts for electron degeneracy, but otherwise is classical. As will be demonstrated in the next section, this equation has important advantages as compared to the standard Vlasov equation.

III. NONLINEAR LANDAU DAMPING: A SPECIFIC EXAMPLE

Next we will illustrate the usefulness of the present approach by considering nonlinear Landau damping of ion-acoustic waves in a plasma where the electrons are fully degenerate and the ions fulfill $\omega/k \gg v_{ti} \gg v_{Fi}$, i.e., they can be assumed to be cold, nonresonant, and nondegenerate. Since the magnetic field can be omitted for ion-acoustic waves, we use Eq. (3) with the spin terms dropped. For one-dimensional (1D) spatial variations along z and electrostatic fields $\boldsymbol{E} = E\hat{\boldsymbol{z}}$ we then obtain

$$\left(\frac{\partial}{\partial t} - \cos\theta_v\tilde{v}\frac{\partial}{\partial z}\right)\tilde{v} - \frac{q_e}{m_e}\frac{E}{\tilde{v}}\sin\theta_v\frac{\partial\tilde{v}}{\partial\theta_v} = -\frac{q_e}{m_e}E\cos\theta_v, \tag{6}$$

which is combined with (4), where the spin integration simply gives 4π and we only need to do the velocity integration, i.e., $d\Omega = \sin\theta_v d\theta_v d\varphi_v$. As a consequence, $f_{\max}$ in Eq. (4) is renormalized to $f_{\max} = m_e^3/4\pi^3\hbar^3$. Contrary to works considering 1D Fermi distributions [22] where the theory becomes hydrodynamic in nature, the dependence on θ_v implies that the problem stays kinetic and that a resonant wave-particle interaction will take place. In the nonlinear regime of Landau damping, the particle distribution may differ strongly from that of linear theory close to the resonance, but still the electromagnetic field may be described by a plane wave ansatz provided the amplitude is not too high. The condition for this is discussed for the classical case in, e.g., Ref. [25], and similar arguments apply to our case. Thus we introduce $E = \tilde{E}(t)\exp[i(kz - \omega t)] + \text{c.c.}$, where c.c. stands for complex conjugate, and the amplitude is slowly varying. For the deformation of the electron velocity surface we make an expansion in harmonics, i.e., we look for solutions of the form $\tilde{v} = v_F + \sum_{n=0}^{\infty}\tilde{v}_n(t,\theta_v)\exp[in(kz - \omega t)] + \text{c.c.}$ Inserting this in (1) gives a hierarchy of coupled equations, one for each harmonic. This hierarchy contains a number of nonlinearities. For $|\tilde{v}_n(t)| \ll v_F$ the ratio $E/\tilde{v}$ can be equated with E/v_F. Furthermore, it can be verified *a posteriori* that the nonlinearities $\propto\cos\theta_v\tilde{v}\partial\tilde{v}/\partial z$ are small compared to the ones $\propto E\sin\theta_v\partial\tilde{v}/\partial\theta_v$ close to the resonance. This follows because the phase velocity $\omega/k \ll v_F$, and hence the resonance, occurs at angles $\cos\theta_v \ll 1$. As a result, close to the resonance the equations for the particle dynamics are

$$\frac{\partial}{\partial t}\tilde{v}_0 = \frac{q_e}{m_e}\frac{E_1^*}{v_F}\frac{\partial\tilde{v}_1}{\partial\theta_v} + \frac{q_e}{m_e}\frac{E_1}{v_F}\frac{\partial\tilde{v}_1^*}{\partial\theta_v}, \tag{7}$$

$$\left[\frac{\partial}{\partial t} - i\delta\omega\right]\tilde{v}_1 = \frac{q_e}{m_e}\left[E\cos\theta_v + \frac{E}{v_F}\frac{\partial\tilde{v}_0}{\partial\theta_v} + \frac{E^*}{v_F}\frac{\partial\tilde{v}_2}{\partial\theta_v}\right], \tag{8}$$

