Stochastic theory of an optical vortex in nonlinear media

Hiroshi Kuratsuji*

Faculty of Science and Engineering, Ritsumeikan University-BKC, Noji-Hill, Kusatsu City, 525-8577, Japan

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A stochastic theory is given of an optical vortex occurring in nonlinear Kerr media. This is carried out by starting from the nonlinear Schrödinger type equation which accommodates vortex solution. By using the action functional method, the evolution equation of vortex center is derived. Then the Langevin equation is introduced in the presence of random fluctuations, which leads to the Fokker-Planck equation for the distribution function of the vortex center coordinate by using a functional integral. The Fokker-Planck equation is analyzed for a specific form of pinning potential by taking into account an interplay between the strength of the pinning potential and the random parameters, diffusion and dissipation constants. This procedure is performed by several approximate schemes.

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I. INTRODUCTION

The propagation of light waves in various refractive media has long been a central subjects in optics and electromagnetic theory [1,2]. In particular, wave propagation in a nonlinear medium has been thoroughly investigated for the last half century (see the general Refs. [3,4]). Among other observations, a notable one is that there appears a soliton-like excitation in the Kerr medium, which was originally pointed out in [5]. Following this seminal idea, a theory of optical vortex was developed [6,7], which is alternatively called the "dark soliton" in defocusing Kerr media. This shares the idea with a superfluid vortex, the vortex that occurs in superfluid He [8]; namely, the basic equation is described by the nonlinear Schrödinger equation. The occurrence of the optical vortex (OV) is of significance from an actual point of view; for example, it may realize a formation of "optical wave guides" in a bright background, without being builtin from the outset (see, e.g., [6]). Besides the spacial formation of an OV [7], its evolutional aspect [9,10] has been also investigated.

Apart from the optical vortex soliton inherent in the nonlinear Schrödinger equation, we also mention the works on the Gaussian-Laguerre optical vortex (or twisted modes) [11], which may be relevant to our attempt that will be discussed below.

In this note we explore a quite different aspect of the evolution of the optical vortex inherent in the nonlinear media. What we are concerned with here is a random behavior of the OV that is naturally expected to occur following the analogy with the random theory of quantum vortex [12]. Indeed, it is generally known that optical substances suffer from various sorts of impurities of a random nature caused by, e.g., the process of fabrication of the substances, which is not well controllable. Thus we need to understand how to deal with such irregularity. So far, a random approach has been investigated in the context of birefringence [13–15], for which stochastic equations (the Langevin and Fokker-Planck equation) for the Stokes parameters are considered. In this way, it is quite intriguing to consider the random effect for the case of optical vortex.

The main purpose here is to develop a stochastic theory of optical vortex in the framework of the Langevin and Fokker-Planck (FP) equations. To carry out this, we start with the nonlinear Schrödinger-type equation, which is derived from the Maxwell-Helmholtz equation using the so-called envelope approximation [16].

The first task is to find out the vortex solution for the nonlinear Schrödinger equation in such a form that it incorporates the vortex center coordinate in a parametric form. By applying the action functional method to this parametric form of solution, the equation of motion of the coordinate of the "vortex center" is constructed in the presence of a pinning potential that represents the interaction between the vortex and embedded impurities.

The next task is to convert the equation of motion to the Langevin equation by adding the random forces. Then this is converted to the Fokker-Planck equation for the distribution function of the *vortex center coordinate* by adopting a functional integral technique [17,18]. The FP equation has a peculiar structure, and a special technique is required to handle it. Taking account of this feature, we try to solve the FP equation while paying special attention to the mutual role between the pinning potential and the stochastic parameters, dissipation and diffusion. As for the random analysis that shares the similar spirit as the present attempt, we mention the article which deals with the reduced random equation starting from the nonlinear Schrödinger equation with additive noises [19] (see also the argument in Sec. IV A).

The paper is organized as follows. In the next section, we set up the nonlinear Schrödinger equation. In Sec. III, the profile of OV is given together with the equation of motion of the vortex center. In Sec. IV, the Langevin and Fokker-Planck equations are constructed. In Sec. V, the analysis of the FP equation is presented. The last section is a summary.

II. NONLINEAR SCHRÖDINGER EQUATION

We start with a description of the light wave traveling through the nonlinear substance. Although the following argument shares the central idea with the preceding papers [6,7], we use several simplifications and refinements in order to develop the stochastic idea in a concise way.

*kra@se.ritsumei.ac.jp

The basic field equation with which we are concerned is the Maxwell-Helmholtz equation for the electric field $\mathbf{E}$ written in a complex form [2]

$$\frac{\partial^2 \boldsymbol{E}}{\partial z^2} + \nabla^2 \boldsymbol{E} + \left(\frac{\omega}{c}\right)^2 \epsilon \boldsymbol{E} = 0. \tag{1}$$

Here we adopt a *slab* geometry; namely, (x,y) means the coordinate that is perpendicular to propagation direction designated by z and $\boldsymbol{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$. The symbol ϵ means the dielectric constant

$$\epsilon = \bar{\epsilon} + g|\mathbf{E}|^2, \tag{2}$$

where $\bar{\epsilon}$ denotes the linear dielectric constant and the second term represents the one induced by the Kerr effect characterized by the coupling constant g. For the case of a weak field, $\boldsymbol{E}$ can be written in the form

$$\boldsymbol{E}(x,y,z) = \psi(x,y,z)\exp[ikn_0 z], \tag{3}$$

namely, ψ gives the modification from the plane wave of the wave number $k = \frac{\omega}{c}$, which propagates along the z direction in the vacuum with the refractive index $n_0 (\equiv \sqrt{\epsilon_0})$. The amplitude ψ is described by a single component, namely, ψ is regarded as a scalar wave. Here we note that ψ is in general a two-component wave that takes into account polarization, but for the case of isotropic medium it is not necessary to consider the polarization effect.

