Random matrix ensemble with random two-body interactions in the presence of a mean field for spin-one boson systems

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For bosons carrying spin-one degree of freedom, we introduce an embedded Gaussian orthogonal ensemble of random matrices generated by random two-body interactions in the presence of a mean field that is spin (S) scalar [called BEGOE($1+2$)-$S1$]. Embedding algebra for the ensemble, for m bosons in Ω number of single-particle levels (each triply degenerate), is U(3Ω) $\supset$ G $\supset$ G1 $\otimes$ SO(3) with SO(3) generating the spin S. A method for constructing the ensemble for a given (Ω, m, S) has been developed. Numerical calculations show that (i) the form of the fixed-(m, S) density of states is close to a Gaussian; (ii) for a strong enough interaction, level fluctuations follow GOE; (iii) fluctuation in energy centroids is large; and (iv) spectral widths are nearly constant with respect to S for $S < S_{\max}/2$. Moreover, we identify two different pairing symmetry algebras in the space defined by BEGOE($1+2$)-$S1$ and numerical results show that random interactions generate ground states with maximal value for the pair expectation value.

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I. INTRODUCTION

In the last several years, bosonic embedded Gaussian orthogonal ensembles of one- plus two-body interactions [denoted by BEGOE($1+2$)] for finite isolated interacting spinless many-boson systems has been analyzed in detail [1–6], as they are generic models for finite isolated interacting many-boson systems. For m bosons in N single-particle (sp) states, in addition to dilute limit (defined by $m \to \infty, N \to \infty, m/N \to 0$), another limiting situation, namely the dense limit (defined by $m \to \infty, N \to \infty, m/N \to \infty$), is also feasible. Such a limiting situation is absent for fermion systems. Hence, the focus was on the dense limit in BEGOE investigations [1–6]. In the strong interaction limit, the two-body part of the interaction dominates over the one-body part and, therefore, BEGOE($1+2$) reduces to BEGOE(2). In the dense limit, the eigenvalue density for these ensembles takes Gaussian form irrespective of the strength of the two-body interaction strength (λ) [1,3,7]. Similarly, as the strength of two-body interaction strength (measured in units of the average spacing between the one-body mean-field sp levels) increases, these ensembles exhibit Poisson to Gaussian transition in level fluctuations (at $\lambda = \lambda_c$) [4] and Breit-Wigner to Gaussian transition in strength functions (at $\lambda = \lambda_F > \lambda_c$) [5]. More recently, it has been shown that they also generate a region of thermalization (at $\lambda = \lambda_t > \lambda_F$) [6].

Going beyond spinless boson systems, very recently BEGOE for two species boson systems with F-spin degree of freedom generated by Hamiltonians that conserve the total F spin of the m-boson systems [called BEGOE($1+2$)-F] is introduced and its spectral properties are analyzed in detail [8]; F spin for bosons is similar to the F spin in the proton-neutron interacting boson model (pnIBM) of atomic nuclei [9,10]. Using numerical calculations, it is shown that for BEGOE($1+2$)-F, the fixed-(m, F) density of eigenvalues is close to a Gaussian in the dense limit and, generically, there is Poisson to GOE transition in level fluctuations as the interaction strength is increased. The interaction strength needed for the onset of the transition is found to decrease with increasing F. Propagation formulas for energy centroids and spectral variances are derived and, using these, covariances in eigenvalue centroids and spectral variances are analyzed. It is also demonstrated that the BEGOE(2)-F ensembles generate ground states with spin $F = F_{\max} = m/2$ and natural F-spin ordering ($F_{\max}, F_{\max}-1, F_{\max}-2, \ldots, 0$, or $\frac{1}{2}$). Going beyond these, pairing symmetry in the space defined by BEGOE($1+2$)-F is introduced and it is shown that, using the expectation values of the pairing Hamiltonian, random interactions generate ground states with maximum value for the pairing expectation value for a given F and, in these, it is largest for $F = F_{\max} = m/2$.

In order to apply BEGOE to a spinor Bose-Einstein condensate (BEC) discussed in [11,12] and to analyze generic structures generated by the 3rd version of the interacting boson model (called IBM-3) of atomic nuclei (here spin S is equivalent to isospin T of the bosons in IBM-3) [13,14], an important extension that needs to be investigated is BEGOE for a system of interacting bosons carrying spin-one degree of freedom. For a system with m number of bosons in Ω number of sp levels, each triply degenerate, we define EGOE of random matrices generated by random two-body interactions that conserve the total spin S. This random matrix ensemble is hereafter denoted by BEGOE(2)-$S1$. In the presence of a mean-field interactions, we have BEGOE($1+2$)-$S1$. It is useful to add that for condensates in an optical trap, the single-mode approximation is often adopted for the studies of spin dynamics and many-body physics, where all spin components are assumed to share the same spatial dependence and only the spin components vary. Thus, in the mean-field approximation, one need not consider orbital angular momentum degree of freedom for spinor condensates [15–17]. However, it

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becomes important to incorporate the effects of orbital angular momentum and its coupling with spin degree of freedom via spin-orbit term for condensates in magnetic or magneto-optical traps [17,18]. In this situation, BEGOE(1 + 2)-$S1$ has to be extended to include additional degrees of freedom like orbital angular momentum; this will need to be addressed in the future. Besides possible applications to spinor BEC and atomic nuclei, the BEGOE(1 + 2)-$S1$ ensemble is also of general interest because embedded ensembles for isolated finite many-body quantum systems are being used as generic models for many-body chaos [19–21]. The purpose of the present paper is to introduce the BEGOE(1 + 2)-$S1$ ensemble and report the results of a first analysis of this ensemble. Now we give a preview.

In Sec. II we introduce the embedded ensemble BEGOE(1 + 2)-$S1$ [also BEGOE(2)-$S1$] for a system of m bosons in Ω number of sp levels that are triply degenerate with total spin S being a good symmetry and give a method for the numerical construction of this ensemble in fixed-(m,S) spaces. The embedding algebra for the ensemble is U(Ω) $\otimes$ [SU(3) $\supset$ SO(3)] and this is described in Sec. III. In Sec. IV we present numerical results for the ensemble averaged eigenvalue density, nearest neighbor spacing distribution (NNSD), width of the fluctuations in energy centroids, and spectral variances. In Sec. V, two types of pairing in BEGOE(1 + 2)-$S1$ space are introduced and some numerical results for ground state structure, vis-à-vis pairing structure, are presented. Finally, Sec. VI gives conclusions and future outlook.

II. DEFINITION AND CONSTRUCTION

A. Definition of the BEGOE(1 + 2) − $S1$ ensemble

Let us consider a system of m ($m > 2$) bosons with spin-one degree of freedom occupying Ω number of sp levels. For convenience, in the remaining part of this section, we use the notation "**s**" for the spin quantum number of a single boson, "s" for the spin carried by a two-boson system, and "S" for $m > 2$ boson system spin. Therefore, $\mathbf{s} = 1$; $s = 0$, 1, 2; and $S = m$, $m - 1$, …, 0. Similarly, the $\hat{S}_z$ ("hat" denoting operator) eigenvalues are denoted by $m_{\mathbf{s}}$, m_s, and M_S, respectively. Also, the space generated by the sp levels $i = 1,2,\ldots,\Omega$ is referred to as orbital space. Then the sp states of a boson are denoted by $|i;\mathbf{s} = 1,m_{\mathbf{s}}\rangle$, with $i = 1,2,\ldots,\Omega$ and $m_{\mathbf{s}} = \pm 1$ and 0. With Ω number of orbital degrees of freedom and three spin ($m_{\mathbf{s}}$) degrees of freedom, the total number of sp states is $N = 3\Omega$. Going further, two-boson (normalized) states that are symmetric in the total orbital $\times$ spin space are denoted by $|(ij);s,m_s\rangle$, with $s = 1 \times 1 = 0$, 1, and 2. For $i \neq j$, all three s values are allowed, while for $i = j$ only $s = 0$ and 2 are allowed. Also, $-s \leqslant m_s \leqslant s$. In terms of boson creation ($b^{\dagger}_{\_\_\_}$) and annihilation ($b_{\_\_\_}$) operators, the sp states are $|i;1,m_{\mathbf{s}}\rangle = b^{\dagger}_{i;s,m_{\mathbf{s}}}|0\rangle$. Similarly, the two-boson states are $|(ij);s,m_s\rangle = \frac{1}{\sqrt{(1+\delta_{ij})}}(b^{\dagger}_{i;1}b^{\dagger}_{j;1})^{s}_{m_s}|0\rangle$. Note that here we are using spin (angular momentum) coupled representation.

In this paper, we consider one- plus two-body Hamiltonians (H) preserving spin S for m-boson systems. Then the Hamiltonian operator $\widehat{H} = \widehat{h}(1) + \widehat{V}(2)$ with the mean-field

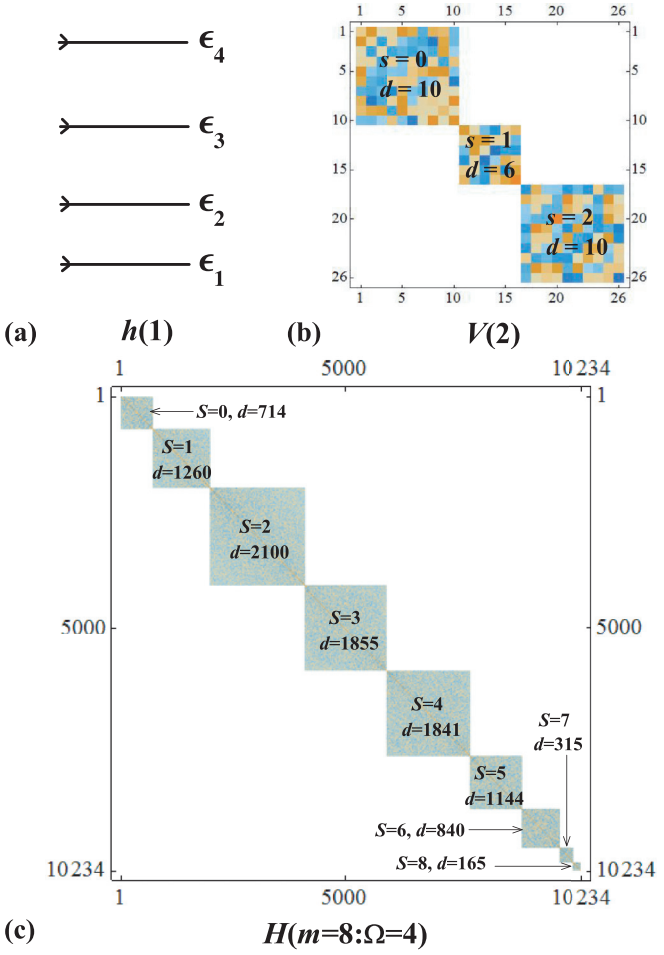

FIG. 1. (Color online) (a) Single-particle levels generated by $\widehat{h}(1)$ for $\Omega = 4$. Here, each level is triply degenerate; i.e., $N = 3\Omega$. (b) Matrix of $\widehat{V}(2)$ in two-boson space for $\Omega = 4$. (c) Decomposition of H matrix in m-particle space into a direct sum of matrices, each with a fixed S value. There is a BEGOE(1 + 2)-$S1$ ensemble in each (m,S) space corresponding to the diagonal blocks in (c). Note that the matrix elements in the off-diagonal blocks in (b) and (c) are all zero. In the matrix plots (b) and (c), generated using MATHEMATICA, we have used one particular member of $V(2)$ and $H(m)$ ensembles that are employed in the numerical calculations reported in Sec. IV A. As the figures are only for illustration, the scale defining the numerical value of the matrix elements is not shown.

