

Akhmediev breathers, Ma solitons, and general breathers from rogue waves: A case study in the Manakov system

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We present explicit forms of general breather (GB), Akhmediev breather (AB), Ma soliton (MS), and rogue wave (RW) solutions of the two-component nonlinear Schrödinger (NLS) equation, namely Manakov equation. We derive these solutions through two different routes. In the forward route, we first construct a suitable periodic envelope soliton solution to this model from which we derive GB, AB, MS, and RW solutions. We then consider the RW solution as the starting point and derive AB, MS, and GB in the reverse direction. The second approach has not been illustrated so far for the two component NLS equation. Our results show that the above rational solutions of the Manakov system can be derived from the standard scalar nonlinear Schrödinger equation with a modified nonlinearity parameter. Through this two-way approach we establish a broader understanding of these rational solutions, which will be of interest in a variety of situations.

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I. INTRODUCTION

We consider the integrable system of two coupled nonlinear Schrödinger equations (NLSEs), namely the Manakov system,

$$\begin{aligned} i q_{1t} + q_{1xx} + 2\mu(|q_1|^2 + |q_2|^2)q_1 &= 0, \\ i q_{2t} + q_{2xx} + 2\mu(|q_1|^2 + |q_2|^2)q_2 &= 0, \end{aligned} \quad (1)$$

where q_1 and q_2 are wave envelopes, x and t are space and time variables, respectively, μ is a real constant, and subscripts denote partial derivatives with respect to the corresponding variables. Equation (1) represent the propagation of an optical pulse in a birefringent optical fiber and in wavelength division multiplexed system [1]. It has been studied widely in the literature [1–7]. The complete integrability of this system of coupled NLSEs was first established by Manakov [1]. It has also been demonstrated that this two-component vector generalization of the focusing NLS equation admits several interesting properties, including (i) infinite number of conservation laws [1], (ii) Lax pair [1], (iii) an infinite-dimensional algebra of noncommutative symmetries [3], (iv) bilinear representation and bright multisoliton solutions [6], and so on. Equation (1) also appear in multicomponent Bose-Einstein condensates [8], biophysics [9], finance [10], and oceanographic studies [11]. Solitons in coupled NLSEs have been the subject of intense study over the past few years because of their interesting collision properties and their robustness against external perturbations. The explicit multibright and dark soliton solutions of Eq. (1) were obtained by Radhakrishnan *et al.* [6,7]. It has also been demonstrated that the soliton solutions of Eq. (1) exhibit a fascinating shape changing collision, resulting in a redistribution of intensity between the modes of the two solitons, which is not observed in the scalar NLSE case [7].

Very recently, a new rational solution called rogue wave (RW) solution has attracted considerable attention [12–37]. RWs, alternatively called freak or giant waves, were first observed in arbitrary depth of ocean circumstances. A wave is classified into this category when its wave height (distance from trough to crest) reaches a value that is at least twice that of the significant wave height [12]. These waves may

arise from the instability of a certain class of initial conditions that tend to grow exponentially and, thus, have the possibility of increasing up to very high amplitudes, due to modulation instability [13]. Over the years, RWs have also been observed in models that arise in the description of multicomponent Bose-Einstein condensates [14], capillary waves [15], multicomponent plasmas [16], and even in finance [17]. Recently, efforts have been made to explain the RW excitation through a nonlinear process. It has been found that the NLS equation can describe many dynamical features of the RW. Certain kinds of exact solutions of the NLS equation have been considered to describe possible mechanisms for the formation of RWs such as Peregrine soliton, time periodic breather or Ma soliton (MS), and space periodic breather or Akhmediev breather (AB) [18,19]. As a consequence, attempts have been made to construct RW solution through different methods for the NLS equation and its higher derivative generalizations. One way of obtaining RW solution or Peregrine soliton for a given system is to first construct a breather solution, either AB or MS. From the latter, the RW solution can be deduced in an appropriate limit.

As far as the system of two coupled NLSEs is concerned, in recent years, the following studies have been undertaken. Breathers and rational breather solutions of multicomponent NLSE are presented in Ref. [20] in a determinantal form, as limiting cases in suitable degenerations of algebrogeometric solutions. Explicit first-, second-, and third-order RW solutions of Eq. (1) have been constructed in Ref. [21] through the modified Darboux transformation method. The authors have also studied some basic properties of multi-rogue wave solutions and their collision structures. In Ref. [22], two types of RW solutions through Darboux transformation method have been derived. The authors have shown that while the first kind of RW solution is similar to the first-order RW solution of NLSE, the second kind of RW behaves differently from that of the first-order rogue wave solution. Recently, an in-depth analysis on the construction of vector Peregrine soliton solution and bright-dark-RW solution of Eq. (1) has been made in Ref. [23]. In all the above cited works, only the explicit forms of RWs are given, and we do not see any simple tractable form of breather solutions.

Since the breather solution plays an important role in the formation of RWs, we aim here to derive the breather solution for the widely studied nonlinear evolution Eq. (1). We divide our analysis into two parts. In the first part, we construct a periodic envelope two-soliton solution through Hirota's bilinearization method. By appropriately restricting the wave number (of one of the solitons to be the complex conjugate of the other), which appears in the two-soliton solution, we obtain the GB form of Eq. (1). From the GB solution, we derive AB, MS, and RW solutions. We note here that through a restricted set of transformations, the GB solution of the Manakov equation can be obtained from the GB solution of the NLS equation with a modified nonlinear parameter. In other words, one can generate the above rational solutions of the Manakov system from the standard nonlinear Schrödinger equation with a modified nonlinear parameter.

In the second part of our work, we analyze the reverse problem: How can one construct an AB or MS or GB from a RW solution? We answer this question by rewriting the RW solution in a factorized form and then generalizing this factorized form in an imbricate series expression [38–40], with certain unknown parameters in this series, following the earlier work of Tajiri and Watanabe [41] for the case of scalar NLS equation, and finally finding these unknown parameters in the imbricate series by substituting it into Eq. (1) and solving the resultant equations. With three different forms of the imbricate series, we derive the AB, MS, and GB solutions from the RW solution of Eq. (1).

The plan of the paper is as follows. In Sec. II, we construct the explicit form of the GB solution of the two coupled NLSE system Eq. (1) through Hirota's bilinearization method. We then explain the method of deriving AB, MS, and RW solutions from the GB solution. The obtained form of RW solution coincides with the ones in the literature. In Sec. III, we discuss the method of constructing the AB solution from the RW solution. In Sec. IV, we demonstrate the construction of MS from RW. In Sec. V, we formulate the imbricate series form for the RWs with certain unknown arbitrary functions in it and then compare this expression with the one derived from the GB in the same way. The comparison provides exact expressions for the unknown arbitrary functions, which appear in the imbricate series of the RW. In this way, we establish a method of constructing GBs from RW. Finally, in Sec. VI, we present our conclusions.

II. GENERAL BREATHER

We seek a periodic envelope solution to the CNLS Eq. (1) with the boundary conditions $|q_i|^2 \rightarrow \tau_i^2$, $i = 1, 2$, as $x \rightarrow \pm\infty$, where τ_1 and τ_2 are real constants. To start with, we bilinearize Eq. (1) through the transformation $q_1 = \frac{g}{f}$ and $q_2 = \frac{h}{f}$, where g and h are complex functions and f is a real function. The resultant bilinearized forms read

$$\begin{aligned} (iD_t + 2ikD_x + D_x^2)g \cdot f &= 0, \\ (iD_t + 2ikD_x + D_x^2)h \cdot f &= 0, \\ (D_t^2 + 2\mu(\tau_1^2 + \tau_2^2))f \cdot f - 2\mu(|g|^2 + |h|^2) &= 0. \end{aligned} \quad (2)$$

In the above, D_t and D_x are Hirota's bilinear operators [6]. Once the nonlinear evolutionary equation has been

bilinearized, with truncated parameter expansion at different levels, a series of solutions, in particular the N -soliton solution, can be obtained. As far as Eqs. (2) are concerned, the N -soliton solution can be obtained with respect to the expansion parameter χ , that is $g = g_0(1 + \chi g_1 + \chi^2 g_2 + \dots)$, $h = h_0(1 + \chi h_1 + \chi^2 h_2 + \dots)$, and $f = (1 + \chi f_1 + \chi^2 f_2 + \dots)$, where g_i 's and h_i 's, $i = 0, 1, 2, \dots, N$, are complex functions of x and t and f_i 's are real variables.