We now adopt the envelope approximation; namely, by substituting Eq. (3) into Eq. (1) and noting the slowly varying nature of ψ, i.e., $|\frac{\partial^2 \psi}{\partial z^2}| \ll k|\frac{\partial \psi}{\partial z}|$, we can derive the equation for the amplitude ψ; namely, we keep only the first-order derivative $\frac{\partial \psi}{\partial z}$ as well as the Laplacian with respect to (x,y). We thus have the Schrödinger-type equation for ψ:

$$i\lambda \frac{\partial \psi}{\partial z} = \left(-\frac{\lambda^2}{n_0}\nabla^2 + m + g|\psi|^2\right)\psi \equiv \hat{H}\psi, \tag{4}$$

with $m = \bar{\epsilon} - n_0^2$ (which is chosen to be positive) and λ is the wavelength divided by 2π. The remaining term, which comes from the linear refractive index, is put aside for the moment. Equation (4) is just the so-called nonlinear Schrödinger equation for which the potential is given by $V = g|\psi|^2$. In what follows we consider the case that g is negative; we put $g \to -g (g > 0)$. This fact corresponds to the dark soliton; namely, the core part of the vortex forms a hollow which looks like a spot in the bright region [6,7]. Here, in connection with the above formulation, we mention typical analytic treatments for two-dimensional vortices (or quasisolitons) in defocusing media [20] and Bose-Einstein condensates [21].

III. EVOLUTION OF VORTEX

A. Action functional form

We now introduce the "quantum" action leading to the Schrödinger-type equation (4):

$$I = \int \left[\frac{i\lambda}{2}\left(\psi^* \frac{\partial}{\partial z}\psi - \text{c.c.}\right) - \psi^* \hat{H}\psi\right] d^2x\, dz$$
$$\equiv \int (L_C - H)\, dz. \tag{5}$$

The nonlinear Schrödinger equation is derived from the variational equation, $\delta I = 0$. By writing the field ψ in a polar form $\psi = F\exp[i\phi]$, the Lagrangian $(L \equiv L_C - H)$ is expressed in a form of fluid-dynamical form. The first term L_C, which is called the canonical term, is given as

$$L_C = \lambda \int F^2 \frac{\partial \phi}{\partial z} d^2x, \tag{6}$$

and the Hamiltonian term consists of the kinetic energy term H_T and the nonlinear interaction potential, which is written in terms of the amplitude F and phase angle ϕ:

$$H = H_T + V, \quad V = \int (mF^2 - gF^4)\, d^2x,$$
$$H_T = \frac{\lambda^2}{n_0}\int \boldsymbol{\nabla}\psi^\dagger \boldsymbol{\nabla}\psi\, d^2x$$
$$= \frac{\lambda^2}{n_0}\int \left[(\nabla F)^2 + F^2(\nabla\phi)^2 + \frac{n_0 m}{\lambda^2}F^2\right] d^2x. \tag{7}$$

The second term is the fluid kinetic energy if we define the "velocity field "as $\boldsymbol{\nabla}\phi = \mathbf{v}$. In this way, if the wave function parametrically includes some *collective coordinate*, the variational equation derives the evolution equation for this coordinate (see below).

B. Vortex profile

Now we shall examine a profile of vortex. Let us put an ansatz for the phase function,

$$\phi = \tan^{-1}\frac{y}{x}. \tag{8}$$

Using this ansatz, it is natural to assume that the density profile is a function of $r = \sqrt{x^2 + y^2}$; $F = F(r)$. Thus the Hamiltonian is written as

$$H = \frac{\lambda^2}{n_0}\int \left[\left(\frac{dF}{dr}\right)^2 + \frac{F^2}{r^2} + \frac{n_0 m}{\lambda^2}F^2 - \frac{gn_0}{\lambda^2}F^4\right] r\, dr$$
$$\equiv \frac{\lambda^2}{n_0}\int \mathcal{H}\, dr. \tag{9}$$

From the variation equation $\frac{d}{dr}(\frac{\partial \mathcal{H}}{\partial F'}) - \frac{\partial \mathcal{H}}{\partial F} = 0$, we obtain the equation for F [8]:

$$\frac{d}{dr}\left(r\frac{dF}{dr}\right) - \frac{F}{r} + \frac{2gn_0}{\lambda^2} r\left(F^3 - \frac{m}{2g}F\right) = 0. \tag{10}$$

This has the asymptotic form for $F(r)$: $F^2(\infty) = \frac{m}{2g} \equiv \rho_0$, which should coincide with the strength of light that is fixed as the condition imposed from the outset. By using the scaling $r = \xi\sqrt{\frac{\lambda^2}{n_0}}, F = \sqrt{m/2g}G$, it follows that [22]

$$\frac{d^2G}{d\xi^2} + \frac{1}{\xi}\frac{dG}{d\xi} - \frac{G}{\xi^2} - G(1 - G^2) = 0. \tag{11}$$

The behavior of the solution will be given by imposing the boundary condition at the origin $G(0) = 0$. The asymptotic behavior at $\xi = 0$ and $\xi = \infty$ is as follows: near $\xi = 0$, we get the modified Bessel function $G(\xi) \sim I_1(\xi)$ if neglecting the nonlinear term. On the other hand, at $\xi = \infty$, by expanding $G = 1 + g$, we obtain $\frac{d^2 g}{d\xi^2} + 2g \simeq 0$, which leads to

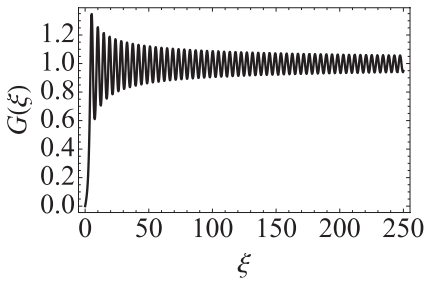

FIG. 1. Vortex profile plotted in a dimensionless form using $[\xi, G(\xi)]$. This indicates an oscillatory convergence to the asymptotic value at infinity.