one-body Hamiltonian $\widehat{h}(1)$ is defined by

$$\widehat{h}(1) = \sum_{i=1}^{\Omega} \epsilon_i \hat{n}_i. \tag{1}$$

The sp energies ϵ_i in (1) are independent of the $m_{\mathbf{s}}$ quantum number and therefore each sp level i is triply degenerate. The $\hat{n}_i$ are number operators and $\hat{n}_i = \sum_{m_{\mathbf{s}}} b^{\dagger}_{i;1,m_{\mathbf{s}}} b_{i;1,m_{\mathbf{s}}}$; the action of $\hat{n}_i$ on the level i gives the number of bosons occupying ith level. Figure 1(a) shows an example of the sp spectrum for $\Omega = 4$. Let us add that, in principle, it is possible to consider $\widehat{h}(1)$ with off-diagonal energies ϵ_{ij}. A two-body Hamiltonian operator $\widehat{V}(2)$ preserving spin S is defined

by

$$\widehat{V}(2) = {\sum_{i,j,k,\ell;s,m_s}}' \frac{V^s_{ijk\ell}}{\sqrt{(1+\delta_{ij})(1+\delta_{k\ell})}} \times (b^\dagger_{i:1} b^\dagger_{j:1})^s_{m_s} \left[(b^\dagger_{k:1} b^\dagger_{\ell:1})^s_{m_s}\right]^\dagger. \quad (2)$$

The "prime" over the summation symbol in (2) indicates that the summation over i, j, k, and ℓ is restricted to $i \geqslant j$ and $k \geqslant \ell$ for $s = 0,2$ and $i > j$ and $k > \ell$ for $s = 1$. The symmetrized (with respect to the total orbital $\times$ spin space) two-body matrix elements $V^s_{ijk\ell} = \langle (ij)s,m_s|\widehat{V}(2)|(k\ell)s,m_s\rangle$ are independent of the m_s quantum number and this ensures that $\widehat{V}(2)$ preserves spin S. It is clearly seen from Eq. (2) that $\widehat{V}(2) = \widehat{V}^{s=0}(2) + \widehat{V}^{s=1}(2) + \widehat{V}^{s=2}(2)$. Then the matrix of $\widehat{V}(2)$ in two-boson spaces will be a direct sum of $s = 0$, 1, and 2 matrices; i.e., it is a 3×3 block matrix and the three diagonal blocks correspond to $s = 0$, 1, and 2, respectively, and the off-diagonal blocks are zero. Dimensions of the $s = 0$, 1, and 2 matrices are $\Omega(\Omega+1)/2$, $\Omega(\Omega-1)/2$, and $\Omega(\Omega+1)/2$ (here m_s is not counted as the states with same s but different m_s will be degenerate). Figure 1(b) shows an example of the $\widehat{V}(2)$ matrix in two-boson spaces. Matrix representation for $\widehat{H} = \widehat{h}(1) + \widehat{V}(2)$ in m-boson spaces with good S can be obtained, for example, using the basis

$$|(\nu_1)^{n_1,S_1},(\nu_2)^{n_2,S_2},\ldots,(\nu_r)^{n_r,S_r};\alpha,S,M_S\rangle. \quad (3)$$

Here ν_i are the occupied sp levels, n_i are number of bosons in the ν_i level, $m = \sum_i n_i$, $S_i = n_i, n_i - 2, \ldots, 0$, or 1, and $S = S_1 \times S_2 \times \ldots S_r$. Also, in (3) "$\alpha$" are additional labels needed for complete specification of the basis states. As $\widehat{H}$ preserves spin S, matrix elements of $\widehat{H}$ in the basis given by Eq. (3) will be independent of M_S quantum number. A method to calculate H matrix dimension $d(\Omega,m,S)$ is discussed in Sec. III. Note that $d(\Omega,m,S)$ gives the number of basis states for a given (Ω,m,S) without counting M_S values, i.e., H matrix dimension for a given (Ω, m,S). Now we are in a position to define the BEGOE$(1+2)$-$S1$ ensemble.

The embedded ensemble BEGOE$(1+2)$-$S1$, in (m,S) spaces, is defined, with { } denoting an ensemble, by the operator

$$\{\widehat{H}(1+2)\}_{\mathrm{BEGOE}(1+2)-S1} = \widehat{h}(1) + \lambda_0\{\widehat{V}^{s=0}(2)\} + \lambda_1\{\widehat{V}^{s=1}(2)\} + \lambda_2\{\widehat{V}^{s=2}(2)\}, \quad (4)$$

with the condition that the $\widehat{V}^{s=0}(2)$, $\widehat{V}^{s=1}(2)$, and $\widehat{V}^{s=2}(2)$ matrices in two-boson spaces [i.e., the three diagonal block matrices in Fig. 1(b)] are independent GOEs with unit variance. This means that all the two-body matrix elements $V^s_{ijk\ell}$ defining the three parts of $\widehat{V}(2)$ are zero centered independent Gaussian variables with variance unity for off-diagonal matrix elements and variance two for the diagonal matrix elements. In Eq. (4), λ_0, λ_1, and λ_2 are the strengths of the $s = 0$, 1, and 2 parts of $\widehat{V}(2)$, respectively. The mean-field Hamiltonian $\widehat{h}(1)$ in Eq. (4) is defined by the sp energies ϵ_i [see Eq. (1)] with average spacing Δ. Without loss of generality, we put $\Delta = 1$ so that λ_0, λ_1, and λ_2 are in the units of Δ. The action of each member of $\{\widehat{H}\}_{\mathrm{BEGOE}(1+2)-S1}$ on (m,S) basis states given by Eq. (3) will give a member of BEGOE$(1+2)$-$S1$ ensemble in a given (m,S) space and the ensemble of all these members then defines BEGOE$(1+2)$-$S1$ ensemble in a given (m,S) space. As shown in Fig. 1(c), the H matrix in m-boson spaces will be a direct sum matrix with the diagonal blocks representing the matrix with fixed S value. Note that the dimension of the diagonal blocks is $d(\Omega,m,S)$. Starting with H defined by Eq. (4) and using the basis given by Eq. (3), there will be an ensemble corresponding to each diagonal block in Fig. 1(c) and this is the BEGOE$(1+2)$-$S1$ ensemble. The simpler BEGOE(2)-$S1$ is defined by Eq. (4) with $\widehat{h}(1) = 0$.

B. A method for constructing the BEGOE$(1+2)$-$S1$ ensemble in many-boson spaces

For numerical analysis, a simple method for constructing the BEGOE$(1+2)$-$S1$ ensemble in a given (m,S) space is to first construct the matrices in good M_S basis [see Eq. (6) ahead] and then employ for spin S projection the $\hat{S}^2$ operator [its eigenvalues are $S(S+1)$], as was done before for spin-$\frac{1}{2}$ fermion [22] and F-spin boson systems [8]. For generating a many-particle basis, first the 3Ω sp states are arranged such that the first Ω number of sp states have $m_{\mathbf{s}} = 1$, the next Ω number of sp states have $m_{\mathbf{s}} = 0$, and the remaining Ω sp states have $m_{\mathbf{s}} = -1$. Then, the sp states are $|r\rangle = |i = r, m_{\mathbf{s}} = 1\rangle$ for $r \leqslant \Omega$, $|r\rangle = |i = r - \Omega, m_{\mathbf{s}} = 0\rangle$ for $\Omega < r \leqslant 2\Omega$, and $|r\rangle = |i = r - 2\Omega, m_{\mathbf{s}} = -1\rangle$ for $2\Omega < r \leqslant 3\Omega$. With this, $r = 1,2\ldots, 3\Omega$. Now, the many-particle states for m bosons can be obtained by distributing m_1 bosons in the $m_{\mathbf{s}} = 1$ sp states, m_2 bosons in the $m_{\mathbf{s}} = 0$ sp states, and, similarly, m_3 bosons in the $m_{\mathbf{s}} = -1$ sp states with $m = m_1 + m_2 + m_3$. Let us denote each distribution of m_1 bosons in $m_{\mathbf{s}} = 1$ sp states by $\mathbf{m}_1 = (m^1_1, m^2_1, \ldots, m^\Omega_1)$, m_2 bosons in $m_{\mathbf{s}} = 0$ sp states by $\mathbf{m}_2 = (m^1_2, m^2_2, \ldots, m^\Omega_2)$, and, similarly, $\mathbf{m}_3 = (m^1_3, m^2_3, \ldots, m^\Omega_3)$ for m_3 bosons in $m_{\mathbf{s}} = -1$ sp states. Note that $m_k = \sum_{i=1}^{\Omega} m^i_k$, $k = 1,2,3$, and m^i_k is the number of bosons in the sp state $|r = i + (k-1)\Omega\rangle$. Each $[\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3]$ defines a m-particle configuration or basis state in occupation number representation with $M_S = (m_1 - m_3)$. To proceed further, the Hamiltonian operator defined by Eq. (4) is converted into M_S representation. Then $\widehat{h}(1) = \sum_{r=1}^{3\Omega} \epsilon'_r \hat{n}'_r$ and the sp energies $\epsilon'_q = \epsilon'_{q+\Omega} = \epsilon'_{q+2\Omega} = \epsilon_r$ for $1 \leqslant q \leqslant \Omega$. Similarly, for changing $\widehat{V}(2)$, first note that two-boson states in M_S representation can be written as $|i,m_{\mathbf{s}};j,m'_{\mathbf{s}}\rangle$; $m_s = m_{\mathbf{s}} + m'_{\mathbf{s}}$. Then the two particle matrix elements are $V'_{i,m_{\mathbf{s}}^{f1};j,m_{\mathbf{s}}^{f2};k,m_{\mathbf{s}}^{i1};\ell,m_{\mathbf{s}}^{i2}}(2) = \langle i,m_{\mathbf{s}}^{f1};j,m_{\mathbf{s}}^{f2}|\widehat{V}(2)|k,m_{\mathbf{s}}^{i1};\ell,m_{\mathbf{s}}^{i2}\rangle$. It is easy to derive formulas for these in terms of $V^s_{ijkl}(2)$ by applying angular momentum algebra to expand $|i,m_{\mathbf{s}};j,m'_{\mathbf{s}}\rangle$ states in terms of the coupled two-particle states $|(ij)s,m_s = m_{\mathbf{s}} + m'_{\mathbf{s}}\rangle$ and normalizing them appropriately. The final formulas are

