Formation of curvature singularities on the interface between dielectric liquids in a strong vertical electric field

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The nonlinear dynamics of the interface between two deep dielectric fluids in the presence of a vertical electric field is studied. We consider the limit of a strong external electric field where electrostatic forces dominate over gravitational and capillary forces. The nonlinear integrodifferential equations for the interface motion are derived under the assumption of small interfacial slopes. It is shown in the framework of these equations that, in the generic case, the instability development leads to the formation of root singularities at the interface in a finite time. The interfacial curvature becomes infinite at singular points, while the slope angles remain relatively small. The curvature is negative in the vicinity of singularities if the ratio of the permittivities of the fluids exceeds the inverse ratio of their densities, and it is positive in the opposite case (we consider that the lower fluid is heavier than the upper one). In the intermediate case, the interface evolution equations describe the formation and sharpening of dimples at the interface. The results obtained are applicable for the description of the instability of the interface between two magnetic fluids in a vertical magnetic field.

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I. INTRODUCTION

As is known [1–3], the free surface of a liquid or the interface between two liquids is unstable in a sufficiently strong vertical electric field. The exponential growth of the amplitude of boundary perturbations in the initial (linear) stage of the instability inevitably leads to a situation where nonlinear effects begin to play a significant role [4–6]. It therefore becomes necessary to take into account the nonlinearity of the corresponding equations of motion.

The general approach to study the behavior of the interface between two liquids or of the free surface of a liquid is to reduce the equations of motion for fluids to lower dimensional equations for the boundary motion. As a rule, the resulting equations are nonlocal (i.e., they contain integrodifferential operators), which seriously hampers their analysis. In most publications, nonlinear waves on the surface of dielectric or conducting fluids are considered under the assumption that the characteristic wavelength is much larger than the depth of the fluid layer [7–15] or that the excited wave packet is spectrally narrow [6,16–23]. Both approaches make it possible to reduce the original problem to the consideration of relatively simple local partial differential equations. The long-wave approximation usually leads to different modifications of the Korteweg–de Vries equation for the elevation of the surface, while the quasimonochromatic approximation gives various modifications of nonlinear Schrödinger or Klein-Gordon equations for the wave envelope.

As demonstrated in Refs. [24,25] for conducting liquids (e.g., liquid metals) and in Refs. [26,27] for dielectric liquids with conducting surfaces (e.g., liquid helium with the electron-charged surface [28]), the nonlinear evolution of the free surface can be effectively studied by analytical methods in the strong-field limit, where the surface motion is completely determined by electrostatic forces. In these papers the long-wave and quasimonochromatic approximations were not used; integrodifferential equations for the boundary motion have been integrated directly.

In the present work we will show that a similar situation arises when we consider the nonlinear dynamics of the interface between two ideal dielectric liquids in a strong vertical electric field (gravity and capillary forces are neglected). In two dimensions, the weakly nonlinear equations of motion derived in the small-angle approximation (see Secs. II, III, and IV) can be solved exactly in two special cases: (i) when $\varepsilon_1/\varepsilon_2 = \rho_1/\rho_2$ and (ii) when $\varepsilon_1/\varepsilon_2 = \rho_2/\rho_1$ (Secs. V and VI). Here ε_1 and ε_2 are the dielectric constants of the lower and upper fluids, respectively, and ρ_1 and ρ_2 are their densities (without loss of generality, we can consider that the lower fluid is heavier than the upper one, i.e., $\rho_1 > \rho_2$). In other cases, i.e., for arbitrary values of dielectric constants and densities, the evolution of a spatially localized perturbation of the interface can be described in the framework of the "local" approximation, based on the use of the leading-order series expansion of the complex velocity in the vicinity of the singular point appearing at the interface (Sec. VII).

The interaction of the electric field with polarization charges at the interface of two fluids causes an explosive growth of the boundary perturbations. As will be demonstrated below, in the generic case this process culminates in the formation of weak root singularities at the interface for which the curvature becomes infinite, while the slope angles remain small. Near the singularity, the interfacial curvature is negative for $\varepsilon_1/\varepsilon_2 > \rho_2/\rho_1$. This condition is satisfied, in particular, for the above-mentioned integrable case (i). For $\varepsilon_1/\varepsilon_2 < \rho_2/\rho_1$, the curvature is positive. In the intermediate case, i.e., in the case (ii), the tendency for the formation of root singularities is absent. The instability leads to the formation and further sharpening of dimples at the interface. Taken together, the obtained results give a rather complete picture of the behavior of the interface under the action of electrostatic forces.

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II. INITIAL EQUATIONS

Let us consider the dynamics of the interface between two deep perfect dielectric (nonconducting) fluids in an external vertical electric field. We assume that both fluids are inviscid and incompressible, and their flows are irrotational (potential). The dispersion relation for linear waves on the interface in the presence of polarization charges (free charges are absent) has the following form [1,2]:

$$\omega^2 = Ag|k| - \frac{\varepsilon_0(\varepsilon_1-\varepsilon_2)A_E E_1 E_2}{\rho_1+\rho_2}k^2 + \frac{\sigma}{\rho_1+\rho_2}|k|^3,$$

where k is the wave number, ω is the frequency, g is the acceleration of gravity, σ is the interfacial tension coefficient, ε_0 is the vacuum permittivity, $A=(\rho_1-\rho_2)/(\rho_1+\rho_2)$ is the Atwood number ($0<A<1$), and $A_E=(\varepsilon_1-\varepsilon_2)/(\varepsilon_1+\varepsilon_2)$ is its analog for the dielectric constants. The electric field strengths in the lower and upper fluids, E_1 and E_2, are related by the expression $\varepsilon_1 E_1=\varepsilon_2 E_2$ (here and below the subscripts "1" and "2" refer to the lower and upper fluids, respectively).

It follows from the dispersion law that if the electric field is strong enough,

$$E_1E_2 \gg \frac{\sqrt{g\sigma(\rho_1-\rho_2)}}{\varepsilon_0(\varepsilon_1-\varepsilon_2)A_E},$$

the second term on the right-hand side of the dispersion relation (it is responsible for the electrostatic forces) dominates for the wave numbers in the range

$$\frac{g(\rho_1-\rho_2)}{\varepsilon_0(\varepsilon_1-\varepsilon_2)A_E E_1E_2} \ll |k| \ll \frac{\varepsilon_0(\varepsilon_1-\varepsilon_2)A_E E_1E_2}{\sigma}. \quad (1)$$

As a consequence, the influence of gravity and capillary forces can be neglected in this range, and the relation between the frequency and the wave number takes the following simple form: $\omega^2 \sim -k^2$.

