Random-matrix-theory approach to mesoscopic fluctuations of heat current

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We consider an ensemble of fully connected networks of N oscillators coupled harmonically with random springs and show, using random-matrix-theory considerations, that both the average phonon heat current and its variance are scale invariant and take universal values in the large N limit. These anomalous mesoscopic fluctuations is the hallmark of strong correlations between normal modes.

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I. INTRODUCTION

The study of heat conduction by phonons in disordered or chaotic structures have attracted recently considerable interest [1–3]. A central issue of these investigations is the dependence of the average heat current J on the system size N. A naive expectation is that disorder or phonon-phonon interactions scatters normal modes and induces a diffusive energy transport that leads to a normal heat conduction described by Fourier's law which states that e.g., in one dimension $J \sim N^{-1}$. Many studies [2–8], however, find that in low-dimensional chains J scales as $J \sim N^{-\alpha}$, where α is usually different from one. In fact, experiments on heat conduction in nanotubes and graphene flakes have reported observations of such anomalous behavior [9–11].

However, many real stuctures such as biological systems [12] and artificial networks in thin-film transistors and nanosensors [13] are not simple one-dimensional or two-dimensional lattices. Rather they are characterized by a complex connectivity that can be easily designed and realized in the laboratory [14–16]. Therefore, understanding the normal and anomalous heat conduction in complex networks is a timely fundamental problem.

The complexity of coherent wave interferences in such networks calls for a statistical treatment of their transport characteristics. Such statistical treatment, based on the random matrix theory (RMT) [17], proved very fruitful in various branches of physics [18–21], ranging from nuclear and atomic physics to mesoscopic physics of disordered and chaotic systems. The RMT approach often uncovers the most fundamental, universal properties of complex systems, and it is the purpose of the present paper to develop this kind of approach for the heat transport problem [22].

At the same time one needs to be aware that the actual networks have various features involved, like sparsity, finite long-range coupling, etc. In our study we do not consider all these important features at all as our primary goal is to point out that the toolbox of wave chaos and RMT modeling can be used for the study of thermal transport. On the other hand one needs to know the importance of these features. The obvious way towards achieving this goal is to solve and understand first the most simple RMT case. Any deviations from its predictions will signify the importance of these other features.

In this paper we address heat transport and the associated sample-to-sample mesoscopic fluctuations of complex networks of N equal masses connected with one another via random harmonic springs. In Sec. II we present the theoretical model associated with a network of N coupled oscillators and express (in the weak coupling limit with the bath) the heat current in terms of the normal modes of the corresponding Hamiltonian. In Sec. III we write down the random matrix corresponding to our model. We show that the standard RMT models cannot explain the scaling form of the variance. Instead, the statistical description of heat transport can be effectively described by an ensemble of random matrices with diagonal elements that fluctuate with a variance N times larger than the corresponding variance of the off-diagonal elements. Using RMT considerations we show that both the average heat current $\langle J\rangle$ and its variance $(\Delta J)^2$ are scale invariant and assume universal values in the large N limit. These anomalous mesoscopic fluctuations are the hallmark of strong correlations between normal modes of the system. In Sec. IV we investigate the effects of boundary conditions. For moderate size networks, with random springs taken from a distribution with variance $\sigma^2 < 1/N$, we find that the heat transport is sensitive to the boundary conditions imposed on the two end sites which are coupled to the thermal baths. A numerical confirmation that our system reach nonequilibrium steady state is given in Sec. V. Our conclusions are given in Sec. VI. We hope that our analysis will motivate the use of RMT models and provide new insight into the mesoscopic fluctuations of heat transport.