$$V'_{i,1;j,1;k,1;\ell,1}(2) = V^{s=2}_{ijkl}(2),$$

$$V'_{i,1;j,0;k,1;\ell,0}(2) = \frac{\sqrt{(1+\delta_{ij})(1+\delta_{k\ell})}}{2} \times \left[V^{s=1}_{ijkl}(2) + V^{s=2}_{ijkl}(2)\right],$$

$$V'_{i,1;j,-1;k,1;\ell,-1}(2) = \frac{\sqrt{(1+\delta_{ij})(1+\delta_{k\ell})}}{6} \times \left[2\,V^{s=0}_{ijkl}(2) + 3\,V^{s=1}_{ijkl}(2) + V^{s=2}_{ijkl}(2)\right],$$

$$V'_{i,0;j,0;k,0;\ell,0}(2) = \left[\frac{1}{3}\,V^{s=0}_{ijkl}(2) + \frac{2}{3}\,V^{s=2}_{ijkl}(2)\right],$$
$$V'_{i,1;j,-1;k,0;\ell,0}(2) = \frac{\sqrt{(1+\delta_{ij})}}{3}\left[V^{s=2}_{ijkl}(2) - V^{s=0}_{ijkl}(2)\right]. \quad (5)$$

All other V' matrix elements follow by symmetries. Now converting $|i,m^i_{\rm s}\rangle$ into $|r\rangle$ states, the V' matrix elements can be written as $\langle r,s\,|\,\widehat{V}(2)\,|t,u\rangle$ matrix elements and then $\widehat{V}(2)$ will be of the form given by Eq. (A.1) of [8]. Now it is easy to see that the construction of a m-particle matrix for $\widehat{h}(1)+\widehat{V}(2)$ in $(\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3)$ basis reduces to the problem of BEGOE(1 + 2) for spinless boson systems, and hence Eq. (A.3) of [8] will give the formulas for the matrix elements. Note that $\widehat{h}(1)$ is diagonal in the $(\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3)$ basis and $\langle(\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3)|\,\widehat{h}(1)\,|(\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3)\rangle = \sum_{r=1}^{\Omega}\epsilon_r(m^r_1+m^r_2+m^r_3)$.

To project out S, we consider the basis

$$|(\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3),\,M_S=0\rangle, \quad (6)$$

as they will contain states with all S values. The dimension of this basis space is $\mathcal{D}(\Omega,m,M_S=0)=\sum_S d(\Omega,m,S)$. In the basis defined by Eq. (6), both the H matrix and the $\hat{S}^2$ matrix are constructed using the procedure described above. Note that for the $\hat{S}^2$ operator, the diagonal two-particle matrix elements $V^s_{ijij}(2)$ are -4, -2, and 2 for $s=0$, 1, and 2, respectively, and they are independent of i and j. Similarly, all the off-diagonal matrix elements are zero and the sp energies are $\epsilon_i=2$ independent of i. Now diagonalizing the $\hat{S}^2$ matrix will give the unitary transformation required to convert the H matrix in the $M_S=0$ basis [given by (6)] into good S basis [we use the fact that $\hat{S}^2$ eigenvalues are $S(S+1)$ with $S=0,1,\ldots,m$ for a given]. This then gives the m-boson matrix as a direct sum of matrices, one each for the allowed S values. Thus, the H matrix will be of the form shown in Fig. 1(c). This procedure has been implemented and computer programs are developed. For completeness, let us add that the BEGOE(1 + 2)-$S1$ ensemble is defined by six parameters $(\Omega,m,S,\lambda_0,\lambda_1,\lambda_2)$ with λ_s's in units of Δ. Before turning to numerical results, we briefly discuss the embedding algebra for the BEGOE(1 + 2)-$S1$ ensemble and, among other things, this gives an easy procedure to calculate $d(\Omega,m,S)$.

III. U(Ω) ⊗ [SU(3) ⊃ SO(3)] EMBEDDING ALGEBRA

Embedding algebra for BEGOE(1 + 2)-$S1$ is not unique and, following the earlier results for the IBM-3 model of atomic nuclei [13,14], it is possible to identify two algebras. They are (i) U(3Ω) ⊃ U(Ω) ⊗ [U(3) ⊃ SO(3)] and (ii) U(3Ω) ⊃ SO(3Ω) ⊃ SO(Ω) ⊗ SO(3). Here we consider case (i) and later in Sec. V we consider (ii) in brief.

First, the spectrum generating algebra U(3Ω) is generated by the $(3\Omega)^2$ number of operators $u^k_q(i,j)$, where

$$u^k_q(i,j) = (b^\dagger_{i;\mathbf{s}=1}\tilde{b}_{j;\mathbf{s}=1})^k_q, \quad k=0,1,2 \quad \text{and} \quad i,j=1,2,\ldots,\Omega. \quad (7)$$

Note that u^k are given in angular momentum coupled representation with $k=\mathbf{s}\times\mathbf{s}=0,1,2$ and $\tilde{b}_{i;1,m_{\rm s}} = (-1)^{1+m_{\rm s}}\,b_{i;1,-m_{\rm s}}$. The quadratic Casimir invariant of U(3Ω) is $\hat{C}_2(\mathrm{U}(3\Omega)) = \sum_{i,j,k}\,u^k(i,j)\cdot u^k(j,i)$. Note that $T^k\cdot U^k = (-1)^k\sqrt{(2k+1)}\,(T^kU^k)^0$. In terms of the number operator $\hat{n}=\sum_{i,m_{\rm s}}\,b^\dagger_{i;1,m_{\rm s}}b_{i;1,m_{\rm s}}$, we have $\hat{C}_2(\mathrm{U}(3\Omega)) = \hat{n}(\hat{n}+3\Omega-1)$. All m-boson states transform as the symmetric irrep $\{m\}$ with respect to U(3Ω) algebra and, therefore,

$$\langle\hat{C}_2(\mathrm{U}(3\Omega))\rangle^{\{m\}} = m(m+3\Omega-1). \quad (8)$$

Using the results given in [23] it is easy to write the generators of the algebras U(Ω) and SU(3) in U(3Ω) ⊃ U(Ω) ⊗ SU(3). The U(Ω) generators are $g(i,j)$, where

$$g(i,j) = \sqrt{3}\,(b^\dagger_{i;\mathbf{s}=1}\tilde{b}_{j;\mathbf{s}=1})^0, \quad i,j=1,2,\ldots,\Omega, \quad (9)$$

and they are Ω^2 in number. Similarly, SU(3) algebra is generated by the eight operators $h^{k=1,2}_q$, where

$$h^k_q = \sum_i\,(b^\dagger_{i;\mathbf{s}=1}\tilde{b}_{i;\mathbf{s}=1})^k_q, \quad k=1,2. \quad (10)$$

It is useful to mention that $(h^0,\,h^1_q,\,h^2_{q'})$ generate U(3) algebra and U(3) ⊃ SU(3). The quadratic Casimir invariants of U(Ω) and SU(3) algebras are $\hat{C}_2(\mathrm{U}(\Omega)) = \sum_{i,j}\,g(i,j)\cdot g(j,i)$ and $\hat{C}_2(\mathrm{SU}(3)) = (3/2)\sum_{k=1,2}\,h^k\cdot h^k$, respectively. The irreps of U(Ω) can be represented by Young tableaux $\{f\}=\{f_1,f_2,\ldots,f_\Omega\}$, $\sum_i f_i = m$. However, as we are dealing with boson systems [i.e., the only allowed U(3Ω) irrep being $\{m\}$], the irreps of U(Ω) and U(3) should be represented by the same $\{f\}$. Therefore, $\{f\}$ will be the maximum of three rows. The U(Ω) and SU(3) equivalence gives a relationship between their quadratic Casimir invariants, $\hat{C}_2(\mathrm{U}(\Omega)) = \hat{C}_2(\mathrm{U}(3)) + (\Omega-3)\,\hat{n}$ and $\hat{C}_2(\mathrm{U}(3)) = \sum_{k=0,1,2}\,h^k\cdot h^k = (2/3)\hat{C}_2(\mathrm{SU}(3)) + (1/3)\hat{n}^2$. Given the U(Ω) irrep $\{f_1\,f_2\,f_3\}$, the corresponding SU(3) irrep in Elliott's notation [24] is given by $(\lambda\mu)$, where $\lambda=f_1-f_2$ and $\mu=f_2-f_3$. Thus,

$$\{m\}_{\mathrm{U}(3\Omega)} \rightarrow [\{f_1\,f_2\,f_3\}_{\mathrm{U}(\Omega)}]\,[(\lambda\,\mu)_{\mathrm{SU}(3)}],$$
$$f_1+f_2+f_3 = m, \quad f_1\geqslant f_2\geqslant f_3\geqslant 0, \quad (11)$$
$$\lambda = f_1-f_2, \quad \mu = f_2-f_3.$$

Using Eq. (11) it is easy to write, for a given m, all the allowed SU(3) and equivalently U(Ω) irreps. Eigenvalues of $\hat{C}_2(\mathrm{SU}(3))$ are given by

$$\langle\hat{C}_2(\mathrm{SU}(3))\rangle^{(\lambda\,\mu)} = C_2(\lambda\,\mu) = [\lambda^2+\mu^2+\lambda\mu+3(\lambda+\mu)]. \quad (12)$$

Let us add that the SU(3) algebra also has a cubic invariant $C_2(\mathrm{SU}(3))$ and its matrix elements are [25]

$$\langle\hat{C}_3(\mathrm{SU}(3))\rangle^{(\lambda\,\mu)} = C_3(\lambda\,\mu) = \tfrac{2}{9}\,(\lambda-\mu)(2\lambda+\mu+3)(\lambda+2\mu+3). \quad (13)$$

The SO(3) subalgebra of SU(3) generates spin S. The spin generators are

$$S^1_q = \sqrt{2}\,h^1_q, \quad \hat{S}^2 = C_2(\mathrm{SO}(3)) = S^1\cdot S^1, \quad \langle\hat{S}^2\rangle^S = S(S+1). \quad (14)$$

Given a $(\lambda\ \mu)$, the allowed S values follow from Elliott's rules [24] and this introduces a "K" quantum number,

$$\begin{aligned} K &= \min(\lambda\ ,\ \mu), \quad \min(\lambda\ ,\ \mu)-2,\ldots,\ 0\ \text{ or }\ 1,\\ S &= \max(\lambda\ ,\ \mu), \quad \max(\lambda\ ,\ \mu)-2,\ldots,\ 0\ \text{ or }\ 1 \quad \text{for}\ \ K=0,\\ &= K, K+1, K+2,\ldots, K+\max(\lambda\ ,\ \mu) \quad \text{for} \quad K\neq 0. \end{aligned} \tag{15}$$