$$\left[\frac{\partial}{\partial t} - in\delta\omega\right]\tilde{v}_n = \frac{q_e}{m_e v_F}\left(E\frac{\partial\tilde{v}_{n-1}}{\partial\theta_v} + E^*\frac{\partial\tilde{v}_{n+1}}{\partial\theta_v}\right), \tag{9}$$

where $\delta\omega \equiv \omega - kv_F\cos\theta$ and $\sin\theta_v \simeq 1$ have been used. Equations (7)–(9) are complemented by Poisson's equation

$$ikE = \frac{2\pi q_e}{\epsilon_0}\int d\theta_v\ \sin\theta_v f_{\max} v_{Fe}^2\tilde{v}_1 - \int d^3v\ f_{i1}, \tag{10}$$

where the ion contribution in the last term is found from the classical linearized Vlasov equation evaluated at zero temperature. Following Ref. [26] we note that the damping time scale will be much longer than the time scale of oscillations induced by the electric field. Hence, only a minor fraction of all particles, those closest to the resonance, participate in the wave-particle interaction. Guided by this, we divide the integration interval in a resonant and a nonresonant region. In the nonresonant region we can linearize the governing equations. Furthermore, we let the standard linear dispersion relation hold by definition, but excluding the resonant region. A crucial issue is the width of the resonant region. However, it can be verified *a posteriori* that the evolution is not critically dependent on this choice. We thus divide the electron integral in (10) as $\int = \int_{nr} + \int_r$, where the index r indicates that the integral only runs over the resonant part $[\theta_{vr} - \Delta\theta, \theta_{vr} + \Delta\theta_v]$, and the resonant angle θ_{vr} fulfills $\omega - k v_{Fe} \cos\theta_{vr} = 0$, with $\Delta\theta_v$ defining the width of the resonant region. The index nr refers to the remaining nonresonant part. In the nonresonant region linear theory is sufficient, where the (weak) time dependence of the amplitudes is treated perturbatively. As a result, Eq. (10) is rewritten as

$$\frac{\partial}{\partial t} E = Q f_{\max} v_{Fe}^2 \int_r d\theta_v \, \tilde{v}_1, \tag{11}$$

where

$$Q \equiv \left[q_e \int_{nr} d\theta_v \, \sin\theta_v \cos\theta_v \frac{f_{\max} v_{Fe}^2}{m_e \delta\omega^2} + 2 \frac{q_e n_0 k}{m_i \omega^3} \right]^{-1}.$$

Equation (11), together with (7)–(9), constitutes a closed system. Assuming that $\tilde{v}_n$ can be neglected at the boundary of the resonant interval, it is straightforward to show that this system is energy conserving, obeying

$$\frac{\partial}{\partial t} \left[-f_{\max} v_{Fe}^2 Q \int_r d\theta_v \frac{1}{\cos\theta_v} \sum_{n=0}^{\infty} |\tilde{v}_n|^2 + \frac{q_e}{m_e} |E|^2 \right] = 0, \tag{12}$$

where the first term is proportional to the energy density of resonant particles (note that $Q < 0$), and the second term is proportional to the wave energy density.

The above system can be solved numerically. A key issue is to first establish a proper value of $\Delta\theta_v$. If the resonant interval is made too small, all resonant particles will not be included, and if it is made too large, the assumption of the model breaks down. The numerical results show that the model is robust for changes of the resonant interval, as long as the limits of the resonance region fulfill $0.1(\omega/kvF) < \Delta\theta_v < \omega/kvF$. The number of harmonics of $\tilde{v}_n$ needed in the simulations depends on the initial amplitude as well as on the length of time of the simulation. In order to validate the simulations we check that the energy content of the highest harmonics included is small compared to that of the lower numbers, and also that the evolution of the electric field amplitude is not affected by the addition of further harmonics.