The motion of fluids can be considered as potential if the electrostatic forces dominate over the viscous forces. This condition can be realized for the wave numbers

$$k^2 \ll \frac{\varepsilon_0(\varepsilon_1-\varepsilon_2)A_E E_1E_2}{\nu^2(\rho_1+\rho_2)}, \quad (2)$$

where ν is the kinematic viscosity of the more viscous liquid. This inequality corresponds to the situation where the Reynolds number ($\mathrm{Re}\sim \omega k^{-2}\nu^{-1}$) is sufficiently high to neglect viscous dissipation (see, e.g., Ref. [15]).

In the unperturbed state, the interface is a horizontal plane (see Fig. 1). We introduce a Cartesian coordinate system in such a way that the x and y axes lay in this plane. The electric field vector is directed along the z axis (i.e., normally to the unperturbed boundary). Let the deviation of the interface from the plane $z=0$ be given by the function $\eta(x,y,t)$, that is, the equation $z=\eta$ defines the shape of the boundary.

The velocity potentials for both fluids, $\Phi_{1,2}$, as well as the electric field potentials, $\varphi_{1,2}$, satisfy the Laplace equations,

$$\nabla^2\Phi_{1,2}=0, \quad \nabla^2\varphi_{1,2}=0.$$

The normal components of the velocities are equal at the interface,

$$\partial_n\Phi_1=\partial_n\Phi_2, \quad z=\eta(x,y,t), \quad (3)$$

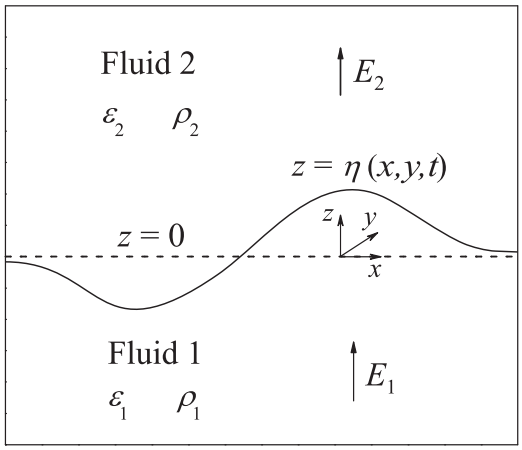

FIG. 1. The geometry of the problem is shown schematically.

where ∂_n denotes the derivative along the normal to the boundary $z=\eta$:

$$\partial_n = \frac{\partial_z - \nabla_\perp\eta\cdot\nabla_\perp}{\sqrt{1+(\nabla_\perp\eta)^2}}.$$

Here $\nabla_\perp$ is the two-dimensional gradient in the $\{x,y\}$ plane. The electric field potentials obey the boundary conditions

$$\varphi_1=\varphi_2, \quad \varepsilon_1\partial_n\varphi_1=\varepsilon_2\partial_n\varphi_2, \quad z=\eta(x,y,t), \quad (4)$$

i.e., the tangential component of the electric field and the normal component of the electric displacement field are continuous at the interface. In addition, the electric field is uniform, and the fluids are at rest at an infinite distance from the interfacial boundary:

$$\varphi_{1,2}\to -E_{1,2}z, \quad \Phi_{1,2}\to 0, \quad z\to\mp\infty.$$

As is known [5,29,30], the equations describing the motion of the interface $z=\eta$ can be represented in the Hamiltonian form,

$$\psi_t=-\delta H/\delta\eta, \quad \eta_t=\delta H/\delta\psi, \quad (5)$$

where η and ψ play the role of generalized coordinate and generalized momentum, respectively. The function ψ is defined by the expression

$$\psi(x,y,t)=\rho_1\,\Phi_1|_{z=\eta}-\rho_2\,\Phi_2|_{z=\eta}.$$

The Hamiltonian H coincides with the total energy of the system:

$$H=\frac{1}{2}\int_{z<\eta}\left[\rho_1(\nabla\Phi_1)^2-\varepsilon_0\varepsilon_1(\nabla\varphi_1)^2+\varepsilon_0\varepsilon_1E_1{}^2\right]d^3r + \frac{1}{2}\int_{z>\eta}\left[\rho_2(\nabla\Phi_2)^2-\varepsilon_0\varepsilon_2(\nabla\varphi_2)^2+\varepsilon_0\varepsilon_2E_2{}^2\right]d^3r. \quad (6)$$

Together, the above relations form a closed system of equations describing the motion of the interface of dielectric liquids under the influence of the electrostatic forces caused by the vertical electric field.

It should be noted that practically the same equations describe the dynamics of the interface between two magnetic fluids in the presence of a vertical magnetic field. In order to get these equations, one needs to replace the electric fields $E_{1,2}$ by the magnetic fields $H_{1,2}$, the dielectric constants $\varepsilon_{1,2}$ by the

relative permeabilities $\mu_{1,2}$, and the electric constant ε_0 by the magnetic constant μ_0.

III. WEAKLY NONLINEAR ANALYSIS

Consider the dynamics of the interface in the approximation of small angles of inclination of the boundary, $|\nabla_\perp \eta| \sim \alpha \ll 1$. For convenience of the further analysis, we switch to dimensionless notations as follows:

$$\Phi_{1,2} \to \frac{\Phi_{1,2} E_1}{k_0}\sqrt{\frac{\varepsilon_0\varepsilon_1}{\rho_1}}, \quad \varphi_{1,2} \to \frac{\varphi_{1,2} E_1}{k_0},$$
$$\psi \to \frac{\psi E_1}{k_0}\sqrt{\varepsilon_0\varepsilon_1\rho_1}, \quad \eta \to \frac{\eta}{k_0},$$
$$t \to \frac{t}{E_1 k_0}\sqrt{\frac{\rho_1}{\varepsilon_0\varepsilon_1}}, \quad \mathbf{r} \to \frac{\mathbf{r}}{k_0},$$

where k_0 is the characteristic wave number.

Expanding the Hamiltonian (6) in powers of η and ψ up to third-order terms (see details in Appendix A), we obtain from (5) the following quadratically nonlinear equations of motion:

$$\psi_t - \frac{2A_E^2}{1-A_E}\hat{k}\eta = \frac{A(1+A)}{4}[(\hat{k}\psi)^2 - (\nabla_\perp\psi)^2] + \frac{A_E^3}{1-A_E}[(\hat{k}\eta)^2 - (\nabla_\perp\eta)^2] + \frac{2A_E^3}{1-A_E}[\hat{k}(\eta\hat{k}\eta) + \nabla_\perp(\eta\nabla_\perp\eta)] + O(\alpha^3), \tag{7}$$

$$\eta_t - \frac{1+A}{2}\hat{k}\psi = -\frac{A(1+A)}{2}[\hat{k}(\eta\hat{k}\psi) + \nabla_\perp(\eta\nabla_\perp\psi)] + O(\alpha^3), \tag{8}$$

where $\hat{k}$ is the two-dimensional integral operator with the kernel whose Fourier transform equals the absolute value of the wave number, i.e.,

$$\hat{k}\, e^{ik_x x + ik_y y} = |k|\, e^{ik_x x + ik_y y}.$$

Here k_x and k_y are the x and y components of the wave vector.