Equation (15) gives $d_{(\lambda\,\mu)}(S)$, the number of times a given S appears in a $(\lambda\ \mu)$ irrep. Similarly the number of substates that belong to a U(Ω) irrep $\{f_1\ f_2\ f_3\}$ are given by $d_\Omega(f_1\ f_2\ f_3)$, where [26]

$$d_\Omega(f_1\ f_2\ f_3) = \begin{vmatrix} d_\Omega(f_1) & d_\Omega(f_1+1) & d_\Omega(f_1+2) \\ d_\Omega(f_2-1) & d_\Omega(f_2) & d_\Omega(f_2+1) \\ d_\Omega(f_3-2) & d_\Omega(f_3-1) & d_\Omega(f_3) \end{vmatrix}. \tag{16}$$

Here $d_\Omega(\{g\}) = \binom{\Omega+g-1}{m}$ and $d_\Omega(\{g\}) = 0$ for $g < 0$. Note that the determinant in Eq. (16) involves only symmetric U(Ω) irreps. Using the U(3Ω) ⊃ U(Ω) ⊗ [U(3) ⊃ SO(3)] algebra, m-boson states can be written as $|m;\{f_1\ f_2\ f_3\}\,\alpha;(\lambda\ \mu)\ K\ S\ M_S\rangle$; The number of α values is $d_\Omega(f_1\ f_2\ f_3)$, K values follow from Eq. (15) and $-S \leqslant M_S \leqslant S$. Note that m and $(\lambda\ \mu)$ give a unique $\{f_1\ f_2\ f_3\}$. Therefore, the H-matrix dimension in fixed-(Ω,m,S) space is given by $d(\Omega,m,S) = \sum_{\{f_1\ f_2\ f_3\}\in m}\ d_\Omega(\{f_1\ f_2\ f_3\})\ d_{(\lambda\,\mu)}(S)$ and they will satisfy the sum rule $\sum_S (2S+1)\ d(\Omega,m,S) = \binom{3\Omega+m-1}{m}$. Also, the dimension $\mathcal{D}(\Omega,m,M_S=0)$ of the H matrix in the basis discussed earlier is $\mathcal{D}(\Omega,m,M_S=0) = \sum_{S\in m} d(\Omega,m,S)$. For example, for ($\Omega=4$, $m=8$), the dimensions $d(\Omega,m,S)$ for $S=$ 0–8 are 714, 1260, 2100, 1855, 1841, 1144, 840, 315, and 165, respectively. Similarly, for ($\Omega=6$, $m=10$), the dimensions for $S=$ 0–10 are 51 309, 123 585, 183 771, 189 630, 178 290, 133 497, 94 347, 51 645, 27 027, 9009, and 3003, respectively. These give $\mathcal{D}(\Omega=4,m=8,M_S=0) = 10\,234$ and $\mathcal{D}(\Omega=6,m=10,M_S=0) = 1\,045\,113$. Because of these very large dimensions, numerical analysis of BEGOE(1 + 2)-$S1$ ensemble is quite difficult.

IV. RESULTS FOR SPECTRAL PROPERTIES

A. Eigenvalue density and NNSD: Numerical results

Using the method described in Sec. II, in some examples the BEGOE(1 + 2)-$S1$ ensemble has been constructed and eigenvalue density and spectral fluctuations are analyzed. We have considered a 100 member BEGOE(1 + 2)-$S1$ ensemble with $m=8$ and $\Omega=4$. The sp energies defining the one-body Hamiltonian $\hat{h}(1)$ in Eq. (4) are chosen to be $\epsilon_i = i + 1/i$, $i = 1,2,\ldots,\Omega$, just as in many of the earlier papers on embedded ensembles [8,22,27–30]. Note that the second term $1/i$ in ϵ_i has been added, as discussed first in [27], to avoid the degeneracy of many-particle states when interaction strength is small. In principle, it is possible to draw the ϵ_i from the eigenvalues of a random ensemble or from the center of a GOE. Let us add that in BEC applications, the use of a harmonic trap suggests equidistant levels with different degeneracies. Similarly, anisotropic traps (elongated or toroidal traps that are uniform along only one direction while still harmonic along the other two directions) [31,32] result in nonequidistant energy levels with different degeneracies. It is possible to carry out BEGOE(1 + 2)-$S1$ calculations with various choices of ϵ_i; however, all the statistical properties discussed in the present paper do not change for sufficiently strong interaction (i.e., for $\lambda > \lambda_C$ as discussed ahead). Turning to the strengths λ_s of the two-body interaction in the $s=0$, $s=1$, and $s=2$ channels, they are all chosen to be equal; i.e., $\lambda_0 = \lambda_1 = \lambda_2 = \lambda$ in Eq. (4). Figure 2 presents the results for the ensemble-averaged fixed-(m,S) eigenvalue density $\overline{\rho^{m,S}(E)}$ for $\lambda = 0.2$. In the construction of the ensemble averaged eigenvalue densities, the spectra of each member of the

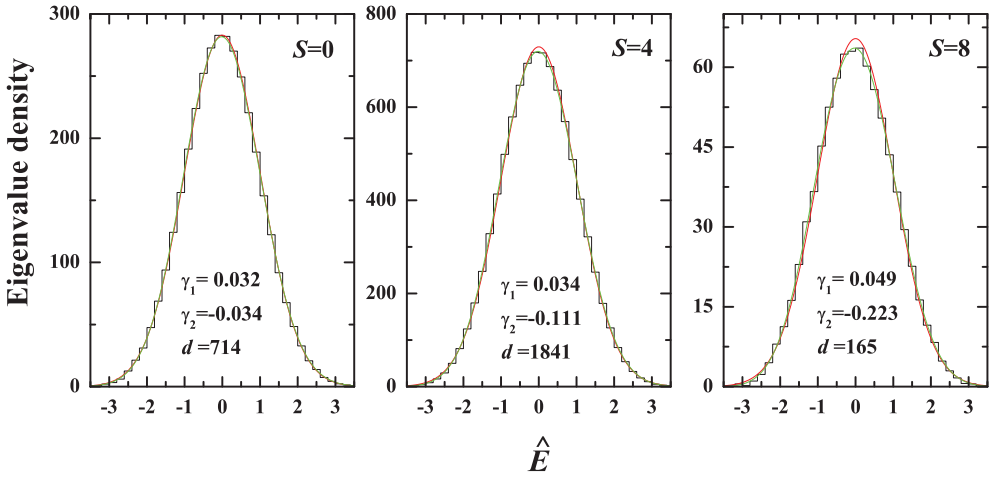

FIG. 2. (Color online) Ensemble averaged eigenvalue density $\rho^{m,S}(\widehat{E})$ vs normalized energy $\widehat{E} = \frac{E-E_c(m,S)}{\sigma(m,S)}$ for a 100 member BEGOE(1 + 2)-$S1$ ensemble with $\Omega=4$, $m=8$, and spins $S=0$, 4, and 8. The strengths of the two-body interaction in the $s=0$, $s=1$, and $s=2$ channels are chosen to be $\lambda_0 = \lambda_1 = \lambda_2 = 0.2$. Note that, in the eigenvalue density $\rho^{m,S}(\widehat{E})$, the $(2S+1)$ degeneracy of the levels is not counted. The eigenvalue densities are compared to Gaussian (red) and Edgeworth (ED) corrected Gaussian (green) forms. The ensemble averaged values of skewness (γ_1) and excess (γ_2) parameters are shown in figure. In the plots, the eigenvalue densities, for a given spin S, are normalized to dimension $d(m,S)$.

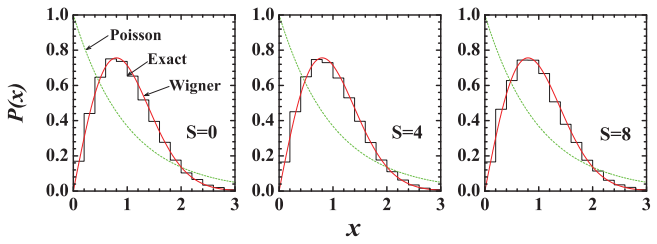

FIG. 3. (Color online) Ensemble averaged NNSD histogram for a 100 member BEGOE(1 + 2)-$S1$ with $m = 8$ and $\Omega = 4$ for two-body interaction strength $\lambda_0 = \lambda_1 = \lambda_2 = 0.2$. Results are shown for the spin values $S = 0$, 4, and 8. Here, x is in the units of local mean spacing. Results are compared with Poisson and GOE (Wigner) forms.

ensemble are first zero centered and scaled to unit width. The eigenvalues are then denoted by $\widehat{E}$. Given are the fixed-(m,S) eigenvalue centroids $E_c(m,S)$ and spectral widths $\sigma(m,S)$, $\widehat{E} = [E - E_c(m,S)]/\sigma(m,S)$. Then the histograms for the density are generated by combining the eigenvalues $\widehat{E}$ from all the members of the ensemble. In the figure, histograms are constructed with a bin size equal to 0.2. Results are shown in Fig. 2 for $S = 0$, 4, and 8 values. It is clearly seen that the eigenvalue densities are close to Gaussian, also the agreements with Edgeworth (ED) corrected Gaussians are excellent.

The NNSD, which gives information about level repulsion, is of GOE type for spinless BEGOE(1 + 2) [1] and BEGOE(1 + 2)-F [8] for a sufficiently strong two-body interaction. In Fig. 3, NNSD results are shown for BEGOE(1 + 2)-$S1$ with $m = 8$ and $\Omega = 4$ using $\lambda = 0.2$ for selected spin values. The NNSDs are obtained by unfolding each spectrum in the ensemble, using the method described in [1], with the smoothed density as a corrected Gaussian with corrections involving up to sixth order moments of the density function. In present calculations, 80% of the eigenvalues (dropping 10% from both ends of the spectrum) from each member are employed. It is clearly seen from the figures that the NNSDs are close to the GOE (Wigner) form.