Next we introduce the normalized initial electric field amplitude $E_n = q_e k E(t=0)/m_e \gamma_L^2$, where $\gamma_L = \pi q_e \omega Q / m_e k^2 v_F^2$ is the linear (Landau) damping rate. In Fig. 1 the evolution of the electric field amplitude is shown for various values of E_n. For $E_n = 0.5$, the evolution cannot be separated

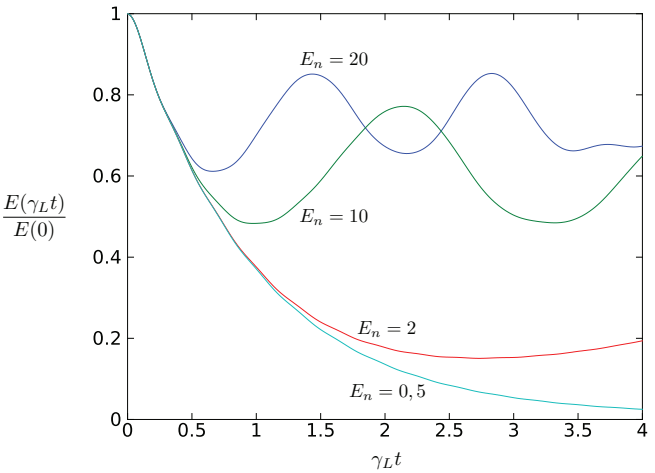

FIG. 1. (Color online) The evolution of the normalized electric field for different initial amplitudes, $E_n = 0.5$, 2, 10, and 20, respectively.

from linear damping, as seen by the lowest lying curve. Increasing the amplitude soon leads to nonlinear behavior, where the initial damping is replaced by amplitude oscillations with a frequency of the order of the bounce frequency $\omega_B = (q_e k E/m_e)^{1/2}$. This is illustrated by the curves for $E_n = 2$, 10, and 20. While our problem of study is weakly nonlinear, in the sense that $\tilde{v}_n \ll v_F$ for all n, it is strongly nonlinear for the resonant particles, since the coupling to the low-frequency perturbation v_0 and the second harmonic v_2 in Eq. (8) can be as large as the linear term in that equation. Similarly in the nonlinear regime $E_n \gtrsim 1$ the coupling strength to the higher harmonics in Eq. (9) decreases rather slowly with the harmonic number. For $E_n = 10$ and $\gamma_L t = 4$ harmonics up to $n = 6$ are needed. More harmonics are needed for a longer simulation.

We stress that the amplitude evolution is similar to the classical case. In particular, the transition from linear damping when $\gamma_L \gg \omega_B$ to amplitude oscillations with a frequency ω_B when $\gamma_L \sim \omega_B$ is almost identical. This can be traced back to the dynamics of the trapped particles, which is the same independently if the background distribution is a Fermi-Dirac or a Maxwellian distribution. The difference between the nondegenerate and the degenerate case lies in the role of the perturbation theory, as will be discussed in some detail below for a general scenario.

Let us take a broader perspective and compare Eq. (3), which is our main result, with the kinetic equation for the distribution function f (i.e., the standard Vlasov equation in case spin effects are neglected). Formally it would seem impossible to apply perturbation theory to the standard Vlasov equation for fully degenerate particles. The problem is that any small perturbation of the Fermi surface violates the condition $f_1 \ll f_0$. However, in reality the situation is not as harsh, and many problems with degenerate electrons can be solved perturbatively starting from the Vlasov equation (see, e.g., Ref. [13]). On the other hand, typically these are the problems that can be solved for an arbitrary equilibrium distribution, including the fully degenerate case. A more difficult class of problems is the one where the (slow) evolution of the background distribution plays an integral part of the dynamics, which for a fully degenerate system means that the nonlinear