II. ZERO-DIMENSIONAL HARMONIC CHAIN

We consider a network of N harmonic oscillators of equal masses $m = m_0$. The system is described by the Hamiltonian [24],

$$\mathcal{H} = \tfrac{1}{2} P^T \hat{M}^{-1} P + \tfrac{1}{2} Q^T \hat{\Phi} Q, \tag{1}$$

where $Q^T \equiv (q_1, q_2, \ldots, q_N)$, $P^T \equiv (p_1, p_2, \ldots, p_N)$, and q_n, p_n are, respectively, the individual oscillator displacements and momenta. The mass matrix is $M_{nm} = \delta_{nm} m_0$, and $\hat{\Phi}$ is the force matrix that also contains information about the boundary conditions (b.c.). For a fully connected network of coupled oscillators with free b.c. $\hat{\Phi}$ takes the form $\Phi_{nm} = (\sum_l k_{nl})\delta_{nm} - k_{nm}$ where k_{nm} are the spring coupling constants. These spring constants k_{nm} are chosen to be symmetric ($k_{nm} = k_{mn}$) and uniformly distributed according to $k_{nm} \in$

$[-\frac{W}{2}+1,\frac{W}{2}+1]$ where the disorder strength parameter W has to be smaller than 2 in order to ensure that $k_{nm}\geqslant 0$. In the case of fixed b.c. $\hat{\Phi}$ has to be modified by considering the coupling of the first and last oscillator to hard walls, i.e., $\Phi^{\rm fix}_{nm}=\Phi_{nm}+(k_{01}\delta_{n1}\delta_{m1}+k_{NN+1}\delta_{nN}\delta_{mN})$. We note that Hamiltonian Eq. (1) describes a scalar phonon model, where the vectorial properties of the modes have not been taken into consideration (and thus the matrix has N and not $3N$ modes) [24].

Next, we want to study the nonequilibrium steady state (NESS) of this network driven by a pair of Langevin reservoirs set at temperatures $T_1^{\rm B}$ and $T_N^{\rm B}$ (we assume $T_1^{\rm B}>T_N^{\rm B}$), and coupled to the first $n=1$ and last $n=N$ masses with a coupling strength γ. The corresponding equations of motion are $\dot{q}_n=\partial\mathcal{H}/\partial p_n$, $\dot{p}_n=-\partial\mathcal{H}/\partial q_n+(-\gamma p_n/m_0+\sqrt{2\gamma T_n^{\rm B}}\zeta_n)(\delta_{n1}+\delta_{nN})$, where $\zeta_n(t)$ is delta-correlated white noise $\overline{\zeta_n(t)\zeta_{n'}(t')}=\delta_{nn'}\delta(t-t')$. The NESS current is evaluated as $\overline{J}=\frac{\gamma}{m_0}(T_1^{\rm B}-T_1)=\frac{\gamma}{m_0}(T_N-T_N^{\rm B})$ where the temperature of the nth oscillator is defined as $T_n\equiv\overline{p_n^2}/m_0$. The notation $\overline{\cdots}$ which will be implicitly assumed from now on, indicates the thermal statistical average. In Sec. V we show some molecular dynamics simulations that confirm that our system reaches a NESS.

In our analysis below we will consider the weak coupling γ limit. In this case, it was shown in Ref. [1,2] that

$$J=\sum_\mu J^{(\mu)};\quad J^{(\mu)}=C_0\frac{I_1^{(\mu)}I_N^{(\mu)}}{I_1^{(\mu)}+I_N^{(\mu)}},\tag{2}$$

where $I_n^{(\mu)}\equiv|\psi_n^{(\mu)}|^2$ and $\psi_n^{(\mu)}$ indicates the nth component of the μth normal mode of the Hamiltonian Eq. (1) and the coefficient $C_0\equiv\frac{\gamma}{m_0}(T_1^{\rm B}-T_2^{\rm B})$. Thus the analysis of heat flux J reduces to the study of the normal modes of Hamiltonian $\mathcal{H}$ given by Eq. (1).