As our aim is to obtain the two-soliton solution, we terminate the expansion at quadratic powers in χ , that is $g = g_0(1 + \chi g_1 + \chi^2 g_2)$, $h = h_0(1 + \chi h_1 + \chi^2 h_2)$, and $f = (1 + \chi f_1 + \chi^2 f_2)$. The resultant two-soliton solution emerges in the form

$$q_1 = \tau_1 e^{i\theta} \frac{g}{f} \quad \text{and} \quad q_2 = \tau_2 e^{i\theta} \frac{h}{f}, \quad \theta = kx - \omega t, \quad (3)$$

where

$$\begin{aligned} g &= h = 1 + e^{\eta_1 + 2i\phi_1} + e^{\eta_2 + 2i\phi_2} + a e^{\eta_1 + \eta_2 + 2i\phi_1 + 2i\phi_2}, \\ f &= 1 + e^{\eta_1} + e^{\eta_2} + a e^{(\eta_1 + \eta_2)}, \quad \eta_j = p_j x - \Omega_j t + \eta_j^0, \\ j &= 1, 2. \end{aligned}$$

In the above p_j , Ω_j , η_j^0 and ϕ_j , $j = 1, 2$, are complex parameters and

$$\begin{aligned} \omega &= k^2 - 2\mu(\tau_1^2 + \tau_2^2), \quad p_j = 2i\sqrt{\mu(\tau_1^2 + \tau_2^2)} \sin \phi_j, \\ \Omega_j &= 2k_j p_j - p_j^2 \cot \phi_j, \quad j = 1, 2, \\ a &= \left[\frac{\sin \frac{1}{2}(\phi_1 - \phi_2)}{\sin \frac{1}{2}(\phi_1 + \phi_2)} \right]^2. \end{aligned} \quad (4)$$

In the above we have chosen $g = h$. One can proceed by assuming $g \neq h$ also. However, in order to obtain the required breather solutions, we find that one has to essentially fix $g = h$.

We have not pursued the possibility of a more general bilinearization than Eq. (2), which will lead to the possibility $g \neq h$ in Eq. (3). We hope to consider such a generalization in future. So in our analysis, we have made this choice in the beginning itself.

Note that due to the choice $g = h$ in Eq. (3), one can effectively make a transformation

$$q_1 = \tau_1 q, \quad q_2 = \tau_2 q, \quad (5)$$

so that

$$q = e^{i\theta} \frac{g}{f} = e^{i\theta} \frac{h}{f} \quad (6)$$

satisfies the scalar NLS equation with the nonlinearity parameter $2\mu(\tau_1^2 + \tau_2^2)$:

$$iq_t + q_{xx} + 2\mu(\tau_1^2 + \tau_2^2)|q|^2 q = 0. \quad (7)$$

Consequently, one can write down the breather solution of the variables q_1 and q_2 equivalently from the breather solution of the above scalar NLS equation as well.

The constants, $\phi_j = \phi_{jR} + i\phi_{jI} \neq 0$, $j = 1, 2$, help us to split the above breather expression into Akhmediev and Ma breathers as we see below. To obtain the breather solution from the above two-soliton solution, we take $\eta_1 = \eta_2^* \equiv \eta$ and $\phi_2 = \phi_1^* \pm \pi$. Substituting these two restrictions in Eq. (3) and considering $\eta = \eta_R + i\eta_I$ and $\phi_1 = \phi_R + i\phi_I$, the exponential

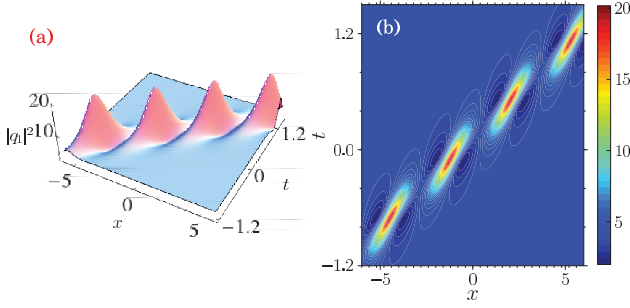


FIG. 1. (Color online) (a) General breather profile of q_1 for the values $\tau_1 = 2$, $\tau_2 = 1$, $\phi_R = 4$, $\phi_I = 1$, $\eta_I^0 = 0.5$, $\eta_R^0 = 0.2$, $k = 0.24$, $\mu = 0.2$ in Eqs. (5) and (8). (b) Corresponding contour plot. Similar profile occurs for q_2 also (not shown here).

functions appearing in Eq. (3) can be rewritten in terms of trigonometric and hyperbolic functions. The resultant expression for q turns out to be

$$q = \cos 2\phi_R e^{i(\theta+2\phi_R)} \left\{ 1 + \frac{1}{\sqrt{a} \cosh(\eta_R + \sigma) + \cos \eta_I} \times \left[\left(\frac{\cosh 2\phi_I}{\cos 2\phi_R} - 1 \right) \cos \eta_I + i \left(\sqrt{a} \tan 2\phi_R \sinh(\eta_R + \sigma) - \frac{\sinh 2\phi_I}{\cos 2\phi_R} \sin \eta_I \right) \right] \right\}, \quad (8)$$

where $\eta_R = p_R x - \Omega_R t + \eta_R^0$, $\eta_I = p_I x - \Omega_I t + \eta_I^0$, $p_I = p_R + i p_I$, $\Omega_I = \Omega_R + i \Omega_I$, η_R^0 , η_I^0 , and σ are constants. The exact forms of p_R , Ω_R , p_I , and Ω_I are given below:

$$\begin{aligned} p_R &= -2\sqrt{\mu(\tau_1^2 + \tau_2^2)} \cos \phi_R \sinh \phi_I, \\ p_I &= 2\sqrt{\mu(\tau_1^2 + \tau_2^2)} \sin \phi_R \cosh \phi_I, \\ \Omega_R &= 2k p_R - \frac{(p_R^2 - p_I^2) \sin 2\phi_R + 2p_R p_I \sinh 2\phi_I}{\cosh 2\phi_I - \cos 2\phi_R}, \\ \Omega_I &= 2k p_I - \frac{(p_R^2 - p_I^2) \sinh 2\phi_I - 2p_R p_I \sin 2\phi_R}{\cosh 2\phi_I - \cos 2\phi_R}. \end{aligned}$$

Consequently, the solution of the Manakov Eq. (1) can be obtained from Eq. (5) as $q_1 = \tau_1 q$ and $q_2 = \tau_2 q$. Equation (8) combined with Eq. (5) constitutes the GB solution of the CNLS Eq. (1). Figure 1 illustrates the behavior of this breather solution, which is periodic both in space and in time. From the GB solution we can derive AB, MS, and RW solutions by restricting the parameters ϕ_R and ϕ_I suitably. In the following, we report the explicit forms of these solutions.

A. AB from GB

To derive the AB solution, we consider the choice $\phi_R \neq 0$ and $\phi_I = 0$ in Eq. (8). This restriction fixes the wave number to be pure imaginary. In this case, we find

$$\begin{aligned} p_R &= 0, \quad p_I = 2\sqrt{\mu(\tau_1^2 + \tau_2^2)} \sin \phi_R, \\ \Omega_R &= p_I^2 \cot \phi_R, \quad \Omega_I = 2k p_I. \end{aligned} \quad (9)$$

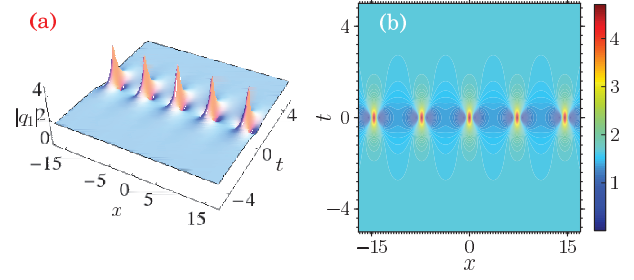


FIG. 2. (Color online) (a) Akhmediev breather profile of q_1 for the values $\tau_1 = 2$, $\tau_2 = 1$, $\phi_R = 0.5$, $\mu = 0.2$, $\eta_I^0 = 0.5$, $\eta_R^0 = 0.1$ in Eq. (10). (b) Corresponding contour plot. Similar profile occurs for q_2 also (not shown here).