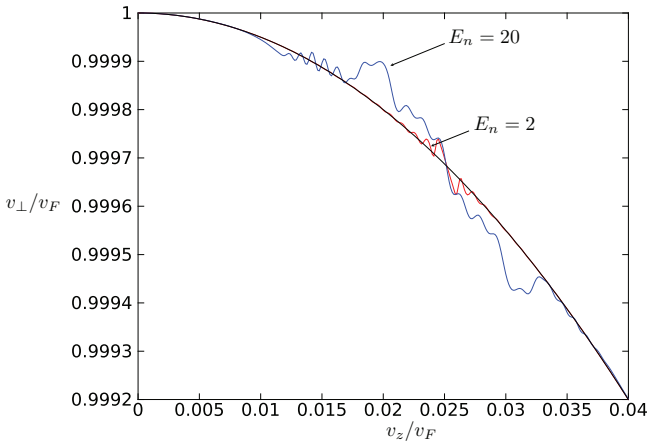

FIG. 2. (Color online) The unperturbed Fermi surface and the Fermi surface for $E_n = 2$ and $E_n = 20$ at $\gamma_l t = 4$.

modifications of the Fermi surface are crucial. A perturbative expansion then runs into trouble. Since the velocity derivatives of f_0 are nonzero only at the *unperturbed* Fermi surface, the perturbed distribution function will be nonzero only at the unperturbed Fermi surface. Furthermore, the inclusion of successive perturbations suffers from the same problem. Using Eq. (3) the situation is dramatically improved. The condition $\tilde{v}_1 \ll v_F$ will hold except for very large amplitudes. For the specific case of nonlinear Landau damping, the direction of the energy flow is crucially dependent on the profile of the background distribution close to the resonance. The low-frequency part of the Fermi surface $\tilde{v}_0(\theta_v,t)$ modifies the spherical background surface with radius v_F. Furthermore, a very small modification is sufficient to alter the dynamics from linear damping to nonlinear oscillations. In Fig. 2 the low-frequency part of the Fermi surface is compared to the unperturbed Fermi surface at $\gamma_L t = 4$ for different initial amplitudes. The unperturbed Fermi level is the smooth curve representing the spherical surface, and the modified curve corresponding to $E_n = 0.5$ cannot be separated from this line, which is consistent with the linear evolution for this initial amplitude as shown in Fig. 1. For $E_n = 2$, the Fermi level is clearly distinct. While $\tilde{v}_0 \ll v_F$ it is clear that $\partial\tilde{v}_0/\partial\theta \sim \cos\theta_v v_F$ close to the resonance, which is sufficient to make the nonlinear terms in Eq. (8) comparable to the linear one. As a result the dynamics is significantly changed from linear damping, in agreement with the second lowest lying curve in Fig. 1. Finally for $E_n = 20$ the strong modifications of the Fermi level apply in a wider region around the resonance. Consequently the transition from linear to nonlinear evolution occurs faster, in agreement with the highest lying curve in Fig. 1.

IV. SUMMARY AND DISCUSSION

In this article we have derived an evolution equation for fully degenerate electrons, applicable on scales much longer than the de Broglie length, including the spin dynamics. When spin effects are omitted, it is a reformulation of the classical Vlasov equation. Such a reformulation has several advantages. First, perturbative expansions are possible under much more general conditions. In particular, nonlinear low-frequency modifications of the Fermi surface can be calculated, as shown in Fig. 2 for the specific case of nonlinear Landau damping. Second, the number of independent variables is reduced since the radial velocity is changed from an independent to a dependent variable, which is very helpful when numerical treatments are made. Furthermore, a numerical treatment of the standard Vlasov equation for fully degenerate systems has the problem that the velocity gradients become infinite. In principle this can be handled by using the initial distribution for small but nonzero temperature T with $T \ll T_F$, which makes the velocity gradients finite. Such an approach has the advantage to include the physical effects of a nonzero temperature, which is always present in a real system. On the other hand, it is numerically costly, as it requires one more dependent variable as compared to Eq. (3), and such a treatment also makes it less straightforward to isolate the physics of the fully degenerate case.

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