It should be noted that for $A_E = 0$ (i.e., in the situation where $\varepsilon_1 = \varepsilon_2$ and there is no polarization charge at the interface) these equations coincide with the system that describes the inertial motion of the boundary between two fluids in the absence of gravity and surface tension [30].

It is convenient to introduce new functions $f = (c\psi + \eta)/2$ and $g = (c\psi - \eta)/2$, where c is the constant given by

$$2c = |A_E|^{-1}\sqrt{(1-A_E)(1+A)}.$$

After the substitution, the linearized equations of motion are separated into two independent equations for the functions f and g,

$$\tau f_t = \hat{k} f, \quad \tau g_t = -\hat{k} g, \tag{9}$$

where τ is the characteristic time of instability development defined by the expression

$$\tau = |A_E|^{-1}\sqrt{(1-A_E)/(1+A)}.$$

The linear equation for f describes the exponential growth of initial periodic perturbations of the interface, while the equation for g describes their attenuation. It is clear that in the study of the instability of the interface, it suffices to consider only the increasing branch of solutions. So we can assume that g is identically zero.

A similar approach can be applied to the nonlinear equations of motion. The analysis is greatly simplified if we neglect the decreasing branch of solutions by putting $g = O(\alpha^2)$. This means that we can set $g = 0$ in the right-hand sides of the equations for f and g. As a result, the equations describing the interface motion take the following form:

$$\tau f_t - \hat{k} f = \frac{A+A_E}{4}[(\hat{k}f)^2 - (\nabla_\perp f)^2] - \frac{A-A_E}{2}[\hat{k}(f\hat{k}f) + \nabla_\perp(f\nabla_\perp f)] + O(\alpha^3), \tag{10}$$

$$\tau g_t + \hat{k} g = \frac{A+A_E}{4}[(\hat{k}f)^2 - (\nabla_\perp f)^2 + 2\hat{k}(f\hat{k}f) + 2\nabla_\perp(f\nabla_\perp f)] + O(\alpha^3). \tag{11}$$

It is important that the nonlinear equation (10) contains only the function f and that Eq. (11) is linear with respect to the function g, i.e., these equations are essentially simpler than the initial equations (7) and (8).

In the following sections, we will study the formation of curvature singularities on the interface in the framework of Eqs. (10) and (11). Note that the linearized equations (9) themselves display a tendency to form singularities due to the explosive growth of small-scale perturbations. However, as shown below, the type of singularity is fully determined by the nonlinear terms of Eqs. (10) and (11).

IV. TWO-DIMENSIONAL CASE: PRELIMINARY ANALYSIS

In the previous section, we have shown that the study of the weakly nonlinear stage of the electrohydrodynamic instability of the interface between two dielectric fluids in a strong vertical electric field reduces to the analysis of the integrodifferential equations (10) and (11).

Let us now assume that interface perturbations have the plane symmetry (all quantities do not depend on the variable y). The operator $\hat{k}$ then can be expressed in terms of the Hilbert transform $\hat{H}$,

$$\hat{k} = -\hat{H}\frac{\partial}{\partial x}, \quad \hat{H}\phi(x) = \frac{1}{\pi}\,\text{p.v.}\int_{-\infty}^{+\infty}\frac{\phi(x')}{x-x'}\,dx'.$$

In plane geometry, Eqs. (10) and (11) take the form

$$\tau f_t + \hat{H} f_x = \frac{A+A_E}{4}[(\hat{H}f_x)^2 - (f_x)^2] - \frac{A-A_E}{2}[\hat{H}(f\hat{H}f_x)_x + (ff_x)_x], \tag{12}$$

$$\tau g_t - \hat{H} g_x = \frac{A+A_E}{4}[(\hat{H}f_x)^2 - (f_x)^2 + 2\hat{H}(f\hat{H}f_x)_x + 2(ff_x)_x]. \tag{13}$$

It is convenient to introduce the functions that are analytic in the upper half-plane of the complex variable x,

$$F = \hat{P} f, \quad G = \hat{P} g,$$

where $\hat{P} = (1 - i\hat{H})/2$ is the projection operator. Since applying the Hilbert transform to a function analytic in the upper half-plane reduces to its multiplication by the imaginary unit, Eqs. (12) and (13) can be rewritten as

$$\tau F_t + iF_x = -(A + A_E)F_x^2/2 - (A - A_E)\hat{P}(F\bar{F}_x)_x, \quad (14)$$

$$\tau G_t - iG_x = -(A + A_E)\left[F_x^2/2 - \hat{P}(F\bar{F}_x)_x\right]. \quad (15)$$

The solution of Eq. (15) with the initial condition $G|_{t=0} = 0$ is given by

$$\begin{gathered} G = \frac{1}{\tau}\int_0^t K(x + it/\tau - i\tilde{t}/\tau, \tilde{t})\, d\tilde{t}, \\ K(x,t) = -(A + A_E)\left[F_x^2/2 - \hat{P}(F\bar{F}_x)_x\right]. \end{gathered} \quad (16)$$

As for the key equation (14), its right-hand side contains two nonlinear terms. In the general case, when all terms are present, it is not possible to solve this equation exactly (approximate solutions will be constructed in Sec. VII). Nevertheless, it can be easily seen that there are two special cases,

$$A_E = A \Leftrightarrow \rho_1/\rho_2 = \varepsilon_1/\varepsilon_2, \quad (17)$$

$$A_E = -A \Leftrightarrow \rho_1/\rho_2 = \varepsilon_2/\varepsilon_1, \quad (18)$$

in which the second or, respectively, the first nonlinear term in Eq. (14) disappears. In these cases the solutions of the evolution equations can be obtained analytically. These situations will be discussed in the two following sections.

The conditions (17) and (18) can be satisfied with acceptable accuracy for some pairs of immiscible dielectric liquids. As an example, let us consider the interface between a liquid organosilicon polymer and mineral or organic oil (see, for instance, Ref. [31]). For polymethylphenylsiloxane ($\rho_1 \approx 1100$ kg/m^3 and $\varepsilon_1 \approx 2.7$) as the lower fluid and transformer oil ($\rho_2 \approx 880$ kg/m^3 and $\varepsilon_2 \approx 2.2$) as the upper fluid, we get $A \approx 0.11$ and $A_E \approx 0.10$, i.e., the first condition (17) holds. Next, if we replace the upper fluid by linseed oil ($\rho_2 \approx 930$ kg/m^3 and $\varepsilon_2 \approx 3.2$), we arrive at $A \approx 0.084$ and $A_E \approx -0.085$, i.e., the second condition (18) becomes true.