Substituting the GB solution Eq. (8) into Eq. (5), the latter provides

$$\begin{aligned} q_1 &= \tau_1 \cos(2\phi_R) e^{i(\theta+2\phi_R)} (1 + L), \\ q_2 &= \tau_2 \cos(2\phi_R) e^{i(\theta+2\phi_R)} (1 + L), \\ L &= \frac{\left[\frac{1}{\cos(2\phi_R)} - 1 \right] \cos \eta_I + i \sqrt{a} \tan(2\phi_R) \sinh(\eta_R + \sigma)}{\sqrt{a} \cosh(\eta_R + \sigma) + \cos \eta_I}. \end{aligned} \quad (10)$$

Here, $\eta_R = -\Omega_R t + \eta_R^0$ and $\eta_I = p_I x - \Omega_I t + \eta_I^0$. We have plotted the solution Eq. (10) in Fig. 2. The solution is periodic in x and localized in t . This spatially periodic breather is nothing but the AB solution.

B. MS from GB

Now we consider the other case, $\phi_R = 0$ and $\phi_I \neq 0$ with $k = 0$ in the GB solution [Eq. (5) with Eq. (8)]. This restriction fixes the imaginary part of the wave number to be zero. This in turn provides another form of the breather solution, which will propagate only in the time direction, that is

$$\begin{aligned} q_1 &= \tau_1 e^{i\theta} (1 + V), \quad q_2 = \tau_2 e^{i\theta} (1 + V), \\ V &= \frac{\cosh(2\phi_I) - 1}{\sqrt{a} \cosh(\eta_R + \sigma) + \cos \eta_I} [\cos \eta_I - i \sinh(2\phi_I) \sin \eta_I], \end{aligned} \quad (11)$$

where

$$\begin{aligned} p_R &= -2\sqrt{\mu(\tau_1^2 + \tau_2^2)} \sinh \phi_I, \quad p_I = 0, \\ \Omega_R &= 2k p_R, \quad \Omega_I = p_R^2 \coth \phi_I, \quad \eta_R = p_R x - \Omega_R t + \eta_R^0, \end{aligned}$$

and

$$\eta_I = -\Omega_I t + \eta_I^0. \quad (12)$$

We depict this solution in Fig. 3. The plot confirms that the solution is periodic in t and localized in x . The wave solution that is temporally breaching and spatially localized is called a Ma breather/MS.

C. RW from GB

To construct the RW solution from the GB expression we consider both ϕ_R and ϕ_I to be nonzero and evaluate the latter in the limit $\epsilon \rightarrow 0$ by implementing a Taylor expansion. Doing so we find $\phi_R = \epsilon \gamma$ and $\phi_I = \epsilon \rho$, where ϵ is a small parameter,

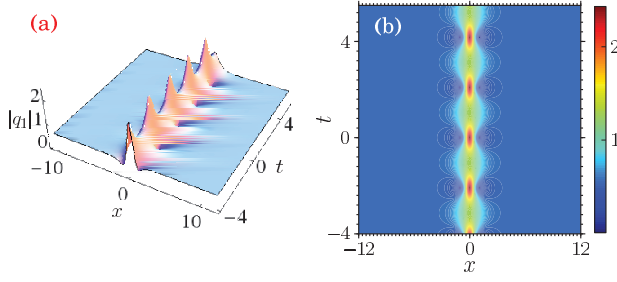


FIG. 3. (Color online) (a) Ma breather profile of q_1 for the values $\tau_1 = 2$, $\tau_2 = 1$, $\phi_I = 0.8$, $\mu = 0.2$, $\eta_I^o = 0.4$, $\eta_R^o = 0.3$ in Eq. (11). (b) Corresponding contour plot. Similar profile occurs for q_2 also (not shown here).

and γ and ρ are constants. Substituting these two expressions in Eq. (4) with the restriction $\eta_2 = \eta_1^*$, $\phi_2 = \phi_1^* + \pi$ and making the Taylor expansion at $\epsilon \rightarrow 0$, we obtain

$$\begin{aligned} p_R &= -2\sqrt{\mu(\tau_1^2 + \tau_2^2)}\rho\epsilon + O(\epsilon^3), \\ p_I &= 2\sqrt{\mu(\tau_1^2 + \tau_2^2)}\gamma\epsilon + O(\epsilon^3), \\ \Omega_R &= [4\mu(\tau_1^2 + \tau_2^2)\gamma - 4k\sqrt{\mu(\tau_1^2 + \tau_2^2)}\rho]\epsilon + O(\epsilon^3), \\ \Omega_I &= [4\mu(\tau_1^2 + \tau_2^2)\rho + 4k\sqrt{\mu(\tau_1^2 + \tau_2^2)}\gamma]\epsilon + O(\epsilon^3), \quad (13) \\ \sqrt{a} &= 1 + \frac{1}{2}(\gamma^2 + \rho^2)\epsilon^2, \\ f &= [(\tilde{\eta}_R^2 + \tilde{\eta}_I^2) + (\gamma^2 + \rho^2)]\epsilon^2 + O(\epsilon^3), \\ g &= [(\tilde{\eta}_R^2 + \tilde{\eta}_I^2) - 3(\gamma^2 + \rho^2) + 4i(\gamma\tilde{\eta}_R + \rho\tilde{\eta}_I)]\epsilon^2 + O(\epsilon^3). \end{aligned}$$

It is also straightforward to check that $\eta_R - \eta_R^0 = \epsilon\tilde{\eta}_R + O(\epsilon^2)$ and $\eta_I - \eta_I^0 = \epsilon\tilde{\eta}_I + O(\epsilon^2)$. Substituting the above expressions, Eq. (13), into the general breather form Eq. (8) and taking the limit $\epsilon \rightarrow 0$ in the resultant expression, we arrive at

$$q_1 = \tau_1 e^{i\theta}(1 - Q) \quad \text{and} \quad q_2 = \tau_2 e^{i\theta}(1 - Q), \quad (14)$$

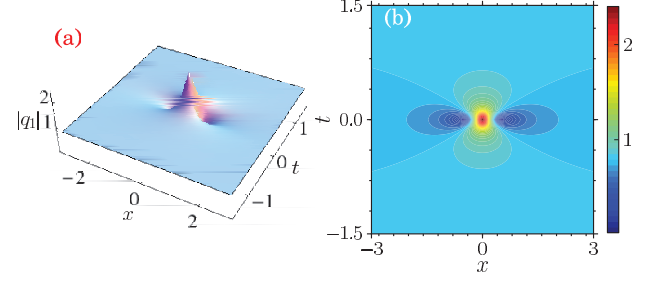


FIG. 4. (Color online) (a) Rogue wave profile of q_1 for the values $\tau_1 = 0.8$, $\tau_2 = 1.5$, $\mu = 1$, $k = 0$ in Eq. (14). (b) Corresponding contour plot. Similar profile occurs for q_2 also (not shown here).

where

$$Q = \frac{4 + 16i\mu(\tau_1^2 + \tau_2^2)t}{1 + 4\mu(\tau_1^2 + \tau_2^2)(x - 2kt)^2 + 16\mu^2(\tau_1^2 + \tau_2^2)^2 t^2},$$

which is nothing but the RW solution of CNLS system, which is localized both in space and time. The RW solution of the CNLS equation matches with the one presented in Ref. [21]. We note that the restriction $\tau_2 = 0$ in Eq. (11) provides the RW solution of the scalar NLS equation. A typical evolution of the RW is shown in Fig. 4.

In the above, we derived AB, MB, and RW solutions from the GB solution. On the other hand, we now point out the interesting possibility that one can also construct the above solutions from the RW solution itself in a reverse way. In the following, we demonstrate this by following the procedure of Tajiri and Watanabe for the case of the scalar NLS equation [41]. To do so we consider the RW solutions in an imbricate series form.