$g(\xi) \simeq \exp[i\sqrt{2}\xi]$ with arbitrary small amplitude. Taking into account this asymptotic behavior, the numerical solution is given and the result is depicted in Figs. 1 and 2. Apart from the numerical profile of the vortex, we can simulate the solution by a simple guess, which may be given as $G(\xi) \simeq 1 - \exp[-\xi]$. This may be used later for the solution of the Fokker-Planck equation.

C. Equation of motion

Having constructed an explicit form for the vortex solution, we now consider the evolutional behavior for a single vortex with respect to the propagation distance z. Following the procedure used in the case of superfluid vortex [23], let us introduce the coordinate of the center of vortex, $\boldsymbol{R}(z) = (X(z), Y(z))$, by which the vortex solution is parameterized such that $F(\boldsymbol{x} - \boldsymbol{R}(z))$ and $\phi(\boldsymbol{x} - \boldsymbol{R}(z))$. By using this parametrization, the canonical term L_C is written as

$$L_C = \lambda \int \rho \mathbf{v} \cdot \dot{\boldsymbol{R}}\, d^2x, \tag{12}$$

where we put $\rho = F^2$ and use the relation $\frac{\partial \phi}{\partial z} = \frac{\partial \phi}{\partial \boldsymbol{R}} \dot{\boldsymbol{R}}, \frac{\partial \phi}{\partial \boldsymbol{R}} = -\nabla\phi$ with $\dot{\boldsymbol{R}} \equiv \frac{d\boldsymbol{R}}{dz}$. With the aid of the Euler-Lagrange equation, we obtain the equation of motion for the center of vortex [23]. The procedure is given in what follows. By differentiating Eq. (12) under the integral symbol, we get

$$\frac{d}{dz}\left(\frac{\partial L_c}{\partial \dot{X}}\right) - \frac{\partial L_c}{\partial X} = \lambda \int \left\{\frac{\partial}{\partial x}(\rho v_y) - \frac{\partial}{\partial y}(\rho v_x)\right\} d^2x \dot{Y},$$

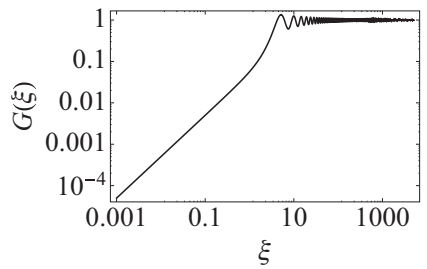

FIG. 2. Vortex profile: log-log plot of Fig. 1. The vortex core is magnified by this plot.

where we use $\frac{\partial(\rho v)}{\partial X} = -\frac{\partial(\rho v)}{\partial x}$. In vector notation, it is written as

$$\left[\int \{\nabla \times (\rho \mathbf{v})\} d^2x\right] \times \dot{\mathbf{R}}. \tag{13}$$

The integral is converted to the line integral

$$\int \{\nabla \times (\rho \mathbf{v})\} d^2x = \int_C \rho \mathbf{v} \cdot d\mathbf{s},$$

which turns out to be, if choosing the contour C as sufficiently large,

$$\int_C \rho \mathbf{v} \cdot d\mathbf{s} = \rho_0 \int_C \mathbf{v} \cdot d\mathbf{s} = \rho_0 \int \{\nabla \times \mathbf{v}\} d^2x. \tag{14}$$

To describe the evolution of the vortex, we need to include the Hamiltonian term $\frac{\partial \tilde{H}}{\partial \mathbf{R}}$ with $\tilde{H} = \int \psi^* \hat{H} \psi d^2x$, where $\hat{H}$ is settled in such a way that the interaction on the vortex is included. (The concrete form will be given below.) In this way, we arrive at the equation of motion [24]:

$$\rho_0 \lambda \sigma (\mathbf{k} \times \dot{\boldsymbol{R}}) = -\frac{\partial \tilde{H}}{\partial \mathbf{R}}, \tag{15}$$

with $\mathbf{k}$ being the unit vector perpendicular to (x,y) plane. Furthermore, σ is defined as

$$\sigma = \int_{R^2} \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) d^2x, \tag{16}$$

which is nothing but the vorticity. The above equation of motion is rewritten in the form

$$\mu \dot{\mathbf{R}} = \mathbf{k} \times \frac{\partial \tilde{H}}{\partial \mathbf{R}}, \tag{17}$$

with $\mu \equiv \rho_0 \lambda \sigma$. Here we have used $\mathbf{k} \cdot \frac{\partial \tilde{H}}{\partial \mathbf{R}} = 0$.

We shall now consider the interaction arising from the "pinning center" embedded in the medium. That is realized as additional linear refractive indices, which give rise to the modification of the wave function. We describe it as a pinning potential, which can be expressed by a localized form; namely, it may be plausible to ideally choose the delta function form $v_{\text{pin}} = v_0 \delta(\mathbf{r} - \mathbf{a})$ with v_0 being the strength of the interaction and from its meaning it should be negative ($v_0 < 0$). This serves as a contribution to the Hamiltonian of the form $\tilde{H} \equiv V_{\text{pin}} = \int |\psi|^2 v_0 \delta(\mathbf{r} - \mathbf{a}) d^2x = v_0 \rho(|\mathbf{R} - \mathbf{a}|)$. Here a slight modification is needed; namely, we have to take into account the fact that $V_{\text{pin}}(\infty) \to 0$. So we modify it such that

$$V_{\text{pin}} = v_0 \tilde{\rho} \equiv v_0 \{\rho(|\mathbf{R} - \mathbf{a}|) - \rho_0\}. \tag{18}$$

Further if there are several pinning centers located at $\mathbf{a}_i (i = 1 \sim n)$, then $v_{\text{pin}} = \sum_{i=1}^n v_{i0} \delta(\mathbf{r} - \mathbf{a}_i)$ leading the potential $V_{\text{pin}} = \sum_{i=1}^n v_{0i} \tilde{\rho}(|\mathbf{R} - \mathbf{a}_i|)$.