III. RANDOM-MATRIX-THEORY FORMULATION

We separate out the random component of the spring constants and rewrite them as $k_{nm}=1-W_{nm}$ where $W_{nm}\in[-\frac{W}{2},\frac{W}{2}]$. The force matrix $\hat{\Phi}$ can be decomposed into a constant matrix $\hat{A}$ and a random part $\hat{R}$:

$$\hat{\Phi}=\hat{A}+\hat{R}\quad\text{where}\quad\hat{A}\equiv N\hat{1}-\hat{U};\quad\hat{R}\equiv\hat{D}+\hat{W},\tag{3}$$

where $\hat{1}$ is the $N\times N$ unit matrix, $\hat{U}$ is a matrix whose all elements are equal to unity, i.e., $U_{nm}=1$, $\hat{D}$ is a diagonal matrix with $D_{nn}=-\sum_{l\neq n}W_{nl}$, and $\hat{W}$ is a random matrix (RM) defined below. The above decomposition allows us to distinguish the various contributions. The matrix $\hat{W}$ can be treated as a "standard" RMT ensemble (note though that it has zero diagonal elements) [25]. It is convenient to rewrite it as $\hat{W}=\sigma\hat{W}_0$ where $\hat{W}_0$ is an RM with elements having unit variance where $\sigma^2\equiv(\Delta W_{nm})^2=W^2/12$. The diagonal matrix $\hat{D}$ has Gaussian distributed random elements with $\langle D_{nn}\rangle=0$ and variance $(\Delta D_{nn})^2=(N-1)\sigma^2$.

A. Standard RMT considerations

The constant matrix $\hat{A}$ can be diagonalized exactly. It has (a) one eigenvalue $\omega_0=0$ with a corresponding eigenvector $(1/\sqrt{N})(1,1,1,\ldots,1)^T$ and (b) $N-1$ degenerate eigenvalues $\omega_\mu=N$ ($\mu=1,2,\ldots N-1$). Now, neglect for the moment $\hat{D}$, and consider adding to $\hat{A}$ the RM $\hat{W}$,

$$\hat{\Phi}'=\hat{A}+\sigma\hat{W}_0.\tag{4}$$

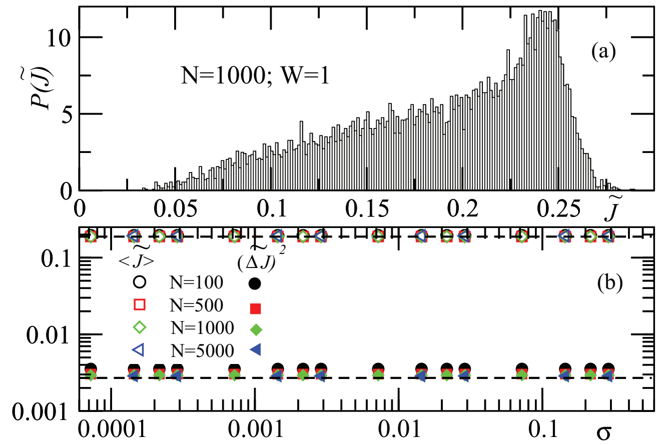

FIG. 1. (Color online) Free boundary conditions. (a) A typical distribution of rescaled heat flux $\tilde{J}\equiv J/C_0$ for a network of $N=10^3$ oscillators and $W=1$. (b) The rescaled average heat current $\langle\tilde{J}\rangle$ (open symbols) and variance $(\widetilde{\Delta J})^2\equiv(\Delta J)^2/C_0^2$ (solid symbols) versus σ. Various system sizes N (indicated in the figure) have been used. The dashed lines are the results of D-RMT ensemble. Both in (a) and (b) we have used Eq. (2) in order to evaluate the heat current J.