III. AB FROM RW

To derive AB from RW solution we first factorize the RW solution Eq. (14) in the following form, namely

$$\begin{aligned} q_1 &= \tau_1 \exp(i\{kx - [k^2 - 2\mu(\tau_1^2 + \tau_2^2)]t\}) \left[1 + \frac{1}{2i\mu(\tau_1^2 + \tau_2^2)t + \frac{1}{2}\sqrt{1 + 4\mu(\tau_1^2 + \tau_2^2)(x - 2kt)^2}} \right] \\ &\times \left[1 + \frac{1}{2i\mu(\tau_1^2 + \tau_2^2)t - \frac{1}{2}\sqrt{1 + 4\mu(\tau_1^2 + \tau_2^2)(x - 2kt)^2}} \right], \\ q_2 &= \tau_2 \exp(i\{kx - [k^2 - 2\mu(\tau_1^2 + \tau_2^2)]t\}) \left[1 + \frac{1}{2i\mu(\tau_1^2 + \tau_2^2)t + \frac{1}{2}\sqrt{1 + 4\mu(\tau_1^2 + \tau_2^2)(x - 2kt)^2}} \right] \\ &\times \left[1 + \frac{1}{2i\mu(\tau_1^2 + \tau_2^2)t - \frac{1}{2}\sqrt{1 + 4\mu(\tau_1^2 + \tau_2^2)(x - 2kt)^2}} \right]. \end{aligned} \quad (15)$$

We note here that one of the remarkable properties displayed by many classical nonlinear evolution equations possessing soliton modes is a nonlinear superposition principle [38–40]. More precisely, an infinite array of solitons placed at equal intervals constitutes an exact periodic solution of the evolution equations. For example, if we take the algebraic soliton solution of the modified Korteweg-de Vries equation in the form $u = u_0 - \frac{4u_0}{4u_0^2(x - 6u_0^2t)^2 + 1}$, we can write a more general solution through a superposition of these algebraic solitons as $u = u_0 - \sum_{m=-\infty}^{\infty} \frac{4u_0}{4u_0^2(x - m\lambda - ct)^2 + 1}$, where λ is the spacing between successive peaks of the sequence of solitary pulses and c is the phase speed of the pattern, which is to be determined. By rewriting the algebraic soliton solution as a hyperbolic cot function and further splitting the latter as sinh and cosh functions, one can get an expression for the periodic solution with an unknown parameter c . This parameter can be derived by substituting the hyperbolic cot function form in the original evolutionary equation (for more details one may refer to Ref. [42] and the references cited therein). The above is indeed an imbricate solution. In fact, a theorem on imbricate series (see for example, Theorem 3.1 in Ref. [38]) asserts that any periodic function $f(x)$ with period L has two series representations. If the usual Fourier series is

$$f(x) = \frac{\alpha}{2\pi} \sum_{n=-\infty}^{\infty} g(\alpha n) e^{i2\pi x n/L}, \quad \alpha > 0, \quad (16)$$

then the alternative expansion is the imbricate series

$$f(x) = \sum_{m=-\infty}^{\infty} G \left[\frac{2\pi}{L\alpha} (x - mL) \right], \quad (17)$$

where $G(k)$ is the Fourier transform of $g(x)$, that is,

$$G(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(x) e^{ikx} dx. \quad (18)$$

However, the imbricate series of rogue waves, which we consider in the following, will be different from the usual way of applying such a series, which is the superposition of the solitary waves. The important point here is that the breather solution Eq. (15) is being constructed by the product of two imbricate series of rogue waves as shown below. For mathematical simplicity, we consider the constant $k = 0$ hereafter (for $k \neq 0$, see below). We consider Eq. (15) in a more general form, that is

$$\begin{aligned} q_1 &= \tau_1 \exp[i(\sigma t + \phi)] \left[1 + b \sum_{n=-\infty}^{\infty} \frac{1}{i\alpha t + v(x) + n} \right] \\ &\times \left[1 + b \sum_{n=-\infty}^{\infty} \frac{1}{i\alpha t - v(x) + n} \right], \\ q_2 &= \tau_2 \exp[i(\sigma t + \phi)] \left[1 + b \sum_{n=-\infty}^{\infty} \frac{1}{i\alpha t + v(x) + n} \right] \\ &\times \left[1 + b \sum_{n=-\infty}^{\infty} \frac{1}{i\alpha t - v(x) + n} \right], \end{aligned} \quad (19)$$

where b is a constant, α , σ , and $v(x)$ are all to be determined. However, to derive the AB solution we consider only this

series. In the above expression, we have grouped the spatial variable x with the real part and the time variable t with the imaginary part. We have superposed the RW solutions in the x direction. Using the trigonometric identity [43] $\cot \pi x = \frac{1}{\pi x} + \frac{x}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{n(x-n)}$, $n \neq 0$, we replace the infinite series by cot function and rewrite Eq. (19) in a more compact form as

$$\begin{aligned} q_1 &= \tau_1 \exp[i(\sigma t + \phi)] (1 + b\pi \cot\{\pi[v(x) + i\alpha t]\}) \\ &\times (1 - b\pi \cot\{\pi[v(x) - i\alpha t]\}), \\ q_2 &= \tau_2 \exp[i(\sigma t + \phi)] (1 + b\pi \cot\{\pi[v(x) + i\alpha t]\}) \\ &\times (1 - b\pi \cot\{\pi[v(x) - i\alpha t]\}). \end{aligned} \quad (20)$$

Our task is to plug Eq. (20) into Eq. (1) and determine the parameters α , σ and the function $v(x)$ consistently. To do so, in the first step, we replace the cot functions in Eq. (20) as $\cos[\pi v(x) \pm i\pi\alpha t] / \sin[\pi v(x) \pm i\pi\alpha t]$ and substitute it into Eq. (1) and rewrite the equations in terms of $\sin[\pi v(x) \pm i\pi\alpha t]$ and $\cos[\pi v(x) \pm i\pi\alpha t]$ and their powers and products. We then simplify these equations using suitable trigonometric identities and rearrange the resultant expressions in the variables $\cos(i\pi\alpha t)$ $\sin(i\pi\alpha t)$ and their powers. By doing so, we have arranged the spatial variable to appear only in the coefficients in the resultant equation. The final expressions for both the equations in Eq. (1) turn out to be one and the same. As a result, we proceed to determine the unknowns σ , μ , and $v(x)$ from the single equation

$$\begin{aligned} r_1 \cos^6(i\pi\alpha t) + r_2 \cos^5(i\pi\alpha t) \sin(i\pi\alpha t) + r_3 \cos^4(i\pi\alpha t) \\ + r_4 \cos^3(i\pi\alpha t) \sin(i\pi\alpha t) + r_5 \cos^2(i\pi\alpha t) \\ + r_6 \cos(i\pi\alpha t) \sin(i\pi\alpha t) + r_7 = 0, \end{aligned} \quad (21)$$