Previously, it was shown that BEGOE(1 + 2) for spinless boson systems [4] as well as BEGOE(1 + 2)-F for two species boson systems [8] generate Gaussian eigenvalue densities in the dense limit, and fluctuations follow GOE with sufficiently strong interaction in the presence of mean field. Therefore, combining these with the results in Figs. 2 and 3, it is plausible to conclude that for finite interacting boson systems the eigenvalue density will be generically of Gaussian form and fluctuations, with sufficiently strong interaction, follow GOE. As discussed in [4,8], with mean field, the interaction strength has to be larger than a critical value for the fluctuations to change from Poisson-like to GOE. In order to verify this, calculations are carried out for various values of λ and the results are shown in Fig. 4. The figure clearly displays the transition of NNSD from the Poisson-like character for small values of λ to the GOE character as the strength λ of the two-body interaction is slowly increased. The transition point (λ_C), from Poisson type to GOE type, is determined empirically, keeping the value of NNSD variance $\sigma^2(0)$ equal to 0.37. For Poisson type, it has the value of 1, and for GOE, it is known to be equal to 0.27. The choice of the critical variance

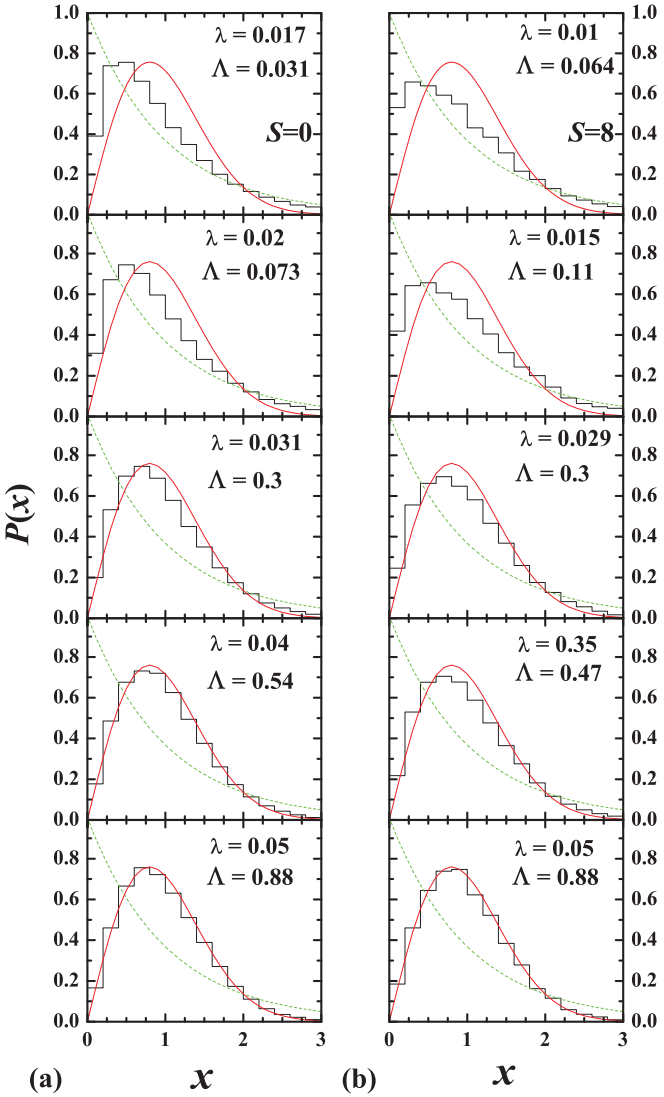

FIG. 4. (Color online) Ensemble averaged NNSD, for various values of λ, with $\Omega = 4$, $m = 8$ using 100 member BEGOE(1 + 2)-$S1$ ensemble. Results are shown for extreme spins: (a) $S = S_{\min} = 0$ and (b) $S = S_{\max} = 8$. Calculated NNSD are compared to the Poisson (green dashed lines) and Wigner (GOE) (red smooth lines) forms. Values of the interaction strength λ and the transition parameter Λ are given in the figure. The values of Λ are deduced as discussed in [22]. The chaos marker λ_C corresponds to $\Lambda = 0.3$ and its values, as shown in the figure, are 0.031 for $S = 0$ and 0.029 for $S = 8$. The bin size is 0.2 for the histograms.

$\sigma^2(0) = 0.37$ of NNSD comes from the (2×2) random matrix model constructed in [33] for the Poisson to GOE transition in NNSD. This model involves a transition parameter Λ and it was shown in [33] that there is an onset of GOE fluctuations at $\Lambda = 0.3$. From the formula for $\sigma^2(0 : \Lambda)$ given in [33], it is easily seen that $\sigma^2(0) = 0.37$ for $\Lambda = 0.3$. In Fig. 4, the values of the Λ parameter are given for different λ values and it is seen that the transition point $\lambda_C = 0.031$ and 0.029 for $S = 0$ and $S = 8$, respectively. In the calculations that correspond to Figs. 2, 3, and 7, $\lambda = 0.2$ is chosen and this is much larger than λ_C. Thus, level fluctuations in all these examples follow GOE.

B. Propagation of energy centroids and spectral variances

With the eigenvalue density being close to Gaussian, it is useful to derive formulas for energy centroids and spectral variances in terms of sp energies ϵ_i and the two-particle $V(2)$ matrix elements V^s_{ijkl}. These also allow us to study, numerically (also analytically if the ensemble averages can be carried out analytically), fluctuations in energy centroids and spectral variances. For a given one- plus two-body Hamiltonian H, a propagation equation for the fixed-(m,S) energy centroids $\langle H\rangle^{m,S}$ in terms of the scalars $\hat{n}$ and S^2 operators [with eigenvalues m and $S(S+1)$, respectively] is not possible. This is easily seen from the fact that for $m \leqslant 2$ bosons, we have five states ($m=0,S=0$; $m=1,S=1$; $m=2,S=0,1,2$) but only four scalar operators ($1,\hat{n},\hat{n}^2,\hat{S}^2$). For the one missing operator, one can use $\hat{C}_2(\mathrm{SU}(3))$, but then only fixed-$(m,(\lambda\,\mu)S)$ averages will propagate [34]. Then, the propagation equation is

$$\begin{aligned}
&\langle\widehat{H}(1+2)\rangle^{m,(\lambda\,\mu),S}\\
&\quad=\langle\widehat{h}(1)+\widehat{V}(2)\rangle^{m,(\lambda,\mu),S}\\
&\quad=m\,\langle\widehat{h}(1)\rangle^{1,(10),1}\\
&\qquad+\left[-\frac{m}{6}+\frac{m^2}{18}+\frac{C_2(\lambda\,\mu)}{9}-\frac{S(S+1)}{6}\right]\langle\widehat{V}(2)\rangle^{2,(20),0}\\
&\qquad+\left[-\frac{5m}{6}+\frac{5m^2}{18}+\frac{C_2(\lambda\,\mu)}{18}+\frac{S(S+1)}{6}\right]\langle\widehat{V}(2)\rangle^{2,(20),2}\\
&\qquad+\left[\frac{m}{2}+\frac{m^2}{6}-\frac{C_2(\lambda\,\mu)}{6}\right]\langle\widehat{V}(2)\rangle^{2,(01),1}. \qquad (17)
\end{aligned}$$

Now summing over all $(\lambda\,\mu)$ irreps that contain a given S will give $\langle\widehat{H}(1+2)\rangle^{m,S}$. This procedure is used to verify the numerical codes we have developed for constructing BEGOE(1+2)-$S1$ members. Just as with energy centroids, it is possible to propagate the variances $\langle[\widehat{H}(1+2)]^2\rangle^{m,(\lambda\,\mu),S}$. However, this requires, besides the quadratic and cubic invariants of SU(3), the so called SU(3) $\supset$ SO(3) integrity basis operators $\hat{X}_3$ and $\hat{X}_4$ that are three and four body in nature, respectively. Thus, the propagation equation for spectral variances $\langle[\widehat{H}(1+2)]^2\rangle^{m,S}$ is more complicated [35]. Alternatively, it is possible to use (m_1,m_2,m_3) configurations introduced in Sec. II. The m-particle space can be decomposed using configurations (m_1,m_2,m_3) with $m_1+m_2+m_3=m$, where m_1 particles are in the unitary orbit with Ω number of sp states all carrying $m_{\mathbf{s}}=+1$ quantum number, m_2 particles in the unitary orbit with $m_{\mathbf{s}}=0$, and m_3 particles in the unitary orbit with $m_{\mathbf{s}}=-1$. The total M_S for such a configuration is (m_1-m_3). Propagation of the traces $\langle\langle H^p\rangle\rangle^{(m_1,m_2,m_3)}$, $p=1,2$, in terms of ϵ_i and $V'_{i,m_{\mathbf{s}}^{f1};j,m_{\mathbf{s}}^{f2};k,m_{\mathbf{s}}^{i1};\ell,m_{\mathbf{s}}^{i2}}(2)$ follow from the results given in [36,37]. The Appendix gives the formulas. These, in turn, will give the traces $\langle\langle H^p\rangle\rangle^{m,M_S}$ $(p=1,2)$,

$$\begin{aligned}
\langle\langle H^p\rangle\rangle^{m,M_S}&=\sum_{\text{fixed }M_S}\langle H^p\rangle^{(m_1,m_2,m_3)}d(\Omega:m_1,m_2,m_3)\\
&=\sum_{S=M_S}^{S_{\max}}d(\Omega,m,S)\langle H^p\rangle^{(m,S)}, \qquad (18)
\end{aligned}$$

with

$$\begin{aligned}
\sum_{\text{fixed }M_S}d(\Omega:m_1,m_2,m_3)&=\mathcal{D}(\Omega,m,M_S)=\sum_{S=M_S}^{S_{\max}}d(\Omega,m,S),\\
d(\Omega,m,S)&=\mathcal{D}(\Omega,m,M_S=S)\\
&\quad-\mathcal{D}(\Omega,m,M_S=S+1).
\end{aligned}$$

Using these, fixed-(m,S) energy centroids and spectral variances are given by

$$\langle H^p\rangle^{m,S}=\frac{\langle\langle H^p\rangle\rangle^{m,M_S=S}-\langle\langle H^p\rangle\rangle^{m,M_S=S+1}}{d(\Omega,m,S)}.$$

It is easy to see that this procedure also gives $d(\Omega,m,S)$ as the dimension of a (m_1,m_2,m_3) configuration is $\binom{\Omega+m_1-1}{m_1}\binom{\Omega+m_2-1}{m_2}\binom{\Omega+m_3-1}{m_3}$. We have implemented this method for numerical calculations of $d(m,S)$, $E_c(m,S)$, and $\sigma^2(m,S)$.

For $(\Omega=3,m=6)$, $(\Omega=4,m=6)$, $(\Omega=4,m=8)$, and $(\Omega=5,m=5)$ examples, choosing $\epsilon_i=i+1/i$ and V^s_{ijkl} as random numbers, the H matrices are explicitly constructed as described in Sec. II and $E_c(m,S)$ and $\sigma^2(m,S)$ are calculated for various S values. These are compared with the numbers obtained using the propagation formulas given in the Appendix and the agreement is found to be exact, as expected. Going beyond this testing, using the propagation formulas we have calculated the variation of the ensemble averaged spectral variances with spin S for $m=8,12,16$ and $\Omega=4$. It should be mentioned that direct matrix construction for $m=12$ and 16 is not possible with the available computational facility. These calculations are possible because of the propagation equations. Results are shown in Fig. 5. It is clearly seen from the figure that the ensemble averaged variances are almost constant for lower spins ($S<S_{\max}/2$) and increase with S close to the $S=S_{\max}$; a similar result is known for fermion systems [38]. It is useful to note that the near constancy of widths is a feature of many-body chaos [27,39,40].