In the analysis of the behavior of the interface between two ferrofluids in the presence of a magnetic field, the dielectric constants $\varepsilon_{1,2}$ in the conditions (17) and (18) should be replaced by the relative permeabilities $\mu_{1,2}$. As is known, ferrofluids are colloidal systems consisting of nanoscale ferromagnetic particles suspended in a carrier fluid (see, e.g., Refs. [32,33]). Usually, the volume fraction of particles in ferrofluids is a few percentages. By varying the concentration of nanoparticles in the base fluids, one can always satisfy the conditions (17) or (18).

We now discuss the existing restrictions on the applicability of Eqs. (14) and (15). Recall that these equations were derived under the assumption of small interfacial slopes that corresponds to the inequality $|\mathrm{Re}\, F_x| \sim \alpha \ll 1$. It is clear that the characteristic slope angles of the interface grow as a result of the instability and, hence, we should take into account that α is a function of time. Let us find the conditions under which $\alpha(t)$ will be small during the entire evolution of the system, starting from the initial time ($t = 0$) up to the moment of singularity formation ($t = t_c$).

Suppose that there initially exists a spatially localized perturbation of the interface,

$$\eta(x,0) = Sa_0/(x^2 + {a_0}^2), \quad (19)$$

where S is a constant responsible for the direction of the perturbation (it is proportional to the area corresponding to the perturbed interface, $S = \pi^{-1}\int \eta\, dx$) and a_0 is the width of the perturbation. According to this expression, at the initial time $t = 0$, the characteristic slope angles are given by $\alpha(0) \approx |S|/{a_0}^2$. It is clear that the small-angle approximation is valid only for small values of the parameter S, i.e., for $|S| \ll 1$. In terms of the functions F and G, the initial condition (19) reads as

$$F(x,0) = \frac{iS/2}{x + ia_0}, \quad G(x,0) = 0, \quad (20)$$

i.e., the function F has the only singularity (a simple pole) at the point $x = -ia_0$.

In the linear approximation, the solution of Eqs. (14) and (15) with initial conditions (20) is the following:

$$F(x,t) = \frac{iS/2}{x + ia_0 - it/\tau}, \quad G(x,t) = 0,$$

i.e., the pole moves with a constant velocity towards the real axis, reaching it at the point $x = 0$ at time $t = t_0 \equiv a_0\tau$. The function F then ceases to be analytic and the interface shape becomes singular. The characteristic angles increase according to the relation

$$\alpha(t) \approx |S|/(a_0 - t/\tau)^2. \quad (21)$$

As will be demonstrated below, the contribution of the nonlinear terms of Eqs. (14) and (15) has a decisive impact on the type of developing singularities. So, in particular, weak root singularities can appear at the interface under the influence of nonlinearity. For them, the slope angles remain small near the singular points. Such behavior differs radically from that predicted by the linear model; according to (21), $\alpha \to \infty$ as $t \to t_0$. Within the framework of the nonlinear equations, the singularity formation time (t_c) differs from its linear estimate (t_0) by the value $\Delta \equiv t_0 - t_c$. Outside the neighborhood of the singular point, the interface dynamics can be well described by linear equations and, consequently, the formula (21) is applicable. According to it, the angles in the periphery at $t = t_c$ are given by

$$\alpha(t_c) \approx |S|\tau^2/\Delta^2. \quad (22)$$

Hence, the small-angle approximation will be valid at the moment of singularity formation only if $\Delta = o(|S|^{1/2})$. As will be shown in Sec. VII, this requirement is really satisfied in the generic case.

It should be noted that the linear term F_x can be eliminated from the key equation (14) by the substitution $x \to x + it/\tau$, that is, by transition to the coordinate system moving together with singularities in the complex x plane. Finally, by switching to the slow time scale, we can obtain an equation in which all terms have the same order of smallness. This explains, in the context of systematic asymptotic expansions, why we keep the terms of different orders of smallness, $O(\alpha)$ and $O(\alpha^2)$, in Eqs. (10) and (11). In a certain sense, this is analogous to the derivation of the Korteweg–de Vries equation for long

waves in shallow water, where the linear translation term is eliminated by transition to the coordinate system moving with the wave (see, e.g., Ref. [34]).

V. DYNAMICS OF THE INTERFACE FOR $A_E = A$

In the present section, we will consider the dynamics of the interface between two dielectric fluids in the presence of a strong vertical electric field in the situation where the ratio of the permittivities of the fluids is equal to the ratio of their densities, i.e., $A_E = A$. The integrodifferential equation (14) turns into a much simpler partial differential equation,

$$\tau F_t + i F_x = -A F_x^2. \tag{23}$$

This equation coincides (up to coefficient A) with that obtained in Ref. [25] for describing the nonlinear dynamics of the free surface of a conducting liquid in a strong external electric field. An equation similar in structure to Eq. (23) has also been considered in Refs. [30,35], where it was used to describe the dynamics of the free surface of an ideal fluid in the absence of external forces. Since the method of solving an equation of the form (23) has been described in detail in the cited references, we will only briefly present the basic relations and the most important features of the solutions.

The governing equation (23) transforms into the complex Hopf equation,

$$\tau V_t + i V_x = -2A V V_x,$$

by the change $V = F_x$. The Hopf equation is solved by the method of characteristics; its solution has the implicit form

$$V = V_0(\tilde{x}), \quad x\tau = \tilde{x}\tau + it + 2A V_0(\tilde{x})t, \tag{24}$$

where $V_0(x) = V|_{t=0}$, and the variable $\tilde{x}$ plays the role of a parameter. The expressions (24) describe the motion of the branch points of the function V on the complex x plane (these points correspond to the condition $\partial x/\partial \tilde{x} = 0$). At some moment $t = t_c$ when one of the branch points reaches the real axis, the analyticity of V is violated and a singularity appears in the solutions of Eq. (23).

As for the equation (15) for the function G, its solution with zero initial condition is given by (16). Note that the highest derivative on the right-hand side of Eq. (15) corresponds to the function $\bar{F}$ analytic in the lower half-plane of the complex variable x. Its influence is suppressed by the action of the projection operator $\hat{P}$. Otherwise, the condition of smallness of g compared to f that was used in deriving (10) and (11) would be violated close to the singularity.