where

$$\begin{aligned} r_1 &= 2\mu(\tau_1^2 + \tau_2^2)(1 + b^2\pi^2 - b^4\pi^4 - b^6\pi^6) - \sigma + b^2\pi^2\sigma, \\ r_2 &= 2b\pi[\sigma - 2\mu(\tau_1^2 + \tau_2^2)(1 + b^2\pi^2)^2], \\ r_3 &= 2\mu(\tau_1^2 + \tau_2^2)[-3\cos^2 A - b^2\pi^2(1 + \cos^2 A) \\ &\quad + b^4\pi^4(2 - \cos^2 A) + 3b^6\pi^6 \sin^2 A] + 3\sigma \cos^2 A \\ &\quad - b^2\pi^2\sigma(1 + \cos^2 A) - 4b^2\pi^4 v'^2(1 - 2\cos^2 A) \\ &\quad + 2b\pi^2\alpha(1 - 2\cos^2 A) + 4b^2\pi^3 v'' \cos A \sin A, \\ r_4 &= 8\mu(\tau_1^2 + \tau_2^2)b\pi[\cos^2 A + b^2\pi^2(1 + b^2\pi^2 \sin^2 A)] \\ &\quad + 4b\pi[-\sigma \cos^2 A + \pi^2 v'^2(2\cos^2 A - 1) \\ &\quad + \pi v'' \cos A \sin A - \frac{1}{2}b\pi^2\alpha(1 - 2\cos^2 A)], \\ r_5 &= 2\mu(\tau_1^2 + \tau_2^2)[3\cos^4 A - b^2\pi^2 \cos^2 A(2 - 3\cos^2 A) \\ &\quad + b^4\pi^4(-1 + 4\cos^2 A - 3\cos^4 A) - 3b^6\pi^6 \sin^4 A] \\ &\quad - 3\sigma \cos^4 A + b^2\pi^2\sigma \cos^2 A(2 - \cos^2 A) \\ &\quad + 2b^2\pi^4 v'^2(1 - 8\cos^2 A + 4\cos^4 A) + 4b\pi^2\alpha \cos^4 A \\ &\quad - 2b^2\pi^3 v'' \cos A \sin A(2\cos^2 A + 1), \\ r_6 &= 4\mu(\tau_1^2 + \tau_2^2)b\pi(-\cos^4 A + 2b^2\pi^2 \cos^2 A \sin^2 A \\ &\quad - b^4\pi^4 \sin^2 A) + 2b\pi\sigma \cos^4 A + 4b\pi^3 v'^2 \cos^2 A \\ &\quad \times (2\cos^2 A - 3) + 4b\pi^2 v'' \cos^3 A \sin A \\ &\quad + 2b^2\pi^3\alpha \cos^2 A(1 - 2\cos^4 A), \end{aligned}$$

$$\begin{aligned}
 r_7 = & 2\mu(\tau_1^2 + \tau_2^2)(-\cos^6 A + 3b^2\pi^2 \cos^4 A \sin^2 A \\
 & - 3b^4\pi^4 \cos^2 A \sin^4 A + b^6\pi^6 \sin^6 A + \sigma \cos^6 A \\
 & - b^2\pi^2\sigma \cos^4 A \sin^2 A + 2b^2\pi^4 v'^2 \cos^2 A(3 - 2\cos^2 A) \\
 & - 2b\pi^2\alpha \cos^4 A + 2b^2\pi^3 v'' \cos^3 A \sin A, \\
 A = & \pi v(x).
 \end{aligned} \tag{22}$$

To solve Eq. (21), we equate the coefficients of various powers of $\cos(i\pi\alpha t) \sin(i\pi\alpha t)$ to zero. This action yields a set of equations $r_i = 0$, $i = 1, 2, \dots, 7$, involving the unknowns σ , α , and $v(x)$. We notice that the coefficient of sixth power of $\cos(i\pi\alpha t)$ gives ($r_1 = 0$)

$$2\mu(\tau_1^2 + \tau_2^2)(1 + b^2\pi^2 - b^4\pi^4 - b^6\pi^6) - \sigma + b^2\pi^2\sigma = 0, \tag{23}$$

from which we can obtain the value of σ , that is

$$\sigma = 2\mu(\tau_1^2 + \tau_2^2)(1 + \pi^2 b^2)^2. \tag{24}$$

The coefficient of $\cos^5(i\pi\alpha t) \sin(i\pi\alpha t)$ also provides the same expression for σ as given in Eq. (24). Equating next the coefficients of $\cosh^3(i\pi\alpha t)$ and $\cos(i\pi\alpha t) \sin(i\pi\alpha t)$ to zero, we get $r_4 = 0$ and $r_6 = 0$. Here we get two equations, which contain the first and second derivatives of the unknown function $v(x)$, namely v' and v'' . Solving these two equations algebraically, we find

$$v'^2 = b^2\mu(\tau_1^2 + \tau_2^2)\{1 - b^2\pi^2 \cot^2[2\pi v(x)]\}, \tag{25}$$

$$\begin{aligned}
 v'' = & 2\mu(\tau_1^2 + \tau_2^2)\pi^3 b^4 \cot[2\pi v(x)] \left\{ \frac{1 + 2\pi^2 b^2}{\pi^2 b^2} \right. \\
 & \left. - \frac{\alpha}{2\mu(\tau_1^2 + \tau_2^2)\pi^2 b^3} + \cot^2[2\pi v(x)] \right\}.
 \end{aligned} \tag{26}$$

From Eqs. (25) and (26) we determine α and $v(x)$ as follows. Differentiating Eq. (25) with respect to x and then replacing the first and second derivatives of $v(x)$, which appear in this equation by Eqs. (25) and (26), respectively, and simplifying the resultant equation we find

$$\alpha = 2\mu(\tau_1^2 + \tau_2^2)(1 + \pi^2 b^2)b. \tag{27}$$

To obtain $v(x)$ we integrate Eq. (25) with respect to x . This action leads us to

$$v(x) = \frac{1}{2\pi} \arccos \left[\frac{1}{\sqrt{1 + \pi^2 b^2}} \cos(\sqrt{2\pi^2 \alpha b x} + v_0) \right], \tag{28}$$

where v_0 is a constant of integration. It is straightforward to check that $v(x)$ satisfies both Eqs. (25) and (26) with α given by Eq. (27). On the other hand, considering the coefficients of $\cos^2(i\pi\alpha t)$ and $\cos^4(i\pi\alpha t)$ and repeating the procedure outlined above we arrive at the same expressions for α and $v(x)$ which are given in Eqs. (27) and (28), respectively.

Finally, equating the coefficient of constant term to zero, $r_7 = 0$, we find that the resultant equation vanishes identically, with the expressions $v(x)$, α , and σ given above. As a result,

we have obtained a compatible set of solutions for α , σ , and $v(x)$, which satisfies all the equations given in Eq. (22). Now substituting the expressions of σ , α , and $v(x)$ into the general form Eq. (20), and after suitable rewriting, we obtain the AB solution in the form

$$\begin{aligned}
 q_1 = & \tau_1(1 + \pi^2 b^2) \exp \{i[2\mu(\tau_1^2 + \tau_2^2)(1 + \pi^2 b^2)^2 t + \phi]\} \\
 & \times \left(1 - \frac{2\pi b}{1 + \pi^2 b^2} M\right), \\
 q_2 = & \tau_2(1 + \pi^2 b^2) \exp \{i[2\mu(\tau_1^2 + \tau_2^2)(1 + \pi^2 b^2)^2 t + \phi]\} \\
 & \times \left(1 - \frac{2\pi b}{1 + \pi^2 b^2} M\right), \\
 M = & \frac{\pi b \cosh 2\pi\alpha t + i \sinh 2\pi\alpha t}{\cosh 2\pi\alpha t - (1/\sqrt{1 + \pi^2 b^2}) \cos(\sqrt{2\pi^2 \alpha b x} + v_0)}.
 \end{aligned} \tag{29}$$

We can note that this solution is periodic in the spatial direction and it grows exponentially fast in the initial stage from the time oscillatory background. After reaching the maximum amplitude at a specific time, it decays exponentially again to the time oscillatory background. These two stages can be called as growing and decaying mode solutions, respectively, as has been done by Tajiri and Watanabe for the case of the scalar NLS equation [41]. A typical AB solution for a suitable set of parametric values is shown in Fig. 2.

We also note here that the Akhmediev breather solution with $k \neq 0$ can also be constructed by the following imbricate series, namely

$$\begin{aligned}
 q_1 = & \tau_1 \exp \{i[2\mu(\tau_1^2 + \tau_2^2)(1 + \pi^2 b^2)^2 t + \phi]\} \\
 & \times \left[1 + b \sum_{n=-\infty}^{\infty} \frac{1}{i\alpha t + v(z) + n}\right] \\
 & \times \left[1 + b \sum_{n=-\infty}^{\infty} \frac{1}{i\alpha t - v(z) + n}\right], \\
 q_2 = & \tau_2 \exp \{i[2\mu(\tau_1^2 + \tau_2^2)(1 + \pi^2 b^2)^2 t + \phi]\} \\
 & \times \left[1 + b \sum_{n=-\infty}^{\infty} \frac{1}{i\alpha t + v(z) + n}\right] \\
 & \times \left[1 + b \sum_{n=-\infty}^{\infty} \frac{1}{i\alpha t - v(z) + n}\right],
 \end{aligned} \tag{30}$$

where $z = x - 2kt$. Substituting this expression into Eq. (1) and repeating the procedure outlined above one can obtain the Akhmediev breather with $k \neq 0$.