IV. STOCHASTIC EQUATIONS

A. Langevin equation and functional integral

We now come to the main body of the paper: the random behavior of the optical vortex. First we start with a modification of the equation of motion. Namely, there is room to include the *dissipative force*. This comes from the fact that the optical substances under consideration suffer from energy absorption due to various causes during the propagation in the absorptive

media; e.g., the resonant absorption of light. In the present context of vortex dynamics, it is sufficient to adopt the phenomenological form [25], because the precise mechanism behind the dissipation is rather complicated to pin down. That may be realized by replacing $\frac{\partial \tilde{H}}{\partial \mathbf{R}} \to \frac{\partial \tilde{H}}{\partial \mathbf{R}} + \eta \dot{\mathbf{R}}$, hence we have two equations which are the modifications of Eqs. (15) and (17); that is,

$$\mu \frac{d\mathbf{R}}{dz} = \mathbf{k} \times \left(\frac{\partial \tilde{H}}{\partial \mathbf{R}} + \eta \frac{d\mathbf{R}}{dz} \right). \tag{19}$$

and

$$\mu \mathbf{k} \times \frac{d\mathbf{R}}{dz} = -\left(\frac{\partial \tilde{H}}{\partial \mathbf{R}} + \eta \frac{d\mathbf{R}}{dz} \right). \tag{20}$$

From the above two equations, $\dot{\mathbf{R}}$ is solved to yield

$$\frac{d\mathbf{R}}{dz} + \mathbf{A} = 0, \tag{21}$$

with

$$\mathbf{A} = \frac{1}{\mu^2 + \eta^2} \left\{ \eta \frac{\partial \tilde{H}}{\partial \mathbf{R}} - \mu \left(\mathbf{k} \times \frac{\partial \tilde{H}}{\partial \mathbf{R}} \right) \right\}. \tag{22}$$

The first term of Eq. (22) represents the "gradient force," whereas the second term gives the gyration term, which is analogous to the Lorentz force acting on a charged particle in the uniform magnetic field $\mathbf{k}$. The gradient force has a meaning of dissipation by considering the quantity

$$\frac{d\tilde{H}}{dz} = \frac{\partial \tilde{H}}{\partial \mathbf{R}} \cdot \frac{d\mathbf{R}}{dz} = -\frac{\eta}{\eta^2 + \mu^2} \left(\frac{\partial \tilde{H}}{\partial \mathbf{R}} \right)^2 \leqslant 0,$$

which suggests the dissipation with respect to the direction of wave propagation.

We now take into account the fluctuations coming from the linear refractive index that is randomly distributed. To derive the equation of motion including the random force, we follow the same argument as the previous section [26]. Namely, we consider the modified action functional of Eq. (5), which includes the random term that should be added to the Hamiltonian:

$$\int H d^2x \equiv \tilde{H} \to \tilde{H} + Z.$$

Here the random term Z is a functional of (ψ, ψ^*), though the explicit form is not given. Then we see that the variation equation leads to the equation of motion including the random force:

$$\mu \dot{\mathbf{R}} = \mathbf{k} \times \left(\frac{\partial \tilde{H}}{\partial \mathbf{R}} + \zeta \right),$$

where $\zeta = \frac{\partial Z}{\partial \mathbf{R}}$. Furthermore, if the dissipative term is added as $\frac{\partial \tilde{H}}{\partial \mathbf{R}} \to \frac{\partial \tilde{H}}{\partial \mathbf{R}} + \eta \dot{\mathbf{R}}$, we arrive at the Langevin equation:

$$\frac{d\mathbf{R}}{dz} + \mathbf{A} = \mathbf{c}, \tag{23}$$

with $\mathbf{c}$ being the reduced random force

$$\mathbf{c} = \frac{1}{\mu^2 + \eta^2} (\eta \zeta - \mu \mathbf{k} \times \zeta). \tag{24}$$

Note that Eq. (23) just corresponds to the Langevin equation for the parameters that are built in the soliton solution, which is derived directly from the nonlinear Schrödinger equation with additive noises [19].

Now we adopt the assumption of white noise for the reduced random force $\mathbf{c} = (c_1, c_2)$, which is given as

$$\langle c_i(z) \rangle = 0, \quad \langle c_i(z) c_j(z+u) \rangle = h \delta_{ij} \delta(u). \tag{25}$$

If the variance of $\mathbf{c}$ is chosen as $\langle \mathbf{c}^2 \rangle = 2h$, the probability distribution of $\mathbf{c}$ is given by the Gaussian functional form

$$P[\mathbf{c}(z)] = \exp\left[-\frac{1}{2h} \int_0^z \mathbf{c}^2(z)\, dz \right]. \tag{26}$$

Here h means the diffusion constant that is a close analogy with the Planck constant in quantum mechanics. Using this distribution, the propagator K between two different points is given by the functional integral [17,18]

$$K[\mathbf{R}(z)|\mathbf{R}(0)] = \int \prod_z \delta[\mathbf{F}(\mathbf{R}(z)) - \mathbf{c}(z)] \times \exp\left[-\int \frac{\mathbf{c}^2(z)}{2h} dz \right] \mathcal{D}\mathbf{R} \mathcal{D}\mathbf{c}(z), \tag{27}$$

with δ the Dirac functional. By carrying out the Gaussian integral over the field $\mathbf{c}(z)$, we obtain the following reduced form of the propagator:

$$K[\mathbf{R}(z)|\mathbf{R}(0)] = \int \exp\left[-\frac{1}{2h} \int_0^z \left(\frac{d\mathbf{R}}{dz} + \mathbf{A} \right)^2 dz \right] \mathcal{D}[\mathbf{R}]. \tag{28}$$

The above form is analogous to the path integral for a particle in vector potential $\mathbf{A}$ together with the scalar potential $V = \frac{\mathbf{A}^2}{2}$. This functional integral can be transcribed to the quantum mechanical path integral by putting $z = i\tau$,

$$K = \int \exp\left[\frac{i}{h} \int \left(\frac{1}{2} \dot{\mathbf{R}}^2 + i \mathbf{A} \cdot \dot{\mathbf{R}} - V \right) d\tau \right] \mathcal{D}[\mathbf{R}]. \tag{29}$$