The shape of the interface is given by the formula $\eta = 2\,\mathrm{Re}(F - G)$. Let us assume that the boundary is symmetric with respect to the point $x = x_c$, where a singularity is formed. It follows from the expressions (16) and (24) that, in the vicinity of the singularity, the interface shape looks like

$$z - z_c \sim -|x - x_c|^{3/2},$$

where $z_c = \eta(x_c,t_c)$. An important feature of such solutions is that they are consistent with the small-angle approximation. Indeed, near the singularity we have

$$\eta_x \sim -(x - x_c)|x - x_c|^{-1/2}.$$

Only the second derivative of the function η (i.e., in fact, the curvature of the interface between two fluids) becomes infinite at the critical time $t = t_c$,

$$\eta_{xx}(x,t_c) \sim -|x - x_c|^{-1/2}, \quad \eta_{xx}(x_c,t) \sim -(t_c - t)^{-1/2}.$$

It should be noted that such behavior of the interface is largely similar to that of vortex sheets, where Moore's curvature singularities appear [36].

It is important that, at time t_c, the slope angles are small not only in the neighborhood of the singularity but also at the periphery. Indeed, the characteristic angles can be estimated using the formula (22) from the shift of the time of singularity formation between linear and nonlinear models, $\Delta = t_0 - t_c$. According to Refs. [24,25], we have $\Delta \sim S^{1/3}$ for the initial conditions (20) with $S > 0$ (see also Sec. VII). Substituting this relation into (22) yields the following estimate: $\alpha(t_c) \sim S^{1/3}$, i.e., the small-angle approximation will be valid if $S \ll 1$.

Thus, if the condition (17) is satisfied, the electrohydrodynamic instability leads to the formation of weak root singularities on the interface between two dielectric fluids. The curvature of the boundary becomes infinite at singular points, while the slope angles remain relatively small. Such behavior of the boundary is similar to that of the free surface of a perfectly conducting liquid in a strong electric field [24,25]. In a certain sense, the results of the present section generalize the results of these works to the case of the interface between two liquids with comparable densities. Indeed, if the density of the upper fluid is much less than that of the lower fluid, $\rho_1 \gg \rho_2$, then it follows directly from (17) that $\varepsilon_1 \gg \varepsilon_2$. In such a case, the electric field does not penetrate into the lower fluid as if it were perfectly conducting.

VI. DYNAMICS OF THE INTERFACE FOR $A_E = -A$

Let us consider the behavior of the interface between two dielectric fluids in a vertical electric field for the case where the ratio of the permittivities of the fluids is equal to the inverse ratio of their densities, i.e., $A_E = -A$. If the condition (18) is satisfied, the system of integrodifferential equations (14) and (15) is split into two independent equations:

$$F_t + iAF_x = -2A^2\hat{P}(F\bar{F}_x)_x, \tag{25}$$

$$G_t - iAG_x = 0, \tag{26}$$

where we have taken into account that $\tau = A^{-1}$ in the considered case. The nonlinear right-hand side of the equation for the function G becomes identically zero. According to this equation, the function G decays to zero, and, as a consequence, one can put $G = O(\alpha^3)$. In terms of the canonical variables η and ψ, this means that

$$2A\eta = (1 + A)\psi \tag{27}$$

at the initial (weakly nonlinear) stage of the interface instability development, i.e., η and ψ are linearly dependent functions.

It is interesting that the relationship (27) between the canonical variables holds not only for the approximation of small surface slopes but also for the original three-dimensional equations of motion. Indeed, in Ref. [37] it was found that if $\varepsilon_1/\varepsilon_2 = \rho_2/\rho_1$, then the special regime of fluid motion can be realized for which the velocity and electric potentials are

linearly related,

$$\varphi_1 = \Phi_1 - z, \quad (1+A)\varphi_2 = (1-A)(\Phi_2 - z). \tag{28}$$

These expressions imply the existence of a coordinate system in which liquids move along the electric field lines, and their velocities are directly proportional to the field strength (see also Ref. [38]). Substituting the formulas (28) into the boundary condition (4), after simple transformations we arrive at (27). Thus, the relation (27) is exact for the increasing branch of the solutions. It follows from Eq. (26) that, at least at the initial (weakly nonlinear) stage of the instability development, the system tends to a state where the relation (27) is true. This suggests the stability of the flow regime corresponding to the expressions (28).

For $G = 0$ the function η, which determines the shape of the boundary, and the complex function F are related by the simple expression $F = \hat{P}\eta$. Equation (25) then will directly govern the boundary motion. The remarkable feature of Eq. (25) is that, in some cases, it can be reduced to ordinary differential equations (ODEs). Therefore, let us represent the unknown function as a sum of simple poles (this corresponds to the case of several (N) local perturbations of the boundary):

$$F(x,t) = \sum_{n=1}^{N} \frac{iS_n/2}{x + p_n(t)}, \quad n = 1,2,\ldots,N,$$

where S_n are real constants and $p_n(t)$ are complex functions that determine the locations of poles in the lower complex half-plane [i.e., Im $p_n(t) > 0$]. Equation (25) then reduces to the following system of ODEs,

$$\frac{dp_n}{dt} = -iA - iA^2 \sum_{j=1}^{N} \frac{S_j}{(p_n - \overline{p}_j)^2}, \quad n = 1,2,\ldots,N.$$

It defines the trajectories of N poles in the complex x plane. In the simplest case, when $N = 1$, these solutions describe the evolution of spatially localized perturbation of the interface with initial condition (20). Then

$$F(x,t) = \frac{iS/2}{x + ia(t)}, \quad a(0) = a_0, \tag{29}$$

where $a(t)$ is a positive function of time which determines the location of the pole on the imaginary axis. This function satisfies the following ODE,

$$da/dt = -A + SA^2/(4a^2). \tag{30}$$

As can be seen, if the pole is at a considerable distance from the real axis (this corresponds to $a^2 \gg |S|$), then the nonlinear term on the right-hand side of Eq. (30) is small compared to the linear term and, hence, the pole moves to the origin with a constant velocity along the imaginary axis. The nonlinear term begins to play a significant role when the pole approaches the real axis.

For negative S (i.e., for a local perturbation of the boundary, which is directed downwards), the nonlinearity accelerates the pole motion, and at the moment $t = t_c$ the function $a(t)$ vanishes (the pole reaches the real axis). Integrating Eq. (30), we obtain for the time of singularity formation

$$t_c = a_0/A - \sqrt{|S|/(4A)}\,\arctan(2a_0/\sqrt{A|S|}). \tag{31}$$

In terms of Sec. IV, this means that the influence of nonlinearity shifts the time of singularity formation by $\Delta \sim |S|^{1/2}$. Note that in the case of such dependence of Δ on the parameter S, even the rough estimate for the surface-slope angles (22) predicts the formation of strong singularities violating the small-angle approximation. In accordance with (29), the boundary shape becomes singular at the moment $t = t_c$: $\eta(x,t_c) = \pi S\delta(x)$, where $\delta(x)$ is the Dirac δ function. Such behavior of the interface fundamentally differs from that considered in the previous section, where only the second derivative of the function η becomes infinite in a finite time.