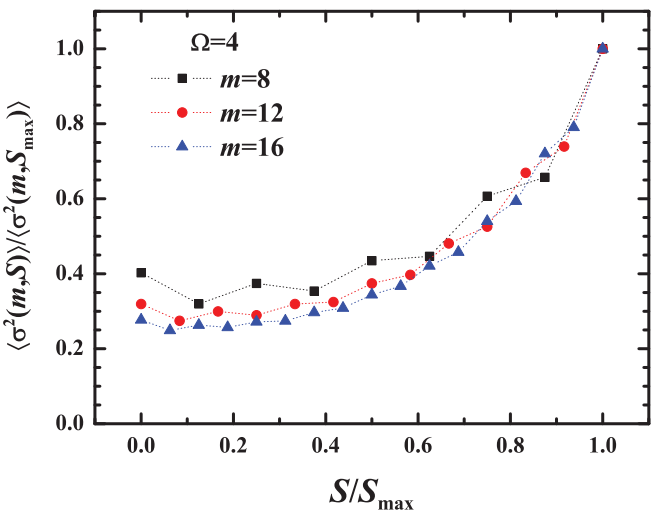

FIG. 5. (Color online) Ensemble averaged variances $\langle\sigma^2(m,S)\rangle/\langle\sigma^2(m,S_{\max})\rangle$ vs $S/S_{\max}$ for a 200 member BEGOE(2)-$S1$ ensemble with $\Omega=4$ and $m=8$, 12, and 16.

C. Fluctuations in energy centroids and spectral variances

Calculation of energy centroids and spectral variances for each member of the ensemble as discussed in Sec. IV B allows us to examine the covariances in these quantities. Normalized covariances are defined by

$$\Sigma_{pp}(m,S:m',S') = \frac{\overline{\langle H^p\rangle^{m,S}\langle H^p\rangle^{m',S'}} - \{\overline{\langle H^p\rangle^{m,S}}\ \ \overline{\langle H^p\rangle^{m',S'}}\}}{\{\overline{\langle H^2\rangle^{m,S}}\ \ \overline{\langle H^2\rangle^{m',S'}}\}^{p/2}}. \quad (19)$$

In Eq. (19), $p=1$ gives normalized covariances in energy centroids and $p=2$ gives normalized covariances in spectral variances. For $(m,S)=(m',S')$, these give information about fluctuations and, in particular, about level motion in the ensemble [1]. For $(m,S)\neq(m',S')$, the covariances (cross correlations) are nonzero for BEGOE while they will be zero for independent GOE representation for the m-boson Hamiltonian matrices with different m or S [with fixed Ω for both (m,S) and (m',S') systems so that the Hamiltonian in two-particle spaces remains the same]. Using the formulas given in the Appendix, we have computed energy centroids and spectral variances for a 200 member BEGOE(2)-$S1$ ensemble for $\Omega=4$–6 and $m=8$–20. In the numerical calculations, we use $\lambda_0=\lambda_1=\lambda_2$. The results for self-correlations in energy centroids ($[\Sigma_{11}(m,S:m,S)]^{1/2}$) and in spectral variances ($[\Sigma_{22}(m,S:m,S)]^{1/2}$) as a function of spin $S/S_{\max}$ are shown in Fig. 6. It is seen that the centroid fluctuations are large for

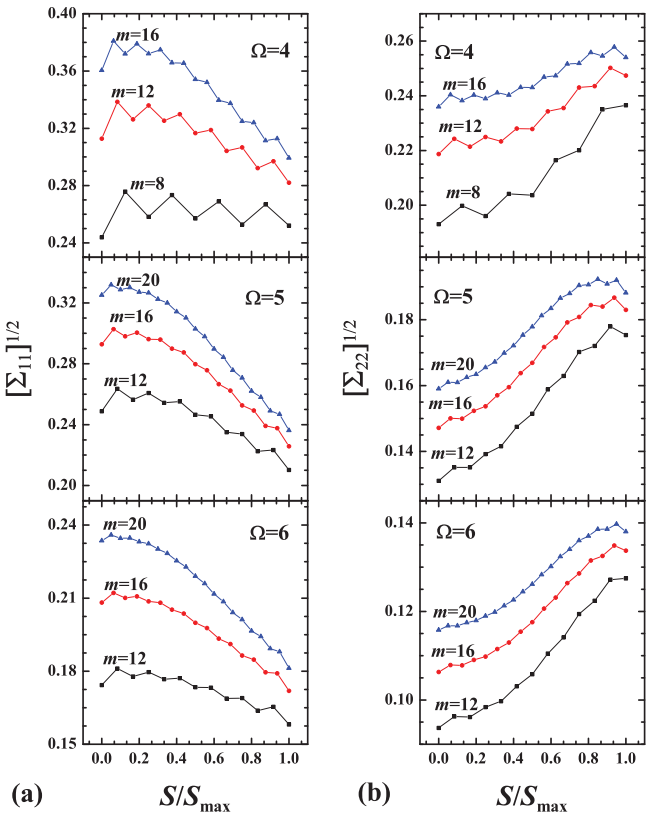

FIG. 6. (Color online) Normalized self-correlations (a) in energy centroids $[\Sigma_{11}]^{1/2}$ and (b) in spectral variances $[\Sigma_{22}]^{1/2}$ as a function of spin S for a 200 member BEGOE(2)-$S1$ ensemble for various values of m and Ω.

$S=0$ with $m\gg\Omega$ and decreases with increase in S value. However, for small m, the variation of $[\Sigma_{11}]^{1/2}$ with spin S is weak. For fixed m, $[\Sigma_{11}]^{1/2}$ decreases with increase in Ω. Also, $[\Sigma_{22}]^{1/2}$ are always smaller than $[\Sigma_{11}]^{1/2}$ just as for BEGOE(2)-F [8]. It is seen from Fig. 6 that for $\Omega=6$, the width of the fluctuations in the variances are 10%–13%. Similarly for large m, with Ω being very small, the widths are quite large but they decrease quickly with increasing Ω. Thus, the width of the fluctuations in spectral widths is found to be much smaller, unlike the width of the fluctuations in energy centroids.

V. PAIRING ALGEBRAS

In the BEGOE(1 + 2)-$S1$ space, it is possible to identify two different pairing algebras (each defining a particular type of pairing) and they follow from the results in [14,23,41]. One of them corresponds to the SO(Ω) algebra in U(3Ω) $\supset$ [U(Ω) $\supset$ SO(Ω)] $\otimes$ [U(3) $\supset$ SO(3)] and we refer to this as SO(Ω)-SU(3) pairing. The other corresponds to the SO(3Ω) in U(3Ω) $\supset$ SO(3Ω) $\supset$ SO(Ω) $\otimes$ SO(3). Note that both the algebras have SO(3) subalgebra that generates the spin S. Below we give some details of these pairing algebras. Pairing structures of the ground states generated by random interactions are discussed in Sec. V C.

A. SO(Ω)-SU(3) pairing

Following the results given in [14,23,41] it is easy to identify the $\Omega(\Omega-1)/2$ number of generators $U(i,j)$, $i<j$ of SO(Ω) in U(3Ω) $\supset$ [U(Ω) $\supset$ SO(Ω)] $\otimes$ [SU(3) $\supset$ SO(3)],

$$U(i,j)=\sqrt{\alpha(i,j)}\,[g(i,j)+\alpha(i,j)\,g(j,i)],\quad i<j,$$
$$|\alpha(i,j)|^2=1,\quad \alpha(i,j)=\alpha(j,i),\quad \alpha(i,j)\alpha(j,k)=-\alpha(i,k). \quad (20)$$

Note that $g(i,j)$ are defined in Eq. (9). The quadratic Casimir invariant of SO(Ω) is

$$\hat{C}_2(\mathrm{SO}(\Omega))=\sum_{i<j}U(i,j)\cdot U(j,i). \quad (21)$$

Applying Eq. (20) now gives

$$\begin{aligned}\hat{C}_2(\mathrm{SO}(\Omega)) &= \sum_{i<j}\alpha(i,j)[g(i,j)\cdot g(i,j)+g(j,i)\cdot g(j,i)\\ &\quad +2\alpha(i,j)\,g(i,j)\cdot g(j,i)]\\ &= \sum_{i\neq j}g(i,j)\cdot g(j,i)+\sum_{i\neq j}\alpha(i,j)\,g(i,j)\cdot g(i,j)\\ &= \hat{C}_2(\mathrm{U}(\Omega))-\sum_{i,j}\beta_i\beta_j\,g(i,j)\cdot g(i,j),\end{aligned}$$
$$\beta_i\beta_j=-\alpha(i,j),\quad \text{for } i\neq j,\quad |\beta_i|^2=1. \quad (22)$$

Here we have introduced β_i's and the $\alpha(i,j)$ are defined in Eq. (20). Now defining the pairing operator $\mathcal{P}^k_q$, $k=0,2$ as

$$\mathcal{P}^k_q=\sum_i\beta_i(b^\dagger_{i;1}b^\dagger_{i;1})^k_q,\quad k=0,2, \quad (23)$$

it is easy to see that

$$H_{\mathcal{P}} = \sum_{k=0,2;q} \mathcal{P}^k_q \left(\mathcal{P}^k_q\right)^\dagger$$
$$= \hat{C}_2(\mathrm{U}(\Omega)) - \hat{C}_2(\mathrm{SO}(\Omega)) - \hat{n}$$
$$= \frac{2}{3}\hat{C}_2(\mathrm{SU}(3)) - \hat{C}_2(\mathrm{SO}(\Omega)) - (\Omega-4)\hat{n} + \frac{\hat{n}^2}{3}. \quad (24)$$

Thus, the pairing Hamiltonian in the $\mathrm{U}(3\Omega) \supset [\mathrm{U}(\Omega) \supset \mathrm{SO}(\Omega)] \otimes [\mathrm{SU}(3) \supset \mathrm{SO}(3)]$ algebra is a sum of $k = 0$ and 2 pairs and it is simply related to the $\mathrm{SO}(\Omega)$ and $\mathrm{SU}(3)$ algebras. It is useful to note that the two-particle matrix elements of the pairing Hamiltonian $H_{\mathcal{P}}$ are $V^{s=0}_{iijj} = 1$, $V^{s=2}_{iijj} = 1$ and all other matrix elements are zero.