For any arbitrary small σ, the $N-1$ degeneracy will be removed and the corresponding eigenvectors will be those of an $(N-1)\times(N-1)$ RM. The $(N-1)$-time degenerate level is broadened into a band of width $\sim\sigma\sqrt{N}$. The perturbation theory applies for $\sigma\sqrt{N}<N$, i.e., for $\sigma<\sqrt{N}$. However, even for larger σ the RMT still applies because then we can simply neglect the matrix $\hat{A}$ in Eq. (4). In short, for small σ we have an RMT for $(N-1)$-rank matrices [the contribution to current of the level with $\omega_0\approx 0$ can be neglected, in comparison to the $(N-1)$ levels], whereas for large σ we have an RMT for N-rank matrices. Thus, in the large N limit we treat Eq. (4) as an ensemble of $N\times N$ GOE matrices [18].

Normalization requires that $\langle I_1^{(\mu)}\rangle=\langle I_N^{(\mu)}\rangle=\langle I_n\rangle=1/N$. Defining a rescaled variable $X_n^{(\mu)}=I_n^{(\mu)}/\langle I_n\rangle$, we can rewrite Eq. (2) as

$$J=\frac{C_0}{N}Z;\quad Z=\sum_{\mu=1}^N z^{(\mu)};\quad z^{(\mu)}\equiv\frac{X_1^{(\mu)}X_N^{(\mu)}}{X_1^{(\mu)}+X_N^{(\mu)}}.\tag{5}$$

According to the standard RMT, and omitting the mode label μ, the joint probability distribution of the rescaled eigenmode intensities X_n is a product of two Porter-Thomas distributions $P(X_1,X_N)=(1/2\pi)(1/\sqrt{X_1X_N})\exp[-(X_1+X_N)/2]$. Assuming further that the various $z^{(\mu)}$ terms appearing in Eq. (5) are statistically independent we get

$$\langle J\rangle=\frac{1}{4}C_0;\quad(\Delta J)^2=\frac{1}{8N}C_0^2.\tag{6}$$

Comparison of these theoretical predictions with a numerical evaluation of the mean and the variance of heat current J via Eq. (2) [see Fig. 1(b)] leads us to conclude that standard RMT considerations describe well the scaling of the *average* current but *not the variance*. This is due to the fact that in the standard

RMT case different eigenvectors (and, thus, different $z^{(\mu)}$'s) are only weakly correlated, due to their orthogonality, i.e.,

$$\langle z^{(\mu)} z^{(\nu)} \rangle = \langle z^{(\mu)} \rangle \langle z^{(\nu)} \rangle + \frac{a}{N} = \langle z \rangle^2 + \frac{a}{N}, \tag{7}$$

with some constant coefficient a. In fact, these weak correlations result in the replacement of the coefficient 1/8 in Eq. (6) by the (numerically evaluated) coefficient 1/16.

Below we show that the matrix D (which is not considered up to now) induces strong correlations between different $z^{(\mu)}$'s, thus, invalidating the assumption which led to Eq. (6) for the variance. To obtain the correct description of the variance, it is necessary to treat the full force matrix, as given in Eq. (3).

B. Strongly fluctuating diagonal elements and D-RMT ensemble

We now consider the ensemble given by Eq. (3). Again for large N, the matrix $\hat{A}$ has no effect, so it is enough to understand the eigenvectors of the random matrix $\hat{R}$. The eigenvalues of $\hat{D}$ are of order $|D_{nn}| \sim \sigma\sqrt{N}$, so that they occupy a band of order $\sigma\sqrt{N}$ and are separated by a typical energy interval $\sigma\sqrt{N}/N = \sigma/\sqrt{N}$. The same is true for the eigenvalues of $\hat{W}$. In this sense $\hat{D}$ and $\hat{W}$ are "of the same strength" and neither can be treated as perturbation to the other. However, the qualitative understanding of the eigenvectors of the combined matrix $\hat{R}$ is along the following lines: The eigenvectors of $\hat{D}$ are localized on the individual sites, i.e., the μth eigenvector is $\psi_n^{(\mu)} = \delta_{n\mu}$. The matrix $\hat{W}$ mixes these eigenvectors, so that eigenvectors of $\hat{R}$ are spread over all sites and resemble those of a standard RMT. Therefore $\langle J \rangle$ is qualitatively the same as the standard RMT result of Eq. (6). The only difference is that the coefficient 1/4 now assumes the numerical value ≈ 0.19 [see Fig. 1(b)].