For positive S (i.e., for a local perturbation of the boundary, which is directed upwards), the motion of the pole slows down. The pole does not reach the real axis and, as a consequence, a singularity does not appear.

Thus, the analysis of the evolution equation for the interface (25) shows that the nonlinearity promotes the development of the instability for boundary perturbations directed from the lighter to the heavier fluid [i.e., for $S < 0$ in terms of the particular solution (29)]; it is responsible for sharpening the interface. It is clear that the applicability of the weakly nonlinear model (10) and (11), based on the small-angle approximation, breaks down for the solutions obtained. In any case, these solutions indicate a tendency to form strong finite-time singularities for which the slope angles increase up to $\pi/2$.

If the lower fluid is much heavier than the upper fluid ($\rho_1 \gg \rho_2$ or, what is the same, $A \approx 1$), then, according to (18), the dielectric constants satisfy the inequality $\varepsilon_1 \ll \varepsilon_2$ (i.e., $A_E \approx -1$). As a consequence, the electric field does not penetrate into the upper fluid. From a mathematical point of view, this is similar to considering a dielectric fluid with a perfectly conducting free surface in the limit where the surface charge completely screens the electric field above the fluid. Such a situation was studied in Refs. [26,27], where exact solutions to the corresponding equations of motion were found. According to the solutions, the singularities (cusps) appear on the surface; its shape in the vicinity of a singular point is defined by the relation

$$z - z_c \sim |x - x_c|^{2/3}.$$

It is natural to assume that such strong singularities can also form on the interface between two dielectric liquids with comparable densities, i.e., for $A \neq 1$.

VII. FORMATION OF SINGULARITIES IN THE GENERAL CASE

In the two previous sections, we considered the special cases, $A_E = \pm A$, in which the governing equation (14) can be solved analytically. In the present section, we will study the dynamics of singularity formation at the interface for arbitrary A_E and A, except for the particular case where $A_E = 0$ and the electrostatic forces are absent (then the characteristic time of instability development τ turns into infinity). It will be demonstrated that, in the general case, it is possible to describe the evolution of a localized perturbation of the interface [for instance, corresponding to the initial conditions of the form (20)].

Let us consider Eq. (14) in the framework of the "local" approximation, which implies that, in studying blow-up phenomena, one should primarily focus on the approach of singularities of the function F to the real axis (one of them touches the axis at $t = t_c$). Hence, it is sufficient to consider a small neighborhood of the closest (to the real axis) singularity, which moves along a trajectory $x = X(t)$ in the lower half-plane of the complex variable x. This allows us to simplify the right-hand side of Eq. (14). Since the influence of singularities of the function $\bar{F}_x$ (all of them are located in the upper half-plane) are suppressed by the projection operator $\hat{P}$ acting on the product $F\bar{F}_x$, it is sufficient to keep only the leading term in the expansion of the derivative $\bar{F}_x$ around the path $x = X(t)$. So, we will assume approximately that

$$\hat{P}(F\bar{F}_x) \approx FR(t), \quad R(t) \equiv \bar{F}_x|_{x=X(t)}. \tag{32}$$

This expression becomes exact when the function F has one simple pole; this situation is similar to that considered in Sec. VI.

Under the approximation (32), the equation of boundary motion (14) rewrites as

$$\tau F_t + iF_x = -(A + A_E){F_x}^2/2 - (A - A_E)R(t)F_x, \tag{33}$$

i.e., we get a nonlinear partial differential equation of the first order instead of the integro-differential equation. In the same way as was done in Sec. V, we introduce the complex velocity $V = F_x$. As a result, we arrive at the equation

$$\tau V_t + Q(t)V_x = -(A + A_E)VV_x,$$

where we have denoted $Q(t) \equiv i + (A - A_E)R(t)$. Its solution can be constructed by the method of characteristics,

$$V = V_0(\tilde{x}), \quad x\tau = \tilde{x}\tau + \int_0^t Q(t)\,dt + (A + A_E)V_0(\tilde{x})t, \tag{34}$$

where $V_0(x) = V|_{t=0}$ and $\tilde{x}$ is a parameter. On the whole, these expressions are only slightly more complicated than those corresponding to the particular case $A_E = A$ (see Sec. V). The appearance of the additional quasilinear term in Eq. (33) leads to the necessity of a self-consistent calculation of the dependence $Q(t)$.

The solution of the equation for G is given by the formula (16). Together, the pair of functions F and G determines the shape of the interface, $\eta = 2\,\mathrm{Re}(F - G)$. Since the function G remains small compared to F up to the moment of singularity formation, we can approximately assume that $\eta \approx 2\,\mathrm{Re}\,F$.

The basic relations concerning the dynamics of singularity formation in the solutions of Eq. (33) are given in Appendix B. From them it follows that, in the generic case, weak curvature singularities appear at the interface ($2\mathrm{Re}\,V_x \approx \eta_{xx} \to \infty$ at singular points).

Let us consider the behavior of the interface for the initial conditions (20), which, in terms of the function V_0, correspond to the expression

$$V_0(\tilde{x}) = -\frac{iS/2}{(\tilde{x} + ia_0)^2}. \tag{35}$$

Assume that $X(t_c) = 0$, i.e., a singularity appears at the point $x = 0$ on the interface. Substituting the expression (35) into the relations describing the motion of singularities (see Appendix B), we find the time t_c in the leading order of the expansion with respect to the small parameter S,

$$t_c \approx a_0\tau - 3\tau[Sa_0(A + A_E)]^{1/3}/2, \tag{36}$$

where the inequality $S(A + A_E) \geqslant 0$ must hold for the parameters of the problem. This inequality gives the necessary condition for singularity formation at the point $x = 0$. One can see that, for $A_E > -A$, the curvature singularity can appear only if $S > 0$, i.e., if the perturbation is directed upwards. The case $A_E = A$ (see Sec. V) also falls into this category. For $A_E < -A$, on the contrary, we have $S < 0$, i.e., the perturbation is directed downwards. In both cases, the curvature becomes infinite in a finite time. The main difference is the sign of the curvature. The curvature is negative for $A_E > -A$ and positive for $A_E < -A$.

As can be seen from the formula (36), $\Delta \sim |S|^{1/3}$ except for the degenerate case where $A_E = -A$. It then follows from the estimate (22) that at $t = t_c$ the small-angle approximation is valid not only in the vicinity of the singular point ($\eta_x = 0$ at $x = 0$) but also in the periphery. Note that the last term in the right-hand side of Eq. (33) provides only minor corrections of order $|S|^{2/3}$ to the estimate for t_c. However, in the case where $A_E = -A$, it is just this term which determines the dynamics of singularity formation. It yields the following nonlinear correction for the blow-up time: $\Delta \sim |S|^{1/2}$ [see also Eq. (31)]. For $A_E = -A$, the interface behavior was studied in detail in Sec. VI. It has been shown that, if $S < 0$, there is a tendency to form strong singularities for which the slope angles are finite.