IV. MS FROM RW

Next we construct the MS solution from the RW solution. To do so, we again rewrite the RW solution given in Eq. (11) in

a slightly different factorized form,

$$\begin{aligned}
 q_1 &= \tau_1 \exp(i\{kx - [k^2 - 2\mu(\tau_1^2 + \tau_2^2)]t\}) \left[1 + \frac{i}{-2\mu(\tau_1^2 + \tau_2^2)t + i\frac{1}{2}\sqrt{1 + 4\mu(\tau_1^2 + \tau_2^2)(x - 2kt)^2}} \right] \\
 &\times \left[1 + \frac{i}{-2\mu(\tau_1^2 + \tau_2^2)t - i\frac{1}{2}\sqrt{1 + 4\mu(\tau_1^2 + \tau_2^2)(x - 2kt)^2}} \right], \\
 q_2 &= \tau_2 \exp(i\{kx - [k^2 - 2\mu(\tau_1^2 + \tau_2^2)]t\}) \left[1 + \frac{i}{-2\mu(\tau_1^2 + \tau_2^2)t + i\frac{1}{2}\sqrt{1 + 4\mu(\tau_1^2 + \tau_2^2)(x - 2kt)^2}} \right] \\
 &\times \left[1 + \frac{i}{-2\mu(\tau_1^2 + \tau_2^2)t - i\frac{1}{2}\sqrt{1 + 4\mu(\tau_1^2 + \tau_2^2)(x - 2kt)^2}} \right]. \tag{31}
 \end{aligned}$$

We can write this equation in the following general form with $k = 0$,

$$\begin{aligned}
 q_1 &= \tau_1 \exp[i(\zeta t + \phi)] \left[1 + ih \sum_{n=-\infty}^{\infty} \frac{1}{\kappa t + i\rho(x) + n} \right] \\
 &\times \left[1 + ih \sum_{n=-\infty}^{\infty} \frac{1}{\kappa t - i\rho(x) + n} \right], \\
 q_2 &= \tau_2 \exp[i(\zeta t + \phi)] \left[1 + ih \sum_{n=-\infty}^{\infty} \frac{1}{\kappa t + i\rho(x) + n} \right] \\
 &\times \left[1 + ih \sum_{n=-\infty}^{\infty} \frac{1}{\kappa t - i\rho(x) + n} \right], \tag{32}
 \end{aligned}$$

where the function $\rho(x)$ and the parameters κ and ζ are to be determined. Here we have superposed the RW in the temporal direction. We have also grouped the temporal variable with the real part and the spatial variable with the imaginary part. We identify the infinite series with the cot hyperbolic function [43], $\coth \pi x = \frac{1}{\pi x} - \frac{ix}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{n(x-in)}$, $n \neq 0$, and rewrite the above expression as

$$\begin{aligned}
 q_1 &= \tau_1 \exp[i(\zeta t + \phi)](1 + h\pi \coth\{\pi[\rho(x) - i\kappa t]\}) \\
 &\times (1 - h\pi \coth\{\pi[\rho(x) + i\kappa t]\}), \\
 q_2 &= \tau_2 \exp[i(\zeta t + \phi)](1 + h\pi \coth\{\pi[\rho(x) - i\kappa t]\}) \\
 &\times (1 - h\pi \coth\{\pi[\rho(x) + i\kappa t]\}). \tag{33}
 \end{aligned}$$

As we did previously, we split the cot hyperbolic function as $\cosh[\pi\rho(x) \pm i\pi\kappa t] / \sinh[\pi\rho(x) \pm i\pi\kappa t]$. We then substitute Eq. (33) into the CNLS Eq. (1) and rewrite the latter in terms of $\cosh[\pi\rho(x) \pm i\pi\kappa t] / \sinh[\pi\rho(x) \pm i\pi\kappa t]$. As before, we simplify this equation further by imposing trigonometric identities and arrive at an equation which is in powers of $\cosh(i\pi\kappa t)$, $\sinh(i\pi\kappa t)$ and their products. In this case, we also find that both the equations in Eq. (1) provide the same expression. As a result, we consider only the following

equation to determine the unknown parameters, that is

$$\begin{aligned}
 &z_1 \cosh^6(i\pi\kappa t) + z_2 \cosh^5(i\pi\kappa t) \sinh(i\pi\kappa t) \\
 &+ z_3 \cosh^4(i\pi\kappa t) \\
 &+ z_4 \cosh^3(i\pi\kappa t) \sinh(i\pi\kappa t) + z_5 \cosh^2(i\pi\kappa t) \\
 &+ z_6 \cosh(i\pi\kappa t) \sinh(i\pi\kappa t) + z_7 = 0, \tag{34}
 \end{aligned}$$

where

$$\begin{aligned}
 z_1 &= 2\mu(\tau_1^2 + \tau_2^2)(-1 + h^2\pi^2 - h^4\pi^4 - h^6\pi^6) + \zeta + h^2\pi^2\zeta, \\
 z_2 &= 4h\pi\mu(\tau_1^2 + \tau_2^2)(1 - h^2\pi^2)^2 - 2h\pi\zeta, \\
 z_3 &= 2\mu(\tau_1^2 + \tau_2^2)[3 \cosh^2 A - h^2\pi^2(1 + \cosh^2 A) \\
 &+ h^4\pi^4(\cosh^2 A - 2) - 3h^6\pi^6 \sinh^2 A] - 3\zeta \cosh^2 A \\
 &- h^2\pi^2\zeta(1 + \cosh^2 A) + 4h^2\pi^4 \rho'^2(1 - 2 \cosh^2 A) \\
 &+ 2h\pi^2\kappa(2 \cosh^2 A - 1) - 4h^2\pi^3 \rho'' \cosh A \sinh A, \\
 z_4 &= 8\mu(\tau_1^2 + \tau_2^2)h\pi(-\cosh^2 A + h^2\pi^2 + h^4\pi^4 \sinh^2 A) \\
 &+ 4h\pi\zeta \cosh^2 A + 2h\pi^3 \rho'^2(-1 + 2 \cosh^2 A) \\
 &+ 4h\pi^2 \rho'' \cosh A \sinh A + 2h^2\pi^3 \kappa(1 - 2 \cosh^2 A), \\
 z_5 &= 2\mu(\tau_1^2 + \tau_2^2)[-3 \cosh^4 A - h^2\pi^2 \cosh^2 A(2 - 3 \cosh^2 A) \\
 &+ h^4\pi^4(1 - 4 \cosh^2 A + 3 \cosh^4 A - 3h^6\pi^6 \cosh^2 A \\
 &\times \sinh^2 A)] + 3\zeta \cosh^4 A - 4h\pi^2\kappa \cosh^4 A \\
 &+ 2h^2\pi^2\zeta \cosh^2 A(1 - 2 \cosh^2 A) - 2h^2\pi^4 \rho'^2 \\
 &\times (1 - 8 \cosh^2 A + 4 \cosh^4 A) - 2h^2\pi^3 \rho'' \cosh A \\
 &\times \sinh A(2 \cosh^2 A + 1), \\
 z_6 &= 4\mu(\tau_1^2 + \tau_2^2)h\pi(\cosh^4 A - 2h^2\pi^2 \cosh^2 A \sinh^2 A \\
 &+ h^4\pi^4 \sinh^4 A) - 2h\pi\zeta \cosh^4 A + 4h\pi^3 \rho'^2 \cosh^2 A \\
 &\times (2 \cosh^2 A - 3) - 4h\pi^2 \rho'' \cosh^3 A \sinh A \\
 &- 2h^2\pi^3 \kappa \cosh^2 A(1 - 2 \cosh^2 A), \\
 z_7 &= 2\mu(\tau_1^2 + \tau_2^2)(\cosh^6 A - 3h^2\pi^2 \cosh^4 A \sinh^2 A \\
 &+ 3h^4\pi^4 \cosh^2 A \sinh^4 A - h^6\pi^6 \sinh^6 A) - \zeta \cosh^6 A \\
 &+ h^2\pi^2\zeta \cosh^4 A \sinh^2 A + 2h^2\pi^4 \rho'^2 \cosh^2 A \\
 &\times (2 \sinh^2 A - 1) + 2h\pi^2\kappa \cosh^4 A \\
 &- 2h^2\pi^3 \rho'' \cosh^3 A \sinh A, \\
 A &= \pi\rho(x). \tag{35}
 \end{aligned}$$