It is possible to enumerate the irreps $[\boldsymbol{\omega}]$ of $\mathrm{SO}(\Omega)$ given a $\mathrm{U}(\Omega)$ [equivalently $\mathrm{SU}(3)$] irrep $\{f\}$ for a given m using the methods given in [41,42] and for some special $\{f\}$ we can also write formulas [14,26]. First, as $\{f\}$ is three rowed, the irrep $[\boldsymbol{\omega}]$ will be maximum three rowed. Then $[\boldsymbol{\omega}] = [\omega_1,\omega_2,\omega_3]$. For symmetric irreps $\{m\}$, we have the simple result

$$\{m\} \to [\omega], \quad \omega = m, m-2, \ldots, 0 \quad \text{or} \quad 1. \quad (25)$$

Reduction formulas for the two row irreps $\{f_1,f_2\} \to [\omega_1,\omega_2]$ and similarly for the three row irreps $\{f_1,f_2,f_3\} \to [\omega_1,\omega_2,\omega_3]$ will be complicated and they are not discussed here. As an example, we show in Table I the $\mathrm{SO}(\Omega)$ irreps contained in all $\mathrm{U}(\Omega)$ irreps for $m = 6$ with $\Omega \geqslant 4$. Given the $[\omega_1,\omega_2,\omega_3]$, the eigenvalues of $\hat{C}_2(\mathrm{SO}(\Omega))$ are given by

$$\langle \hat{C}_2(\mathrm{SO}(\Omega))\rangle^{[\omega_1,\omega_2,\omega_3]} = \omega_1(\omega_1+\Omega-2) + \omega_2(\omega_2+\Omega-4) + \omega_3(\omega_3+\Omega-6). \quad (26)$$

Equations (24), (26), and (13) will give $\langle H_{\mathcal{P}}\rangle^{\{f_1,f_2,f_3\},[\omega_1,\omega_2,\omega_3]}$. It is useful to mention that, for $\Omega \geqslant 6$, $\omega = \omega_1 + \omega_2 + \omega_3$ is the seniority quantum number (see also Sec. V B). Finally, for the $\mathrm{U}(\Omega) \supset \mathrm{SO}(\Omega)$, the complimentary pairing algebra is the noncompact $sp(6)$ algebra generated by the 21 operators $\mathcal{P}^k_q$, $(\mathcal{P}^k_q)^\dagger$, h^1_q, h^2_q, and $\hat{n}$. This complicated algebra will be discussed elsewhere; for a recent review on complimentary algebras, see [43].

Before going further, it is useful to mention that the Majorana operator $(\hat{M})$ that changes the space labels (i,j) in a two-particle space without changing the spin labels m_s is related in a simple manner to $\hat{C}_2(\mathrm{U}(3))$. Denoting the spin labels by $\alpha,\beta,\ldots$, we have

$$\hat{M} = \sum_{i,j;\alpha,\beta} b^\dagger_{j,\alpha} b^\dagger_{i,\beta} (b^\dagger_{i,\alpha} b^\dagger_{j,\beta})^\dagger = \hat{C}_2(U(3)) - 3\hat{n}. \quad (27)$$

Also the relation $\hat{S}^2 = \frac{4}{3}\hat{C}_2(\mathrm{SU}(3)) - 2h^2 \cdot h^2$ from Sec. III shows that the $\hat{S}^2$ operator can be generated by a combination of the Majorana operator and the quadrupole-quadrupole operator $h^2 \cdot h^2$.

TABLE I. $\mathrm{U}(\Omega) \supset \mathrm{SO}(\Omega)$ reductions for $\Omega \geqslant 4$. In the table, $[\omega]^2$ implies that the irrep $[\omega]$ appears twice in the reduction.

$\{f\}$	Ω	$[\omega]$
{6}	$\Omega \geqslant 4$	[6],[4],[2],[0]
{5,1}	$\Omega \geqslant 4$	[1,1],[2],[3,1],[4],[5,1]
{4,2}	$\Omega \geqslant 4$	[0],[2]^2,[2,2],[3,1],[4],[4,2]
{4,1,1}	$\Omega \geqslant 6$	[1,1],[2,1,1],[3,1],[4,1,1]
	$\Omega = 4$	[1,1],[2],[3,1],[4]
	$\Omega = 5$	[1,1],[2,1],[3,1],[4,1]
{3,3}	$\Omega \geqslant 4$	[1,1],[3,1],[3,3]
{3,2,1}	$\Omega \geqslant 6$	[1,1],[2],[2,1,1],[2,2],[3,1],[3,2,1]
	$\Omega = 4$	[1,1],[2]^2,[2,2],[3,1]
	$\Omega = 5$	[1,1],[2],[2,1],[2,2],[3,1],[3,2]
{2,2,2}	$\Omega \geqslant 6$	[0],[2],[2,2],[2,2,2]
	$\Omega = 4$	[0],[2]
	$\Omega = 5$	[0],[2],[2,2]

B. SO(3Ω) pairing

The second pairing algebra follows from the recognition that $\mathrm{U}(3\Omega)$ admits $\mathrm{SO}(3\Omega)$ subalgebra. Also, the pairing here is generated by $k = 0$ pairs $b^\dagger_i \cdot b^\dagger_i$ alone, which is seen below. Following the results in [23] the generators of $\mathrm{SO}(3\Omega)$ are easy to identify and they are

$$u^{k=1}_q(i,i), \quad i = 1,2,\ldots,\Omega,$$
$$V^k_q(i,j) = \sqrt{(-1)^k \alpha(i,j)} \times \left[u^k_q(i,j) + \alpha(i,j)(-1)^k\, u^k_q(j,i)\right], \quad i < j,$$
$$|\alpha(i,j)|^2 = 1, \quad \alpha(i,j) = \alpha(j,i),$$
$$\alpha(i,j)\alpha(j,k) = -\alpha(i,k). \quad (28)$$

The operators u^k_q are defined by Eq. (7). Carrying out angular momentum algebra the following relation between the quadratic Casimir invariants $\hat{C}_2(\mathrm{SO}(3\Omega))$ and $\hat{C}_2(\mathrm{U}(3\Omega))$, of $\mathrm{SO}(\Omega)$ and $\mathrm{U}(3\Omega)$, can be established using Eqs. (28),

$$\hat{C}_2(\mathrm{SO}(3\Omega)) = 2\sum_i u^1(i,i)\cdot u^1(i,i) + \sum_{i<j;k} V^k(i,j)\cdot V^k(i,j)$$
$$= \hat{C}_2(\mathrm{U}(3\Omega)) - \sum_{i,k}(-1)^k u^k(i,i)\cdot u^k(i,i) + \sum_{i\neq j;k}(-1)^k \alpha(i,j) u^k(i,j)\cdot u^k(i,j). \quad (29)$$

Introducing the pairing operator P_+,

$$P_+ = \sum_i \gamma_i\, P_+(i) = \frac{1}{2}\sum_i \gamma_i\, b^\dagger_{i;1}\cdot b^\dagger_{i;1}, \quad P_- = (P_+)^\dagger, \quad (30)$$

we can prove the following relationship between $\hat{C}_2(\mathrm{SO}(3\Omega))$ and the pairing Hamiltonian $H_P = 4P_+P_-$,

$$4\,H_P = 4P_+P_- = -\hat{n} + \hat{C}_2(\mathrm{U}(3\Omega)) - \hat{C}_2(\mathrm{SO}(3\Omega))$$
$$= \hat{n}(\hat{n}+3\Omega-2) - \hat{C}_2(\mathrm{SO}(3\Omega)),$$
$$\gamma_i\gamma_j = -\alpha(i,j), \quad \text{for } i\neq j, \quad |\gamma_i|^2 = 1. \quad (31)$$

The $\beta \leftrightarrow \alpha$ relation is needed for the correspondence between H_P and $\hat{C}_2(\mathrm{SO}(3\Omega))$, the important point now being that the three operators P_+, P_-, and $P_0 = (\Omega+\hat{n})/2$ will form a SU(1,1) algebra complimentary to $\mathrm{SO}(3\Omega)$. Thus, the $\mathrm{SO}(3\Omega)$ pairing is much simpler. With $\mathrm{U}(3\Omega)$ irreps being $\{m\}$, the $\mathrm{SO}(3\Omega)$ irreps are labeled by the seniority quantum number ω, where

$$\omega = m, m-2, \ldots, 0 \quad \text{or} \quad 1, \quad (32)$$

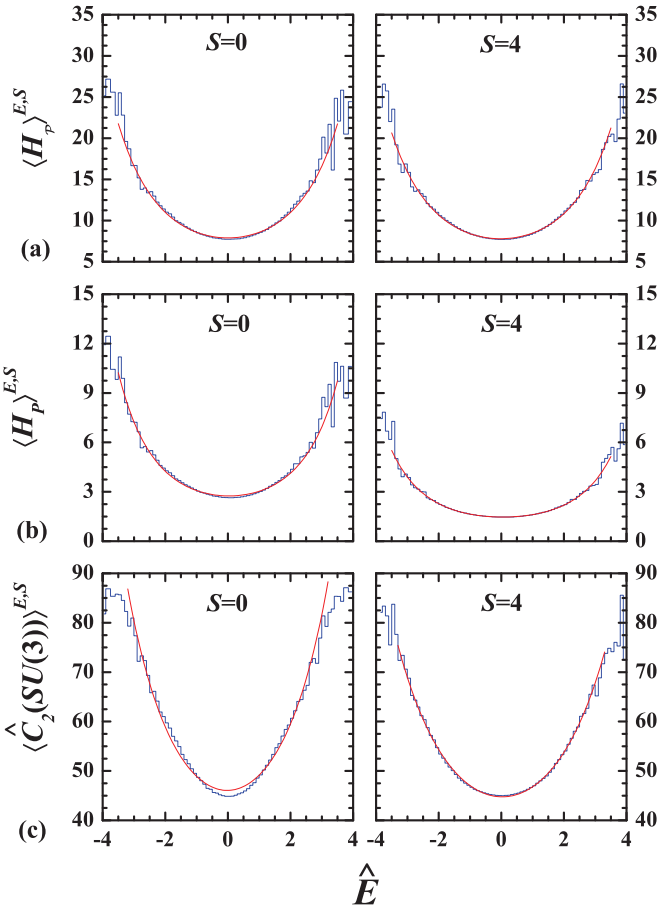

FIG. 7. (Color online) Expectation values of the two pairing Hamiltonians and $\hat{C}_2(\mathrm{SU}(3))$ vs $\widehat{E}$ for a 100 member BEGOE(1 + 2) ensemble with H defined by Eq. (4) and ($\Omega = 4, m = 8$). Results are shown for spins $S = 0$ and $S = 4$. (a) Expectation values of H_P, (b) expectation values of H_P, and (c) expectation value of $\hat{C}_2(\mathrm{SU}(3))$. Ensemble averaged results are shown by histograms, while continuous curves are ratios of Gaussians given by EGOE theory [8]. See text for further details.

and H_P eigenvalues are

$$\langle H_P \rangle^{m,\omega} = \tfrac{1}{4}(m - \omega)(m + \omega + 3\Omega - 2). \qquad (33)$$

The two-particle matrix elements of H_P are simply $V^{s=0}_{iijj} = 1$ and all other matrix elements are zero. Let us mention that to obtain complete classification of states we need irrep reductions for $\mathrm{SO}(3\Omega) \supset \mathrm{SO}(\Omega) \otimes \mathrm{SO}(3)$ and this will be discussed in future.