To conclude this section, we note that the results obtained on the basis of the approximation (32) are fully consistent with the exact results corresponding to the special cases $A_E = \pm A$. This confirms the quality of the model equation (33).

VIII. CONCLUDING REMARKS

In the present work, the behavior of the interface between two liquid dielectrics in a strong vertical electric field has been studied. In the framework of the Hamiltonian formalism, the quadratic nonlinear integrodifferential equations (10) and (11) describing the interface dynamics were derived in the approximation of small surface slopes. Our analysis has shown that there exist two special cases, (17) and (18), where the equations of motion can be solved exactly for the plane geometry of the problem (all functions depend only on one spatial variable). In the first case, where the ratios of the permittivities and of densities of the two liquids coincide with each other ($\varepsilon_1/\varepsilon_2 = \rho_1/\rho_2$), the development of the instability results in the formation of root singularities on the interface. In the second case, where the ratio of the permittivities of the liquids equals the inverse ratio of their densities ($\varepsilon_1/\varepsilon_2 = \rho_2/\rho_1$), a tendency for the formation of strong singularities (cuspidal dimples) on the interface has been demonstrated.

Recall that the conditions (17) and (18) correspond to considering two different nonlinear terms existing in the governing equation (14). In the general case, i.e., for arbitrary values of dielectric constants and densities, both these terms affect the dynamics of singularity formation. For this

situation, it was proposed to describe the evolution of a localized perturbation of the boundary in the framework of the approximate equation (33). Its analysis has shown that weak root singularities, which do not violate the small-angle approximation, appear on an initially smooth interface for almost any initial conditions. At the singular points, the curvature tends to minus infinity for $\varepsilon_1/\varepsilon_2 > \rho_2/\rho_1$ (this also includes the special case where $\varepsilon_1/\varepsilon_2 = \rho_1/\rho_2$) and to plus infinity for $\varepsilon_1/\varepsilon_2 < \rho_2/\rho_1$. The stronger singularities with finite interface slopes appear in the intermediate (degenerate) case, where $\varepsilon_1/\varepsilon_2 = \rho_2/\rho_1$.

It is clear that if the boundary curvature increases infinitely in a finite time, the capillary forces inevitably begin to play an essential role. According to (1), this happens on the scale of $k \geqslant k_1 \sim \varepsilon_0 E^2/\sigma$, where the interface dynamics is simultaneously governed by capillary and electrostatic forces (here E is the characteristic electric field strength). If the permittivities of the liquids differ considerably, then the self-similar regime of the boundary motion for which the scale decreases as $(t_c - t)^{2/3}$ can be realized [39,40]. The instability development then can result in the formation of conical tips of size k_1^{-1} on the boundary, for which the interfacial curvature increases as $(t_c - t)^{-2/3}$. In the formal limit of infinitely strong electric field, the size of conical formations turns into zero, $k_1^{-1} \to 0$, i.e., they are compressed into the points corresponding to root singularities. From this we can conclude that there exists a nonlinear mechanism for energy transfer from the large scale k_0^{-1} to the small scale k_1^{-1} related to the formation of 3/2-power singularities. These root singularities generate more strong singularities, whose consideration is beyond the scope of weakly nonlinear analysis.

Next, let us discuss the applicability of the irrotational flow approximation. This approximation is valid for large Reynolds numbers [see the condition (2)] but violated for $k \geqslant k_2 \sim E\nu^{-1}(\varepsilon_0/\rho_1)^{1/2}$. In the formal limit $E \to \infty$ we have $k_2^{-1} \to 0$, i.e., the flow can be assumed to be potential up to the moment of singularity formation. For a finite value of the scale k_2^{-1}, our approach, based on the Hamiltonian formalism and, therefore, neglecting dissipation, is not applicable in the neighborhood of the developing singularity.

Let us estimate the spatial scales k_1^{-1} and k_2^{-1} where capillary and, respectively, viscous forces should be taken into account. For simplicity, we consider the situation where the upper fluid is absent. Let $E = E_2 = 10^7$ V/m, i.e., the electric field strength considerably exceeds the threshold for the onset of surface instability ($\sim 2.5 \times 10^6$ V/m for deionized water). Further, let the wavelength of the initial surface perturbation $(2\pi/k_0)$ be ~ 0.01 m. For water ($\rho = 10^3$ kg/m^3, $\sigma \approx 0.073$ N/m, and $\nu \approx 10^{-6}$ m^2/s) we get $2\pi/k_1 \approx 5 \times 10^{-4}$ m and $2\pi/k_2 \approx 7 \times 10^{-6}$ m, i.e., the conditions (1) and (2) are satisfied. Moreover, it is seen that the effect of viscosity can be neglected in comparison with capillary and electrostatic forces in the vicinity of the singularity. The same situation takes place for other low-viscosity liquids, such as ethyl alcohol, benzene, nitrobenzene, acetone, and so on. Similar estimates for the role of viscosity in the formation of Taylor cones on the surface of liquid metal in the presence of an electric field are given in Ref. [41].

The viscosity of organic or mineral oils is much greater than the viscosity of water. For estimations we take linseed oil ($\sigma \approx 0.025$ N/m and $\nu \approx 5 \times 10^{-5}$ m^2/s). The scales $2\pi/k_1 \approx 2 \times 10^{-4}$ m and $2\pi/k_2 \approx 3 \times 10^{-4}$ m remain small as compared to $2\pi/k_0$, but now they are close to each other. On the one hand, this ensures the validity of our weakly nonlinear analysis. On the other hand, this implies that viscous forces, as well as capillary forces, have a significant importance at the strongly nonlinear stages of the instability. Note that the interfacial tension is always less than the surface tensions of the liquid pair used; it can take the relatively small values of 10^{-4}–10^{-3} N/m (see, e.g., Ref. [31]). This can lead to a decrease in the relative contribution of capillary forces.

Finally, for high-viscosity liquids (e.g., glycerol and some liquid organosilicon polymers), the applicability conditions of our model are violated. Indeed, for glycerol ($\sigma \approx 0.06$ N/m, $\nu \approx 10^{-3}$ m^2/s, and $\rho \approx 1260$ kg/m^3) we obtain $2\pi/k_1 \approx 4 \times 10^{-4}$ m and $2\pi/k_2 \approx 8 \times 10^{-3}$ m, i.e., the scales k_0^{-1} and k_2^{-1} are comparable, and, consequently, the role of viscosity is essential throughout the evolution of the system. Usually, viscosity delays the formation of singularities without qualitatively changing the behavior of the system (see, e.g., the recent paper [42] where the singularity formation on the viscous vortex sheet caused by the Kelvin-Helmholtz instability has been studied). It then can be assumed that the main conclusion of the present work, namely that the nonlinearity determines the tendency to the formation of singularities where the interfacial curvature blows up, and its sign coincides with the sign of the expression $(\varepsilon_2\rho_2 - \varepsilon_1\rho_1)$, may also be valid.