B. Fokker-Planck equation

Using the analogous way with a procedure to obtain the Schrödinger equation from the path integral expression, we can derive the Fokker-Planck equation. Namely, let us consider the probability distribution $P(\mathbf{R})$, which satisfies the integral equation:

$$P(\mathbf{R}, z) = \int K(\mathbf{R}, z | \mathbf{R}', 0) P(\mathbf{R}', 0)\, d\mathbf{R}'. \tag{30}$$

Hence following the usual procedure of path integral, we get

$$ih \frac{\partial P}{\partial \tau} = \left[\frac{h}{2} \left(-ih \frac{\partial}{\partial \mathbf{R}} - i\mathbf{A} \right)^2 + V \right] P, \tag{31}$$

which is just the Schrödinger equation for a particle moving in the vector potential and the potential $(\mathbf{A}, V)$. Now going back to the real variable and by noting the relation $i \frac{\partial}{\partial \tau} = -\frac{\partial}{\partial z}$, we obtain

$$\frac{\partial P}{\partial z} = \frac{h}{2} \frac{\partial^2 P}{\partial \mathbf{R}^2} + \frac{\partial (\mathbf{A}(\mathbf{R}) P)}{\partial \mathbf{R}}. \tag{32}$$

This is the Fokker-Planck equation of the standard form [27]. This can be rewritten in a form of continuity equation for the

probability current

$$\frac{\partial P}{\partial z} + \mathrm{div}\mathbf{J} = 0, \tag{33}$$

with the current

$$\mathbf{J} = -\frac{h}{2}\frac{\partial P}{\partial \mathbf{R}} - \mathbf{A}P. \tag{34}$$

Here we look for an equilibrium state for the distribution function; for this purpose we put it in a form of the Boltzmann distribution: namely, $P = \exp[-\beta H]$ (β means an inverse of *effective* temperature), and substituting this form into Eq. (33), we have

$$\left(\frac{h\beta}{2} - \frac{\eta}{\eta^2+\mu^2}\right)\left\{\frac{\partial^2 \tilde{H}}{\partial \mathbf{R}} - \beta\left(\frac{\partial \tilde{H}}{\partial \mathbf{R}}\right)^2\right\} = 0. \tag{35}$$

For this relation to be held for arbitrary $\tilde{H}$, it follows the equation

$$\frac{h\beta}{2} = \frac{\eta}{\eta^2+\mu^2}. \tag{36}$$

This relation is just the fluctuation-dissipation relation. From this relation, one can define *effective temperature* in terms of the diffusion and dissipation parameters as well as vortex charge. The effective temperature is used as a comparison parameter to characterize the strength of the vortex pinning as will be seen below.

V. ANALYSIS OF THE FOKKER-PLANCK EQUATION

As is seen from the above argument, the structure of the FP equation has a peculiar form and we need to invoke a special technique in order to extract the solutions. For this aim, we pay attention to a mutual interplay between the stochastic parameters and the strength of pinning potential.

In what follows we consider the case that the pinning center is placed at the origin; $\mathbf{a} = 0$. From the geometrical aspect, this forms a "pillar" penetrating through the origin of the (x,y) plane. The vortex drifts around this pillar driven by the random effect (see Fig. 3).

We here use polar coordinates $X = R\cos\theta, Y = R\sin\theta$. As a functional form of $\rho(R)$, we choose this in such a form that it extrapolates between $\rho(0) = 0$ and $\rho(\infty) = \rho_0$ (instead of the one that is obtained by numerical solution), namely, $\rho(R) = \rho_0 U(R)$ with

$$U(R) = 1 - \exp[-\alpha R], \tag{37}$$

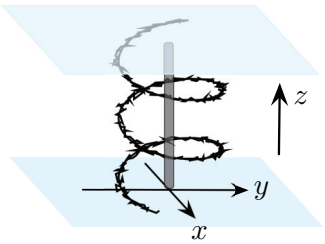

FIG. 3. (Color online) Conceptual image of fluctuation of vortex around a pinning center penetrating through the origin of (x,y) plane (indicated by shaded area).

which leads to

$$\mathbf{A} = -\frac{v_0\eta\alpha}{\eta^2+\mu^2}\exp[-\alpha R]\hat{r} + \frac{2\alpha v_0\mu}{\mu^2+\eta^2}\exp[-\alpha R](\mathbf{k}\times\hat{r}).$$

It is easy to see that the distribution function P is written by a function of radial variable only, hence we have

$$\mathrm{div}\mathbf{J} = -\frac{1}{R}\frac{\partial}{\partial R}(RJ_r), \tag{38}$$

where J_r is the radial component, and the FP equation turns out to be

$$\frac{\partial P}{\partial z} = \frac{1}{R}\frac{\partial}{\partial R}\left[R\left(\frac{h}{2}\frac{\partial P}{\partial R} + \frac{v_0\eta}{\eta^2+\mu^2}U'(R)P\right)\right] \tag{39}$$

with $U'(R) \equiv \frac{dU}{dR}$. Now we try to look for solutions of the eigenvalue problem of the form;

$$P(R,z) = \exp[-Ez]f(R). \tag{40}$$

Substituting this into Eq. (33), it follows that

$$\frac{d}{dR}\left[R\left(\frac{df}{dR} + \kappa U'(R)f\right)\right] = -ERf. \tag{41}$$

Here we make the scaling $E \to \frac{2E}{h}$ and introduce the parameter

$$\kappa = \frac{2v_0\,\rho_0\,\eta}{h(\eta^2+\mu^2)} \equiv v_0\,\rho_0\,\beta, \tag{42}$$

This means the ratio between the effective temperature and the pinning strength, where we use the fluctuation-dissipation relation (36).

A. Weak and strong coupling approximation

We examine the eigenvalue equation (41). To carry out this procedure, we note the role of the parameters κ that has been introduced above. Namely, according to the magnitude of κ, we have two approaches: the weak and strong coupling schemes; the former can be treated by perturbation, whereas the latter by a nonperturbational way.