As far as the variance $(\Delta J)^2$ is concerned we get results that are qualitatively different from the standard RMT result of Eq. (6). It turns out that in this case each eigenvector of $\hat{R}$ "remembers" the set $(D_{11}, \ldots, D_{NN})$ of the eigenvalues of $\hat{D}$ so that correlations between different eigenvectors of $\hat{R}$ are significantly stronger than those for the standard RMT. Namely the mode-mode correlations between the different $z^{(\mu)}$'s of the matrix $\hat{R}$ are described by

$$\langle z^{(\mu)} z^{(\nu)} \rangle = \langle z^{(\mu)} \rangle \langle z^{(\nu)} \rangle (1 + \epsilon) = \langle z \rangle^2 (1 + \epsilon), \tag{8}$$

where ϵ is a constant. Using Eq. (8) we calculate the variance $(\Delta Z)^2$ of the random variable Z [see Eq. (5)]:

$$(\Delta Z)^2 = N^2 \epsilon \langle z^2 \rangle + O(N). \tag{9}$$

Expressing $(\Delta J)^2$ in terms of Z via Eq. (5) we get

$$(\Delta J)^2 = C_0^2 \epsilon \langle z \rangle^2, \quad \langle z \rangle \approx 0.19. \tag{10}$$

Direct numerical evaluation of the variance $(\Delta J)^2$ based on Eq. (2) confirms the above theoretical estimates. In Fig. 1(b) we show some of our numerical results for rescaled variance $(\widetilde{\Delta J})^2 \equiv (\Delta J)^2 / C_0^2$. The data indicate that $(\widetilde{\Delta J})^2$ is scale invariant for any disorder strength σ. Further $1/N$ numerical analysis allows us to extract the asymptotic value $\epsilon \approx 0.075$.

Thus, the large anomalous current fluctuations originate from the large variances of the matrix elements D_{nn}, in combination with the RMT W (D alone, being diagonal in n, cannot produce any current).

We have also checked that correlations between the matrices $\hat{D}$ and $\hat{W}$ do not play a role in our arguments. Detail numerical analysis indicates that if instead of the actual $\hat{D}$ (i.e., $D_{nn} = -\sum_i W_{ni}$) we consider a diagonal random matrix completely independent of $\hat{W}$ so that

$$(\Delta R_{nm})^2 = \sigma^2 [1 + (N-1)\delta_{nm}], \tag{11}$$

we still obtain the same behavior for $\langle J \rangle$ and $(\Delta J)^2$ [dashed lines in Fig. 1(b)]. We remark that this kind of D-RMT ensembles have previously appeared in the context of mescoscopic physics [26].

IV. FIXED BOUNDARY CONDITIONS

Finally we investigate the effect of b.c. on the statistics of heat flux. We consider the other limiting case of fixed b.c. We assume that the first and the last oscillator are coupled to the left and right walls with spring constants $k_{01} = 1 - W_{01}$ and $k_{NN+1} = 1 - W_{NN+1}$, respectively, which are taken from the same ensemble of random springs as the ones in the bulk of the network. The random components are then included in the matrix elements D_{11} and D_{NN}, respectively. The constant matrix $\hat{A}$ also changes to $\hat{A}^{\text{fix}} = \hat{A} + \hat{C}$ where $C_{nm} = \delta_{n1}\delta_{m1} + \delta_{nN}\delta_{mN}$. This results in a slight shift of the zero mode $\omega_0 = 0$ of the matrix $\hat{A}$ together with a "deformation" of the $(1,1,\ldots,1)^T$ eigenvector. Contribution of this level to the total current is of order $1/N$, and it is disregarded below.