Equating the various powers of $\cosh(i\pi\kappa t) \sinh(i\pi\kappa t)$ to zero, we obtain $z_i = 0, i = 1, 2, \dots, 7$. We then solve these equations and determine $\zeta, \varrho(x)$, and κ as follows. The coefficient of $\cosh^6(i\pi\kappa t)$ gives

$$2\mu(\tau_1^2 + \tau_2^2)(-1 + h^2\pi^2 + h^4\pi^4 - h^6\pi^6) + \zeta + \zeta h^2\pi^2 = 0, \quad (36)$$

from which we fix

$$\zeta = 2\mu(\tau_1^2 + \tau_2^2)(1 - \pi^2 h^2)^2. \quad (37)$$

We also obtain the same expression for ζ by equating the coefficient of $\cosh^5(i\pi\kappa t) \sinh(i\pi\kappa t)$ to zero. We proceed to consider the coefficients of $\cosh^3(i\pi\kappa t) \sinh(i\pi\kappa t)$ and $\cosh(i\pi\kappa t) \sinh(i\pi\kappa t)$, namely, $z_4 = 0$ and $z_6 = 0$. We consider these two expressions to determine the unknown $\varrho(x)$. Solving these two equations algebraically, we find

$$\begin{aligned} \varrho'^2 &= h^2\mu(\tau_1^2 + \tau_2^2)\{1 - h^2\pi^2 \coth^2[2\pi\varrho(x)]\}, \\ \varrho'' &= 2\mu(\tau_1^2 + \tau_2^2)\pi^3 h^4 \coth[2\pi\varrho(x)] \left\{ \frac{1 - 2\pi^2 h^2}{\pi^2 h^2} \right. \\ &\quad \left. + \frac{\kappa}{2\mu(\tau_1^2 + \tau_2^2)\pi^2 h^3} + \coth^2[2\pi\varrho(x)] \right\}. \end{aligned} \quad (38)$$

We solve these two equations in the same manner as we did previously. Our result shows that

$$\kappa = -2\mu(\tau_1^2 + \tau_2^2)(1 - \pi^2 h^2)h \quad (39)$$

and

$$\varrho(x) = \frac{1}{2\pi} \cosh^{-1} \left[\frac{1}{\sqrt{1 - \pi^2 h^2}} \cosh(\sqrt{-2\pi^2 \kappa h x} + \varrho_0) \right], \quad (40)$$

where ϱ_0 is a constant. We can also obtain the same expression for $\varrho(x)$ and κ from the coefficients of $\cosh^2(i\pi\kappa t)$ and $\cosh^4(i\pi\kappa t)$ by solving the resultant equations algebraically in the same manner.

Inserting the obtained expressions of ζ, ϱ , and κ in the final determining equation, we find that it vanishes trivially. With these expressions, the general form of Eq. (32) now becomes

$$\begin{aligned} q_1 &= \tau_1(1 - \pi^2 h^2) \exp \{ i[2\mu(\tau_1^2 + \tau_2^2)(1 - \pi^2 h^2)^2 t + \phi] \} \\ &\quad \times \left(1 + \frac{2\pi h}{1 - \pi^2 h^2} M \right), \\ q_2 &= \tau_2(1 - \pi^2 h^2) \exp \{ i[2\mu(\tau_1^2 + \tau_2^2)(1 - \pi^2 h^2)^2 t + \phi] \} \\ &\quad \times \left(1 + \frac{2\pi h}{1 - \pi^2 h^2} M \right), \\ M &= \frac{\pi h \cos 2\pi\kappa t - i \sin 2\pi\kappa t}{\cos 2\pi\kappa t - (1/\sqrt{1 - \pi^2 h^2}) \cosh(\sqrt{-2\pi^2 \kappa h x} + c)}, \end{aligned} \quad (41)$$

which is nothing but the Ma breather solution. This solution is periodic in the temporal direction and localized in space. It grows and decays recurrently in time oscillate background as in the case of the NLS equation [41]. The Ma breather solution of CNLS equations for a set of parametric values is shown in Fig. 3.

The RW solution can also be obtained as the limiting case of Ma breathers. This can be done by imposing the limit $h \rightarrow 0$ and incorporating the Taylor series expansion.

We note here that the Ma breather solution with $k \neq 0$ can also be constructed by the following imbricate series,

$$\begin{aligned} q_1 &= \tau_1 \exp \{ i[2\mu(\tau_1^2 + \tau_2^2)(1 - \pi^2 h^2)^2 t + \phi] \} \\ &\quad \times \left[1 + ih \sum_{n=-\infty}^{\infty} \frac{1}{\kappa t + i\varrho(z) + n} \right] \\ &\quad \times \left[1 + ih \sum_{n=-\infty}^{\infty} \frac{1}{\kappa t - i\varrho(z) + n} \right], \\ q_2 &= \tau_2 \exp \{ i[2\mu(\tau_1^2 + \tau_2^2)(1 - \pi^2 h^2)^2 t + \phi] \} \\ &\quad \times \left[1 + ih \sum_{n=-\infty}^{\infty} \frac{1}{\kappa t + i\varrho(z) + n} \right] \\ &\quad \times \left[1 + ih \sum_{n=-\infty}^{\infty} \frac{1}{\kappa t - i\varrho(z) + n} \right], \end{aligned} \quad (42)$$

where $z = x - 2\kappa t$. Substituting this expression into Eq. (1), and repeating the procedure given above, one can obtain the Ma soliton with $k \neq 0$.

V. GB AS IMBRICATE SERIES OF RW

It is very difficult to derive the GB solution from the RW solution in the same fashion as we did in the previous two cases. This is mainly because in the present analysis we have to include two arbitrary functions (both of which are functions of t and x), one with real part and another with imaginary part in the imbricate series. The determining equations that come out from the imbricate series are difficult to solve, unlike the earlier two cases. To overcome this difficulty, we adopt the following methodology. We show that the absolute square of the modulus of RW solution of Eq. (1) can be written in terms of the second derivative of a logarithmic function, which involves product of two imbricate series [see Eq. (45) given below]. We then rewrite this expression in a more compact form, which involves trigonometric functions that also contain these two arbitrary functions. Unlike the earlier two cases, we do not substitute this series into Eq. (1) and determine these two unknown arbitrary functions (as it is very difficult to solve the determining equation). Instead, we also rewrite the absolute square of the modulus of the general breather solution as the second derivative of a logarithmic function that contains product of two functions [see Eq. (49) given below]. At this stage, since both the GB expression and the RW solution have been written in the same form. We compare the arguments inside the logarithmic function and fix the exact expression of the two unknown arbitrary functions. This in turn confirms that the GB can also be derived from the RWs. In the following, we present the exact mathematical details of this procedure.