(i) First we consider the *weak coupling* scheme, for which $|\kappa| \ll 1$ holds, namely, $v_0\,\rho_0 \ll (\beta)^{-1}$. The physical meaning of this inequality is that the strength of pinning multiplied by the power of light ρ_0 is very weak relative to the effective temperature. For this case we can use perturbation: Let us consider an expansion with respect to κ, i.e., $f(R) = f_0(R) + f_1(R) + \cdots$ and $E = E_0 + E_1 + \cdots$. The 0th order satisfies

$$\frac{d^2 f_0}{dR^2} + \frac{1}{R}\frac{df_0}{dR} = E_0 f_0,$$

which gives the Bessel function of 0th order: $f_0(R) = J_0(\sqrt{E_0}R)$. Here the boundary condition is imposed such that f_0 vanishes at $R = a$. Here a is a sort of "regularization parameter" [28]. Hence $J_0(\sqrt{E}a) = 0$, from which the eigenvalues E_0^n are obtained as well as the corresponding eigenfunctions $f_0^{(n)} \equiv J_0(\sqrt{E_n}R)$. The first-order term is determined by

$$\frac{d}{dR}\left(R\frac{df_1}{dR}\right) + \kappa\frac{d}{dR}(RU'(R)f_0) = E_0 f_1 + E_1 f_0. \tag{43}$$

Here we select the 0th order function as $f_0(R) = J_0(\sqrt{E_n}R) \equiv f_0^{(n)}(R)$ for a particular label n. By multiplying $f_0^{(n)}(R)$ for both sides of Eq. (43) and integrating by part, we obtain the first-order correction for E

$$E_1 = \frac{\kappa \int_0^a f_0^{(n)}(R)\frac{d}{dR}\left(RU'(R)f_0^{(n)}\right)dR}{\int_0^a f_0^{(n)}(R)^2 dR}. \tag{44}$$

The first-order correction for the eigenfunction can be expanded as $f_1(R) = \sum_m a_m f_0^{(m)}(R)$ with the expansion parameter:

$$a_m = \int_0^a f_0^{(m)} f_1(R)\, dR = \frac{\kappa \int_0^a f_0^{(m)}\frac{d}{dR}\left(RU'(R)f_0^{(n)}\right) dR}{E_0^m - E_0^n}$$

for $m \neq n$. In this way the procedure may be continued to the higher orders up to one's desire though the step becomes more and more complicated.

(ii) Next we consider the *strong coupling* scheme, for which $|\kappa| \geqslant 1$ holds. The physical meaning of this is that the value $v_0\,\rho_0$ is comparable to or of the same order as the effective temperature, in contrast to the weak coupling case. First we try to find the eigenfunction with zero eigenvalue $E = 0$ for Eq. (41), is written as

$$\frac{df}{dR} + \kappa U'(R) f = 0 \tag{45}$$

where we choose the integral constant is set to be zero. Hence we get

$$f_0(R) = \exp[-\kappa U(R)], \tag{46}$$

which is nothing but the Boltzmann distribution. In this way, the Boltzmann distribution becomes the eigenfunction correspond to the zero eigenvalue. This is consistent with the equilibrium distribution given in the above.

Having this form in mind, we write the solutions of Eq. (41) in the form

$$f(R) = \exp[-\kappa U(R)]g(R), \tag{47}$$

where $g(R)$ satisfies

$$\frac{d}{dR}\left(R\exp[-\kappa U]\frac{dg}{dR}\right) = ER\exp[-\kappa U]g. \tag{48}$$

For $E = 0$, we have $\frac{dg}{dR} = 0$ leading to $g_0(R) = k$ (constant). This settles that $E_0 = 0$ is the lowest eigenvalue.

We examine how to get the higher-order eigenfunctions in a successive way. To carry this out, we adopt the variational procedure. By multiplying g to both sides of Eq. (48) and integrating up to the boundary $R = a$, and using an integration by part, we obtain

$$\int_0^a R\exp[-\kappa U]\left(\frac{dg}{dR}\right)^2 dR = E\int_0^a R\exp[-\kappa U]g^2\, dR,$$

where the boundary condition for $g(R)$ is imposed: $g(a) = 0$. Hence we settle the functional

$$J = \int_0^a \left\{\left(\frac{dg}{dR}\right)^2 - Eg(R)^2\right\}\exp[-\kappa U]R\, dR, \tag{49}$$

for which we apply the variation principle $\delta J = 0$. We look for a trial function for $g(r)$ as a polynomial of R. The conditions to be satisfied for g are twofold: (i) the normalization $\int g^2(R)\exp[-\kappa U(R)]R\, dR = 1$ and (ii) the orthogonality relation holds for two eigenfunctions g and g', that is, $\int_0^a g(R)g'(R)\exp[-\kappa U(R)]R\, dR = 0$. The eigenfunctions are graded according to the order of polynomial.

Here we try to find the first order that is next to the zeroth-order $g_0 = k$. Let us put $g_1(R) = A + BR + CR^2$. To determine these coefficients, we first use the orthogonality condition $\int g_0(R)g_1(R)\exp[-\kappa U]R\, dR = 0$, which turns out to be the linear relation between A, B, C; hence if eliminating C, the remaining terms A, B are left to be determined. Next using the normalization, one has a quadratic relation for A, B. Finally, from the variation equation $\delta J = 0$, we obtain two equations for A, B as well as for E. In this way one can obtain the first-order eigenfunction. The solutions of higher order may be obtained by continuing the similar procedure, which may form the orthonormal set $g_n(R)$.

Having obtained the sequence of eigenfunctions, we can derive the distribution function in the following way:

$$P(R,z) = \sum_n C_n(z)g_n(R)\exp[-\kappa U].$$

The expansion coefficients $C_n(z)$ satisfy $dC_n/dz = -EC_n$ with the initial condition $C_n(0)$ which is determined by the initial condition

$$P(R,0) = \sum_n C_n(0)g_n(R)\exp[-\kappa U]$$

and hence $C_n = \int P(R,0)g_n(R)\exp[\kappa U]R\, dR$.