In addition, two new levels emerge from the $N-1$ degenerate subspace of the matrix $\hat{A}$. The first one has the highest energy $\omega_{N-1} = N + 1$ with a corresponding eigenmode $\psi^{(N-1)} = (1/\sqrt{2})(1,0,\ldots,-1)^T$. This is an exact eigenvalue and eigenvector of $\hat{A}^{\text{fix}}$. The second level is slightly lower than $N+1$ (approximately by $2/N$) and its eigenvector is symmetric, i.e., $\omega_{N-2} \approx N + 1 - 2/N$ with $\psi^{N-2} \approx (1/\sqrt{2})(1 - 1/N, -2/N, \ldots, -2/N, 1 - 1/N)^T$. Below we refer to these states as "surface" modes.

It turns out that for a network described by the constant force matrix $\hat{A}^{\text{fix}}$, most of the current is carried by the two surface modes. Using Eq. (2) we find that $J^{(N-1)} = J^{(N-2)} = \frac{1}{4}C_0$. At the same time, the remaining $N-3$ degenerate modes *do not contribute* to the current (in the large N limit). Since any of these eigenvectors $\psi^{(\mu)}$ is orthogonal to both $\psi^{(N-1)}$ and $\psi^{(N-2)}$ we get that $\psi_1^{(\mu)} = \psi_N^{(\mu)}$ and $\psi_1^{(\mu)} = -\psi_N^{(\mu)}$. These constraints are satisfied simultaneously only if $\psi_1^{(\mu)} = \psi_N^{(\mu)} = 0$ for any $\mu = 1, \ldots, N-3$. Thus the total heat current is

$$J = \sum_{\mu=1}^{N} J^{(\mu)} \approx \frac{1}{2} C_0. \tag{12}$$

Equation (12) will hold as long as the RM $\hat{R}$ does not destroy the pair of states $\psi^{(N-1)}$ and $\psi^{(N-2)}$. As σ increases we observe a coupling of the two states towards a linear combination, i.e., $(1/\sqrt{2})[\psi^{(N-1)} \pm \psi^{(N-2)}]$. The origin of this reorganization is traced to the matrix $\hat{D}$ which in the $\{\psi^{(N-1)}, \psi^{(N-2)}\}$ subspace, would produce a pair of eigenvalues separated by a distance of order $\sigma\sqrt{N}$. This has to be compared to the separation of order $1/N$ between the surface mode eigenvalues ω_{N-1} and ω_{N-2} of the matrix $\hat{A}^{\text{fix}}$. When σ reaches a value $\sigma_c \sim N^{-3/2}$ the two "surface" eigenstates are destroyed giving rise to a set of new modes that have components $(0,\ldots,0,1)^T$ and $(1,0,\ldots,0)^T$. Consequently, the average current will drop to approximately a zero value. As σ continues to increase, the matrix $\hat{W}$ lifts the degeneracy of the $N-3$ levels centered around $\omega = N$ and creates a spectral band of size $\delta_W \sim \sigma\sqrt{N}$. For some critical value of $\sigma = \sigma_{\rm RMT} \sim 1/\sqrt{N}$ the bandwidth δ_W becomes as broad as the gap that separates the degenerate states from the surface states. The latter now merge with the states in the band, and the RMT results are recovered.

For disorder strength such that the dominant contribution comes only from the two surface states, a quantitative description of the heat transport can be achieved by considering a two-level system described by a 2×2 Hamiltonian $\hat{\Phi}^{(2)} = \hat{A}^{(2)} + \hat{D}^{(2)}$. The diagonal matrix $\hat{D}^{(2)}$ has elements $D^{(2)}_{nm} = \sqrt{N}W_{nm}\delta_{nm}$ where $W_{nm}\in[-W/2,W/2]$. The perfect system is described (in the site representation) by the 2×2 matrix $\hat{A}^{(2)} = -\frac{1}{N}\hat{U}^{(2)}$ where $U^{(2)}_{nm} = 1$. The matrix $\hat{A}^{(2)}$ has eigenvalues $\omega_1 = 0, \omega_2 = -2/N$ and corresponding eigenvectors $\psi^{(1)} = (1/\sqrt{2})(1,-1)^T$ and $\psi^{(2)} = (1/\sqrt{2})(1,1)^T$. We can diagonalize $\hat{\Phi}^{(2)}$ and get the corresponding eigenvectors. Using Eq. (2) we obtain $J_2 = \frac{2C_0}{4+N^3(W_{11}-W_{22})^2}$. The corresponding average heat current is