C. Pairing expectation values

Expectation values of the pairing Hamiltonians in the eigenstates generated by random interactions gives information on regular structures generated by random forces and also they are a measure of chaos [30,44]. Results for the expectation values of the two pairing Hamiltonians and also $\hat{C}_2(\mathrm{SU}(3))$ in the eigenstates of the BEGOE(1 + 2) Hamiltonian defined by Eq. (4) are shown in Fig. 7. We have chosen the parameters in the region of chaos (see Fig. 3), i.e., $\lambda_0 = \lambda_1 = \lambda_2 = \lambda = 0.2$, so that fluctuations in the expectation values will be minimal [30]. It is seen that the expectation values are largest near the ground states and then decrease as we move towards the center of the spectrum. Due to finiteness of the model space, the curves are essentially symmetric around the center. The calculated results are in good agreement with the prediction [8] that, for boson systems (just as it was well verified for fermion systems [30]), expectation values will be ratios of Gaussians; see Sec. VI and Eq. (43) in [8]. Results in the figure show that with repulsive pairing, ground states will be dominated by low seniority structure (small value for ω or $\omega_1 + \omega_2 + \omega_3$). Also, with random interactions, there is no clear distinction between the two different pairing structures. This difference might become evident for larger systems (Ω and m large) by analyzing pairing expectation values as a function of λ, but these numerical examples are currently not feasible due to large matrix dimensions. In addition, results in Fig. 7(c) show that random interactions give ground states with large value for the expectation value of $\hat{C}_2(\mathrm{SU}(3))$, which implies that ground states will be dominated by the SU(3) irrep $(\lambda\mu) = (m0)$. Note that $\langle \hat{C}_2(\mathrm{SU}(3)) \rangle^{(m0)} = m^2 + 3m$ and, therefore, $\langle \hat{C}_2(\mathrm{SU}(3)) \rangle^{(80)} = 88$. As seen from Fig. 7(c), this is close to the pairing expectation value near the ground state. This result is important for the IBM-3 model of atomic nuclei [13].

VI. CONCLUSIONS AND FUTURE OUTLOOK

We have introduced in this paper the embedded Gaussian orthogonal ensemble of random matrices generated by random two-body interactions in the presence of a mean field for spin-one boson systems, and a method for constructing this ensemble for numerical calculations is described. We have presented analytical formulation and some numerical results for this ensemble. Using numerical calculations, it is shown that BEGOE(1 + 2)-$S1$ generates a Gaussian eigenvalue density in the dense limit and exhibits Poisson to GOE transition in level fluctuations as a function of the interaction strength λ. Moreover, covariances in energy centroids and spectral variances (these are lowest two moments of the two point function) are also studied. Preliminary aspects of one of the embedding algebras $\mathrm{SU}(\Omega) \otimes \mathrm{SU}(3)$ and also two pairing algebras in the space defining BEGOE(1 + 2)-$S1$ are discussed in the paper. More detailed study of the effects of random interactions in presence of the two pairing interactions will be useful for gaining new insights into IBM-3 model of atomic nuclei [13,14] and this will be discussed in future. It is important to mention that the numerical results in the present paper are for systems with $\mathcal{D}(\Omega, m, M_S = 0) \lesssim 10\,000$ [dimension of the basis defined by Eq. (6)]. It is possible to deal with larger systems [$\mathcal{D}(\Omega, m, M_S = 0) \gtrsim 10\,000$] by directly constructing the H matrix in a good S basis using angular momentum algebra. This is being attempted and results for systems with larger (Ω, m) values will be reported in the future. Extension of BEGOE(2)-$S1$ to BEGUE(2)-$S1$ (note that BEGUE stands for bosonic embedded Gaussian unitary ensemble) and to the more restricted BEGUE(2)-SU(3) with H preserving SU(3) symmetry for spin-one boson systems are possible; see [45] for preliminary results for BEGUE(2)-SU(3). Finally, applications of the BEGOE(1 + 2)-$S1$ ensemble to spin-one BEC would be possible in future.

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APPENDIX

Given Ω number of sp orbitals, each with spin $\mathbf{s} = 1$, the number of sp states is $N = 3\Omega$. Now arrange these sp states in such a way that the first Ω states have $m_\mathbf{s} = 1$ (these are labeled $\alpha = 1$ states), the next Ω states have $m_\mathbf{s} = 0$ (labeled $\alpha = 2$ states), and the remaining Ω having $m_\mathbf{s} = -1$ (labeled $\alpha = 3$ states) so that a state $|r\rangle = |i = r, m_\mathbf{s} = 1\rangle$ for $r \leqslant \Omega$, $|r\rangle = |i = r - \Omega, m_\mathbf{s} = 0\rangle$ for $\Omega < r \leqslant 2\Omega$, and $|r\rangle = |i = r - 2\Omega, m_\mathbf{s} = -1\rangle$ for $r > 2\Omega$. To proceed further, the two-body Hamiltonian defined by $V^{s=0,1,2}_{i_1 j_1 k_1 \ell_1}(2)$ matrix elements is converted into the $|i, m_\mathbf{s} = 0, \pm 1\rangle$ basis by changing to $V_{ijkl}(2)$, where $H = 1/4 \sum_{ijkl}^{N=3\Omega} V_{ijkl}(2)\, A_i A_j B_k B_l$. Here $A(B)$ are the boson creation (destruction) operators. Given two-body matrix elements in $V_{ijkl}(2)$ form, it is possible to write propagation formulas, in configuration space $\widetilde{m} \equiv (m_1, m_2, m_3)$, for $\langle H \rangle^{\widetilde{m}}$ and $\langle H^2 \rangle^{\widetilde{m}}$ for each member of BEGOE(2)-$S1$. Note that m_1, m_2, and m_3 are numbers of bosons in the unitary orbits with $\alpha = 1$, 2, and 3, respectively, and $m = m_1 + m_2 + m_3$. The propagation formula for energy centroid is given by

$$\langle H \rangle^{\widetilde{m}} = \sum_{\alpha}^{3} w_{\alpha\alpha} \frac{m_\alpha (m_\alpha - 1)}{2} + \sum_{\alpha<\beta}^{3} w_{\alpha\beta}\, m_\alpha m_\beta. \quad \text{(A1)}$$

Here $w_{\alpha\beta}$ are the average two-particle matrix elements for the two particles in orbits α and β and they are given by

$$w_{\alpha\beta} = \sum_{i\in\alpha, j\in\beta}^{N} \frac{V_{ijij}}{\Omega(\Omega + \delta_{\alpha\beta})}. \quad \text{(A2)}$$

The spectral variance is can be written as

$$\langle H^2 \rangle^{\widetilde{m}} = [\langle H \rangle^{\widetilde{m}}]^2 + \langle [H(\nu = 1)]^2 \rangle^{\widetilde{m}} + \langle [H(\nu = 2)]^2 \rangle^{\widetilde{m}}. \quad \text{(A3)}$$

Propagation formulas for the variances generated by the irreducible $\nu = 1$ and $\nu = 2$ parts can be written as

$$\langle [H(\nu = 1)]^2 \rangle^{\widetilde{m}} = \sum_{\alpha\beta} \left\{ \frac{m_\alpha}{\Omega} + \frac{m_\alpha (m_\alpha - \delta_{\alpha\beta})}{\Omega(\Omega + \delta_{\alpha\beta})} \right\} \sum_{i\in\alpha, j\in\beta} \left(\varepsilon^{\alpha\beta}_{ij}\right)^2 \quad \text{(A4)}$$

and

$$\langle [H(\nu = 2)]^2 \rangle^{\widetilde{m}} = \sum_{\alpha\beta\gamma\eta} \left\{ \frac{m_\alpha (m_\beta - \delta_{\alpha\beta})}{4\Omega(\Omega + \delta_{\alpha\beta})} + \frac{m_\alpha (m_\beta - \delta_{\alpha\beta})(m_\gamma - \delta_{\gamma\alpha} - \delta_{\gamma\beta})}{2\Omega(\Omega + \delta_{\alpha\beta})(\Omega + \delta_{\gamma\alpha} + \delta_{\gamma\beta})} + \frac{m_\alpha (m_\beta - \delta_{\alpha\beta})(m_\gamma - \delta_{\gamma\alpha} - \delta_{\gamma\beta})(m_\eta - \delta_{\eta\alpha} - \delta_{\eta\beta} - \delta_{\eta\gamma})}{4\Omega(\Omega + \delta_{\alpha\beta})(\Omega + \delta_{\gamma\alpha} + \delta_{\gamma\beta})(\Omega + \delta_{\eta\alpha} + \delta_{\eta\beta} + \delta_{\eta\gamma})} \right\} \sum_{i\in\alpha, j\in\beta, k\in\gamma, l\in\eta} \left(V^{\nu=2}_{ijkl}\right)^2. \quad \text{(A5)}$$

The induced sp energies $\varepsilon^{\alpha\beta}_{ij}$ in Eq. (A4) are given by $\varepsilon^{\alpha\beta}_{ij} = \sum_\beta \xi^{\alpha\beta}_{ij}\, (m_\beta - \delta_{\alpha,\beta})$, where $\xi^{\alpha\beta}_{ij} = \sum_k V^{\nu=1+2}_{ikjk}/(\Omega + 2\delta_{\alpha\beta})$. The $\nu = 1 + 2$ part of $V(2)$ with respect to the $\widetilde{m}$'s is defined by $V^{\nu=1+2}_{ijkl} = V_{ijkl} - \delta_{ik}\, \delta_{jl}\, (1 + \delta_{ij})\, w_{\alpha\beta}$ $_{i\in\alpha, j\in\beta}$. Similarly, the $\nu = 2$ part with respect to the $\widetilde{m}$'s is defined by $V^{\nu=2}_{ijkl} = V^{\nu=1+2}_{ijkl} - \delta_{ik}\xi^{\alpha\beta}_{jl}$ $_{j,l\in\alpha; i,k\in\beta}$ $- \delta_{jl}\xi^{\alpha\beta}_{ik}$ $_{i,k\in\alpha; j,l\in\beta}$ $- \delta_{il}\xi^{\alpha\beta}_{jk}$ $_{j,k\in\alpha; i,l\in\beta}$ $- \delta_{jk}\xi^{\alpha\beta}_{il}$ $_{i,l\in\alpha; j,k\in\beta}$ and these appear in Eq. (A5).

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