Remark on the approximate scheme. Here we discuss the physical significance of the above two approximate procedures. The strong coupling approximation is applicable to the case when the equilibrium state is established, which is realized by the Boltzmann distribution, the eigenfunction with zero eigenvalue. The existence of this fixed eigenfunction implies a *stability* of the strong coupling scheme. In other words, to maintain the equilibrium, the strength of the pinning potential (multiplied by light strength) is needed to keep enough magnitude.

On the other hand, the weak coupling is *unstable* in the sense that the choice of starting unperturbed eigenvalues as well as eigenstates [i.e., $E_n^0, f_n(R)$] is ambiguous. However, this is valid even for the case when there is no equilibrium state. Namely, there is *uncontrolled* fluctuation leading to an instability that may prevent the distribution from the equilibrium state.

B. Small diffusion limit

Now we consider the other approximate scheme on the basis of the asymptotic analysis of a functional integral. This uses the small diffusion limit $h \sim 0$, for which we can use an analogous way with the semiclassical approximation that is based on the asymptotic expansion with respect to the Planck constant.

In the limit of $h \sim 0$, the functional integral (28) becomes $K_{\rm sc} = \exp[-\frac{S_{\rm cl}}{h}]$ with the *classical* action written in terms of

the polar coordinate:

$$S_{\rm cl} = \int \left[\frac{1}{2}\left(\dot{R} + \frac{v_0\,\eta}{\eta^2+\mu^2}\frac{dU}{dR} \right)^2 + \frac{1}{2}R^2\left(\dot{\theta} + \frac{v_0\,\rho_0\,\mu}{\eta^2+\mu^2}\frac{dU}{dR} \right)^2 \right] dz. \tag{50}$$

Noting that θ is a cyclic coordinate and the conjugate momentum to θ is the constant of motion $p_\theta = \dot{\theta} + \mu\frac{d\tilde{H}}{dR} = c$, the Lagrangian is given by

$$L_{\rm cl} = \frac{1}{2}\left(\dot{R} + \frac{v_0\,\rho_0\,\eta}{\eta^2+\mu^2}\frac{dU}{dR} \right)^2 + \frac{c^2}{R^2}. \tag{51}$$

On the other hand, the equation of motion is derived by the "Rouse function": $R = c\dot{\theta} - L$ [29]. Though this looks rather complicated, we note that c can be chosen arbitrarily small. By virtue of this feature, the equation of motion for R becomes simple; namely, we can set the equation such the first term of Eq. (51) vanishes to result in the so-called the instanton equation

$$\dot{R} = -\frac{v_0\,\rho_0\,\eta}{\eta^2+\mu^2}\frac{dU}{dR} = -\alpha\frac{v_0\,\rho_0\,\eta}{\eta^2+\mu^2}\exp[-\alpha R],$$

hence the transition amplitude $K_{\rm sc}$ turns out to be

$$K_{\rm sc} = \exp\left[-\frac{c^2}{h}\int_{z_i}^{z}\frac{dz}{R^2} \right]. \tag{52}$$

Using the form of $U(R)$, it follows that

$$\frac{dz}{dR} = \frac{\eta^2+\mu^2}{v_0\,\rho_0\,\eta\alpha}\exp[\alpha R]$$

together with the parameter κ [Eq. (42)]. Here noting that v_0 is negative, we obtain

$$K_{\rm sc}(R,z|R_i,z_i) = \exp\left[-\frac{2}{|\kappa|\alpha}\left(\frac{c}{h}\right)^2\int_{R_i}^{R}\frac{\exp[\alpha R]}{R^2}\,dR \right] \tag{53}$$

($v_0 = -|v_0|$). The probability distribution function is calculated from the well-known relation

$$P(R,z) = \int K_{\rm sc}(R,z|R_i,z_i)P(R_i,z_i)\,dR_i\,dz_i.$$

If choosing $P(R_i) = \delta(R_i - R_0)\delta(z_i - z_0)$ (delta function) and putting $t = \alpha R$, we finally obtain

$$P(t,z) = \exp\left[-\frac{2}{|\kappa|}\left(\frac{c}{h}\right)^2 K(t) \right], \tag{54}$$

with

$$K(t) = \int_{R_0\alpha}^{R\alpha}\frac{\exp[t]}{t^2}dt. \tag{55}$$

Here we note that the upper and lower limits depend on the parameter α which is the size of the vortex. The remaining task is to estimate the expression numerically. To give a rough estimate, we adopt the factor in front of the integral: $\frac{2}{\kappa}(\frac{c}{h})^2 \sim 1$.

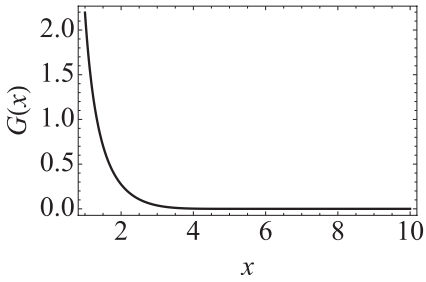

FIG. 4. Profile of the distribution function $\exp[-K(x)] \equiv G(x)$, where x runs through $1 \sim 10$.

The only remaining parameter is α, for which we choose such that $R_0\alpha \sim 1$. Noting that $\frac{1}{\alpha}$ means the size of the vortex, this relation suggests that the initial vortex position R_0 is chosen to be the order of the vortex size. The numerical result is plotted in Fig. 4 for the function $P(R,z) \sim \exp[-K(R\alpha)]$.

VI. SUMMARY

A stochastic theory has been presented for the optical vortex with the aim of exploring its behavior in the presence of random fluctuations of the refractive index. The theory is formulated in the framework of Langevin and Fokker-Planck equations by using the functional integral. The attempt would open a different facet for the traditional subject of the optical vortex in nonlinear Kerr media [6,7,9].