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APPENDIX A

With the use of the Green's first theorem, the Hamiltonian (6) can be represented as an integral over the interface,

$$H = \frac{1}{2}\int_S [\Phi_1 \partial_n \Phi_1 - (\rho_2/\rho_1)\Phi_2 \partial_n \Phi_2 - \tilde{\varphi}_1 \partial_n \tilde{\varphi}_1 + (\varepsilon_2/\varepsilon_1)\tilde{\varphi}_2 \partial_n \tilde{\varphi}_2]_S dS, \quad \text{(A1)}$$

where $dS \equiv \sqrt{1 + (\nabla_\perp \eta)^2}\, dxdy$ is the surface differential and $\tilde{\varphi}_{1,2}$ are the perturbations of the dimensionless electric potentials,

$$\tilde{\varphi}_1 = \varphi_1 + z, \quad \tilde{\varphi}_2 = \varphi_2 + (\varepsilon_1/\varepsilon_2)z.$$

Note that, for the auxiliary potentials $\tilde{\varphi}_{1,2}$, the boundary conditions (4) take the form

$$\tilde{\varphi}_1 = \tilde{\varphi}_2 + (1 - \varepsilon_1/\varepsilon_2)\eta, \quad \partial_n \tilde{\varphi}_1 = (\varepsilon_2/\varepsilon_1)\partial_n \tilde{\varphi}_2, \quad z = \eta(x,y,t). \quad \text{(A2)}$$

In order to derive the equations of motion for the interface, we should expand the integrand of the Hamiltonian (A1) in powers of the canonical variables η and ψ. Let us take into account that the harmonic functions decaying at $z \to \mp\infty$ obey the relations

$$\phi_{1,2}|_{z=\eta} = \sum_{n=0}^{\infty} \frac{(\pm\eta)^n \hat{k}^n}{n!} \phi_{1,2}|_{z=0},$$
$$\partial_z \phi_{1,2}|_{z=0} = \pm\hat{k}\phi_{1,2}|_{z=0}.$$

Applying them to the potentials $\tilde{\varphi}_{1,2}$, $\Phi_{1,2}$, and their derivatives, we can express all the functions contained in (A1) in terms of η and ψ. In particular, we find for the potentials $\tilde{\varphi}_{1,2}$ and $\Phi_{1,2}$ from the boundary conditions (3) and (A2),

$$\tilde{\varphi}_{1,2}|_{z=\eta} = \pm\frac{A_E(1 \mp A_E)}{1 - A_E}\,\eta - A_E(1 - A_E)\hat{k}^{-1}[\hat{k}(\eta\hat{k}\eta) + \nabla_\perp(\eta\nabla_\perp\eta)] + O(\alpha^3), \tag{A3}$$

$$\Phi_{1,2}|_{z=\eta} = \pm\frac{1 + A}{2}\,\psi + \frac{(1 + A)(1 \mp A)}{2}\hat{k}^{-1} \times [\hat{k}(\eta\hat{k}\psi) + \nabla_\perp(\eta\nabla_\perp\psi)] + O(\alpha^3). \tag{A4}$$

In deriving the relations (A3) and (A4), the quadratic nonlinearity is only taken into account.

Substituting (A3), (A4), and also the expansions for the normal derivatives of the potentials into (A1), we arrive after simple transformations at the following expression for the Hamiltonian of the system,

$$H = \frac{1 + A}{4}\int (\psi\hat{k}\psi - A\eta[(\hat{k}\psi)^2 - (\nabla_\perp\psi)^2])dxdy - \frac{A_E^2}{1 - A_E}\int (\eta\hat{k}\eta + A_E\eta[(\hat{k}\eta)^2 - (\nabla_\perp\eta)^2])dxdy.$$

It contains only second- and third-order terms in the integrand.

APPENDIX B

The expressions (34) give the solution of Eq. (33) in the implicit form. The problem of finding the explicit solution reduces to analyzing the map $x \to \tilde{x}$ specified by Eq. (34). This map ceases to be one-to-one at points where

$$\partial x/\partial\tilde{x} = 1 + (A + A_E)V_0'(\tilde{x})t/\tau = 0. \tag{B1}$$

Here V_0' defines the derivative with respect to the argument. Solution of (B1) gives a trajectory $\tilde{x} = \tilde{X}(t)$ in the complex $\tilde{x}$ plane. The motion of the branch point of the function V then is given by

$$X(t)\tau = \tilde{X}(t)\tau + \int_0^t Q(t)\,dt + (A + A_E)V_0(\tilde{X}(t))t. \tag{B2}$$

In terms of V, the time function $Q(t)$ is determined the auxiliary equation

$$Q(t) = i + (A - A_E)\bar{V}|_{x=X(t)}.$$

At the moment $t = t_c$ when the branch point touches the real axis, i.e., $\mathrm{Im}\, X(t_c) = 0$, the solution of Eq. (33) becomes singular.

Let us examine the behavior of the solution near a singularity. Expanding (34) in a small neighborhood of the point $t = t_c$, $x = x_c \equiv X(t_c)$, and $\tilde{x} = \tilde{x}_c \equiv \tilde{X}(t_c)$ and taking into account (B1) and (B2), in the leading order we get

$$V \approx V_0(\tilde{x}_c) + V_0'(\tilde{x}_c)\,\delta\tilde{x},$$
$$\tau\delta x \approx Q(t_c)\,\delta t + (A + A_E)V_0(\tilde{x}_c)\,\delta t + (A + A_E)V_0''(\tilde{x}_c)\,t_c(\delta\tilde{x})^2/2,$$

where $\delta t = t - t_c$, $\delta x = x - x_c$, and $\delta\tilde{x} = \tilde{x} - \tilde{x}_c$. Excluding the parameter $\delta\tilde{x}$ from these expressions, we obtain close to the singularity,

$$V \approx V_0(\tilde{x}_c) + V_0'(\tilde{x}_c)\left[\frac{\tau\delta x - \delta t\,(Q(t_c) + (A + A_E)V_0(\tilde{x}_c))}{(A + A_E)V_0''(\tilde{x}_c)\,t_c/2}\right]^{1/2}. \tag{B3}$$

One can see that the derivatives V_x and V_t become infinite for $\delta t \to 0$, which corresponds to the formation of 3/2-power singularities on the interface. The exception is the special case where $A_E = -A$, and the expansion (B3) loses its meaning. This situation is considered in Sec. VI.

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