$$\langle J_2\rangle = 2C_0\frac{w\arctan\left[\frac{w}{2}\right] - \log\left[1+\left(\frac{w}{2}\right)^2\right]}{w^2};\quad w = N^{3/2}W, \tag{13}$$

while for the variance we get

$$(\Delta J_2)^2 = \frac{w^3\arctan\left[\frac{w}{2}\right] - 8\left(\log\left[1+\left(\frac{w}{2}\right)^2\right] - w\arctan\left[\frac{w}{2}\right]\right)^2}{2w^4} \times C_0^2. \tag{14}$$

These theoretical predictions are compared in Figs. 2 and 3 with the numerically evaluated average heat current and variance via Eq. (2) for various system sizes N and disorder strength W. Obviously Eqs. (13) and (14) do not apply for $\sigma \geqslant \sigma_{\rm RMT} \sim N^{-1/2}$ when RMT dominates the transport.

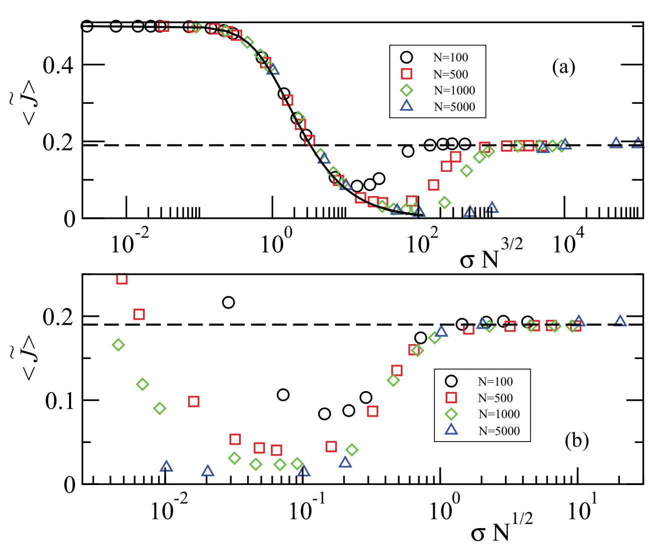

FIG. 2. (Color online) Fixed boundary conditions. The rescaled average heat current $\langle\tilde{J}\rangle \equiv \langle J\rangle/C_0$ vs disorder strength for various system sizes N. In (a) we scale the x axis as $\sigma N^{3/2}$ while in (b) (RMT domain) we scale it as $\sigma N^{1/2}$. The dashed lines are the predictions of D-RMT while the solid lines represent Eqs. (13) and (14).