To begin with, we rewrite the RW solution Eq. (14) in the form

$$|q_1|^2 = \tau_1^2(1 - Q)(1 - Q^*), \quad |q_2|^2 = \tau_2^2(1 - Q)(1 - Q^*), \quad (43)$$

where Q is the same expression given in Eq. (14) and the star denotes complex conjugate of it. The above expressions can be rewritten as the second derivative of a logarithmic function, namely

$$|q_1|^2 = \tau_1^2 - \frac{\tau_1^2}{\mu(\tau_1^2 + \tau_2^2)} \frac{\partial^2}{\partial x^2} \ln \left(\frac{1}{S} \times \frac{1}{T} \right),$$

$$|q_2|^2 = \tau_2^2 - \frac{\tau_2^2}{\mu(\tau_1^2 + \tau_2^2)} \frac{\partial^2}{\partial x^2} \ln \left(\frac{1}{S} \times \frac{1}{T} \right),$$
(44)

where

$$S = \left[\frac{1}{2} \sqrt{1 + 4\mu(\tau_1^2 + \tau_2^2)(x - 2kt)^2 + 2i\mu(\tau_1^2 + \tau_2^2)t} \right]^2,$$

$$T = \left[\frac{1}{2} \sqrt{1 + 4\mu(\tau_1^2 + \tau_2^2)(x - 2kt)^2 - 2i\mu(\tau_1^2 + \tau_2^2)t} \right]^2.$$

Now we consider Eq. (44) in a more general form as

$$|q_1|^2 = \tau_1^2 - \frac{\tau_1^2}{\mu(\tau_1^2 + \tau_2^2)} \times \frac{\partial^2}{\partial x^2} \ln \left\{ \sum_{n=-\infty}^{\infty} \frac{1}{[\phi(x,t) - i\psi(x,t) - n]^2} \times \sum_{n=-\infty}^{\infty} \frac{1}{[\phi(x,t) + i\psi(x,t) - n]^2} \right\},$$

$$|q_2|^2 = \tau_2^2 - \frac{\tau_2^2}{\mu(\tau_1^2 + \tau_2^2)} \times \frac{\partial^2}{\partial x^2} \ln \left\{ \sum_{n=-\infty}^{\infty} \frac{1}{[\phi(x,t) - i\psi(x,t) - n]^2} \times \sum_{n=-\infty}^{\infty} \frac{1}{[\phi(x,t) + i\psi(x,t) - n]^2} \right\},$$
(45)

where $\phi(x,t)$ and $\psi(x,t)$ are arbitrary functions of x and t , which need to be determined. In the above expression, we considered the superposition of RWs in both space and time directions. Using the trigonometric identity [43] $\csc^2(\pi x) = \frac{1}{\pi^2} \sum_{k=-\infty}^{\infty} \frac{1}{x-k^2}$, the above expression can be rewritten in the form,

$$|q_1|^2 = \tau_1^2 - \frac{\tau_1^2}{\mu(\tau_1^2 + \tau_2^2)} \frac{\partial^2}{\partial x^2} \ln \{ \pi^2 \csc^2[\pi(\phi - i\psi)] \times \pi^2 \csc^2[\pi(\phi + i\psi)] \},$$

$$|q_2|^2 = \tau_2^2 - \frac{\tau_2^2}{\mu(\tau_1^2 + \tau_2^2)} \frac{\partial^2}{\partial x^2} \ln \{ \pi^2 \csc^2[\pi(\phi - i\psi)] \times \pi^2 \csc^2[\pi(\phi + i\psi)] \}.$$
(46)

We further simplify the expression on the right-hand side by using the relation $\pi^4 \csc^2[\pi(\phi - i\psi)] \csc^2[\pi(\phi + i\psi)] = \frac{4\pi^4}{\cosh 2\pi\psi - \cos 2\pi\phi}$. As a result, Eq. (46) can be brought to the form

$$|q_1|^2 = \tau_1^2 + \frac{\tau_1^2}{\mu(\tau_1^2 + \tau_2^2)} \frac{\partial^2}{\partial x^2} \ln[\cosh 2\pi\psi - \cos 2\pi\phi],$$

$$|q_2|^2 = \tau_2^2 + \frac{\tau_2^2}{\mu(\tau_1^2 + \tau_2^2)} \frac{\partial^2}{\partial x^2} \ln[\cosh 2\pi\psi - \cos 2\pi\phi].$$
(47)

As we mentioned in the beginning of this section, we do not substitute this solution into Eq. (1) and determine the form of ψ and ϕ . Instead of this, we compare Eq. (47) with the GB, which is rewritten in the same form. For this purpose, we rewrite the GB solution Eq. (5) with Eq. (8) in the form

$$|q_1|^2 = \tau_1^2 + \frac{\tau_1^2}{\mu(\tau_1^2 + \tau_2^2)} \frac{\partial^2}{\partial x^2} \ln f,$$

$$|q_2|^2 = \tau_2^2 + \frac{\tau_2^2}{\mu(\tau_1^2 + \tau_2^2)} \frac{\partial^2}{\partial x^2} \ln f,$$
(48)

where $f = 1 + 2e^{\eta_R} \cos \eta_I + ae^{2\eta_R}$, $\eta_R = p_R x - \Omega_R t + \eta_R^0$, and $\eta_I = p_I x - \Omega_I t + \eta_I^0$. To compare this with the one derived from the RW solution, we rewrite f as $f = 2e^{\eta_R} [\sqrt{a} \cosh(\eta_R + \sigma) - \cos(\eta_I + \theta)]$ with the η_R and η_I as given above. The resultant expression now turns out to be

$$|q_1|^2 = \tau_1^2 + \frac{\tau_1^2}{\mu(\tau_1^2 + \tau_2^2)} \frac{\partial^2}{\partial x^2} \ln [\sqrt{a} \cosh(p_R x - \Omega_R t + \sigma) - \cos(p_I x - \Omega_I t + \theta)],$$

$$|q_2|^2 = \tau_2^2 + \frac{\tau_2^2}{\mu(\tau_1^2 + \tau_2^2)} \frac{\partial^2}{\partial x^2} \ln [\sqrt{a} \cosh(p_R x - \Omega_R t + \sigma) - \cos(p_I x - \Omega_I t + \theta)],$$
(49)

where $\sigma = \eta_R^0 + \frac{1}{2} \ln a$ and $\theta = \eta_I^0 + \pi$.

Now let us compare the two expressions $|q_1|^2$ and $|q_2|^2$, the one derived from RW solutions [vide Eq. (47)] and the other derived from the GB solution [vide Eq. (49)]. Doing so, we find

$$\cosh 2\pi\psi = \sqrt{a} \cosh(p_R x - \Omega_R t + \sigma),$$

$$\cos 2\pi\phi = \cos(p_I x - \Omega_I t + \theta),$$
(50)

or

$$\cosh 2\pi\psi = \cosh(p_R x - \Omega_R t + \sigma),$$

$$\cos 2\pi\phi = \frac{1}{\sqrt{a}} \cos(p_I x - \Omega_I t + \theta).$$
(51)

From these two sets of equations, we find two different expressions for ψ and ϕ , namely

$$\psi = \frac{1}{2\pi} \ln [\sqrt{a} \cosh(p_R x - \Omega_R t + \sigma) + \sqrt{a \cosh^2(p_R x - \Omega_R t + \sigma) - 1}],$$

$$\phi = \frac{1}{2\pi} (p_I x - \Omega_I t + \theta)$$
(52)

and

$$\psi = -\frac{1}{2\pi} (p_R x - \Omega_R t + \sigma),$$

$$\phi = \frac{1}{2\pi} \arccos \left[\frac{1}{\sqrt{a}} \cos(p_I x - \Omega_I t + \theta) \right].$$
(53)

An exact imbricate series of RW solution for breather solution of CNLS equations can be displayed by substituting Eq. (52) or Eq. (53) into Eq. (45). The solution is periodic in both space and time.

VI. CONCLUSION

During the past five years or so, rogue wave solutions have been studied intensively in different physical contexts and several applications have been proposed. Rogue waves in an array of optical wave guides is one of the examples [44]. Optical rogue wave has already been suggested for application in enhancing supercontinuum generation, and several authors have elucidated key aspects of the underlying nonlinear dynamical processes [45]. In this work, we have constructed a class of nonlinear waves, namely GB, AB, MS, and RW for the well-known two-coupled NLSEs Eq. (1). To derive these solutions, we followed two different paths. By following the conventional procedure, we first brought out the explicit form of a GB solution from which we derived the other forms of rational solutions, namely AB, MS, and RW solutions. We then deviated from this conventional approach and derived AB, MS, and GB from the RW solution as the starting point. The expressions obtained in both the directions match with

each other. Our study on the coupled NLSEs will be useful in the study of rogue waves in birefringent optical fibers, multicomponent Bose-Einstein condensates, multicomponent plasmas, and so on. We also hope to derive higher-order breather solutions in both the directions discussed in this paper by extending the procedure.

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