In particular the Fokker-Planck equation was analyzed while paying attention to the role of the pinning potential. This is based on a peculiar form of the FP equation, containing the parameter κ, the ratio between the strength of the pinning potential and the effective temperature. According to the magnitude of this parameter, the solutions of the FP equation can be categorized into weak and strong coupling regimes.

We have also examined an asymptotic behavior leading to a compact form of the distribution function. Here as for the asymptotic analysis of path integral, which somehow shares a similar idea with the present work, the following attempts are mentioned: the viscous instanton for Burgers turbulence [30] and the Gaussian vortex approximation to two-dimensional turbulence [31], both of which are based on the general theory of statistical dynamics of classical systems [32].

The present theory is restricted to the case that a light wave has a single component; that is, no effect of polarization has been considered. It is straightforward to extend the formulation to the case that polarization is taken into account [33,34]. Indeed a preliminary study of an optical vortex incorporating polarization has been made [35], for which the vortex for a polarized wave results in the so-called *coreless vortex*. Further study of optical vortex incorporating polarization will be investigated elsewhere.

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[1] M. Born and E. Wolf, *Principle of Optics* (Pergamon, Oxford, 1975).

[2] L. Landau and E. Lifschitz, *Electrodynamics of Continuous Media*, Course of Theoretical Physics Vol. 8 (Pergamon, Oxford, 1968), Chap. 11.

[3] N. Bloembergen, *Nonlinear Optics* (Benjamin, New York, 1965).

[4] N. Bloembergen, Rev. Mod. Phys. **54**, 685 (1982).

[5] R. Y. Chiao, E. Gamire, and C. H. Townes, Phys. Rev. Lett. **13**, 479 (1964).

[6] A. W. Snyder, L. Polarian, and D. J. Mitchell, Opt. Lett. **17**, 789 (1992).

[7] G. A. Swartzlander, Jr. and C. T. Law, Phys. Rev. Lett. **69**, 2503 (1992).

[8] E. M. Lifschitz and L. P. Pitaevski, *Statistical Physics, Part 2*. Course of Theoretical Physics Vol. 9 (Butterworth-Heinemann, New York, 1980).

[9] J. Christou, V. Tikhonento, Y. S. Kivshar, and B. L. Davies, Opt. Lett. **21**, 1649 (1996).

[10] D. Rozas, Z. S. Sacks, and G. A. Swartzlander, Jr., Phys. Rev. Lett. **79**, 3399 (1997).

[11] For example, L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw, and J. P. Woerdman, Phy. Rev. A **45**, 8185 (1992); M. J. Padget and L. Allen, Opt. Commun. **121**, 36 (1995); F. Flossman, U. T. Schwartz, and M. Maier, *ibid.* **250**, 218 (2005). As a general text for the Gaussian beam, see A. Yarifv, *Optical Electronics in Modern Communications*, 2nd ed. (Clarendon, Oxford, 1997), Chap. 2.

[12] A. T. Dorsey, Phys. Rev. B **46**, 8376 (1992).

[13] T. Ueda and W. L. Kath, J. Opt. Soc. Am. B **11**, 818 (1994).

[14] C. R. Menyuk and P. K. A. Wai, J. Opt. Soc. Am. B **11**, 1288 (1994).

[15] K. A. Wai and C. R. Meynyuk, Opt. Lett. **19**, 1517 (1994).

[16] See, e.g. S. A. Akhmanov, *Physical Optics* (Clarendon, Oxford, 1997).

[17] M. Fibich and E. Helfand, Phys. Rev. **183**, 265 (1969).

[18] J. Zittarz and J. S. Langer, Phys. Rev. **148**, 741 (1966).

[19] G. E. Falkovich, I. Kolokolov, V. Lebedev, and S. K. Turitsyn, Phys. Rev. E **63**, 025601(R) (2001).

[20] E. A. Kuznetsov and J. J. Rasmussen, Phys. Rev. E **51**, 4479 (1995).

[21] L. A. Smirnov and V. A. Mironov, Phys. Rev. A **85**, 053620 (2012).

[22] This equation just coincides with the single vortex profile for the superfluid (see [8]), except just the opposite sign of the last two terms, namely, in [8], this is written as $+(G - G^3)$.

[23] H. Kuratsuji, Phys. Rev. Lett. **68**, 1746 (1992).

[24] H. Lamb, *Hydrodynamics* (Cambridge University, Cambridge, England, 1935).

[25] The dissipative force given here is similar with the one that was used in the magnetic system; L. D. Landau and E. M. Lifschitz, Phys. Z. Sowjetunion **8**, 153 (1935).

[26] The procedure given here corresponds to the nonlinear Schrödinger equation with additive noise [19]; namely, the variation equation $\delta(I - Z) = 0$ with respect to the field ψ just leads to the nonlinear Schrödinger equation with the random noise.

[27] R. Kubo, M. Toda, and N. Hashitsume, *Statistical Physics II*, 2nd ed. (Springer-Verlag, New York, 1985).

[28] The introduction of the cutoff radius is required in order to derive a proper result, which is chosen to be a size of the background wave field.

[29] See, e.g., L. D. Landau and E. M. Lifschitz, *Mechanics*, Course of Theoretical Physics, 3rd ed. (Butterman-Heinemann, New York, 1976).

[30] E. Balkovsky, G. Falkovich, and I. Kolokolov, Int. J. Mod. Phys. B **11**, 3223 (1997).

[31] K. Kleineberg and R. Friedrich, Phys. Rev. E **87**, 033007 (2013).

[32] P. C. Martin, E. Siggia, and H. Rose, Phys. Rev. A **8**, 423 (1973).

[33] H. Kuratsuji and S. Kakigi, Phys. Rev. Lett. **80**, 1888 (1998).

[34] A. W. Snyder, S. J. Hewlett, and D. J. Mitchell, Phys. Rev. Lett. **72**, 1012 (1994).

[35] H. Kuratsuji, in *Topology in Ordered Phases*, Proceedings of the First International Symposium on TOP, Sapporo, Japan, 2005, edited by S. Tanda *et al.* (World Scientific, Singapore, 2006).