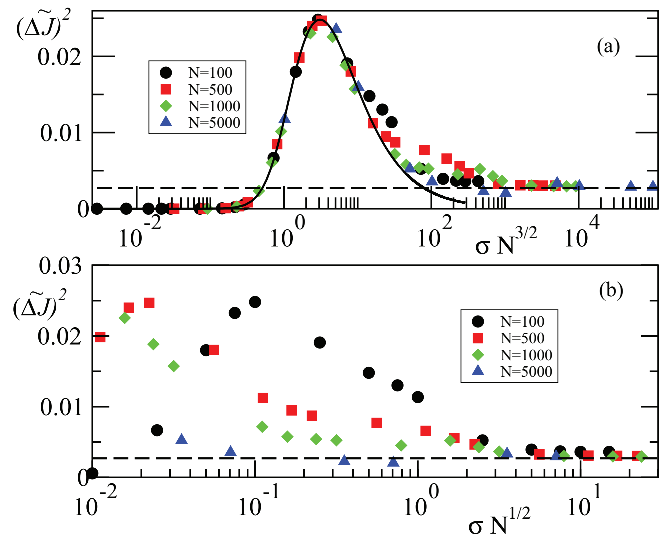

FIG. 3. (Color online) Fixed boundary conditions. The rescaled variance $(\widetilde{\Delta J})^2 \equiv (\Delta J)^2/C_0^2$ vs disorder strength for various system sizes N. In (a) we scale the x axis as $\sigma N^{3/2}$ while in (b) (RMT domain) we scale it as $\sigma N^{1/2}$. The dashed lines are the predictions of D-RMT while the solid lines represent Eqs. (13) and (14).

V. NESS FOR A FULLY CONNECTED NETWORK

In order to establish that the fully connected network of harmonic oscillators Eq. (1) reaches the NESS, we have also performed independent molecular dynamics (MD) simulations for both free and fixed boundary conditions. Since these simulations are time consuming we confine ourselves to moderate N sizes. In Fig. 4 we report such representative simulations for a case of a fully connected network of $N=5$ coupled oscillators with random springs k_{nm} taken from a uniform distribution $k_{nm}\in[1-W/2;1+W/2]$ and

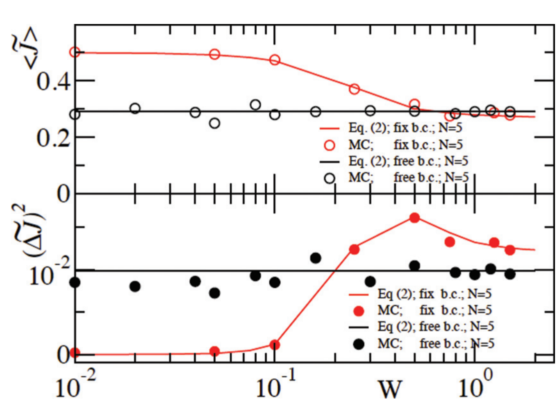

FIG. 4. (Color online) Molecular dynamics (MD) simulations (open and solid symbols) for the case of a network of $N=5$ fully connected oscillators. The results from the MD are compared with the results coming from Eq. (2). A nice agreement, both for the mean heat current (upper) and the variance (lower) is observed, indicating that our system can reach a NESS.

compare these results with the ones coming from a direct diagonalization of the associated force matrix $\hat{\Phi}$ with the use of Eq. (2).

In Fig. 4 open symbols correspond to the average heat current and full symbols to its variance evaluated from the MD simulations, while the solid lines are the results of the diagonalization method that makes use of Eq. (2). For the MD simulations we have used typically 100 disorder realizations (this has to be compared to the diagonalization method where typically we had more than 5000 realizations). An additional time average (over the last 20 time units) was performed in order to average out the oscillations of the chain elements. In order to check the convergence of the MD simulations, we have compared the flux J for two different times (the time t is measured in units of mean inverse frequency). A convergence towards the theoretical results of Eq. (2) is evident indicating that our system reached a NESS.

VI. CONCLUSIONS

In conclusion, we have employed RMT modeling as a valuable tool for the analysis of mesoscopic fluctuations of heat current J in complex (chaotic) networks. For the most basic chaotic system consisting of a fully connected network of random springs we have found that both the average heat current and its variance are scale invariant. For large N limit, these quantities assume a universal value which is independent of the specific boundary conditions. Our analysis indicated that the statistical properties of J are affected by the existence of correlations between normal modes. It would be interesting to investigate the statistical properties of heat current for other more realistic network geometries where finite long-range coupling [27] and/or sparsity are considered, or anharmonicity is present [1,28] and establish analogies with mesoscopic phenomena observed in the realm of electron transport.

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