Scattering approach to fidelity decay in closed systems and parametric level correlations

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Based on an exact analytical approach to describe scattering fidelity experiments [Köber *et al.*, Phys. Rev. E **82**, 036207 (2010)], we obtain an expression for the fidelity amplitude decay of quantum chaotic or diffusive systems under arbitrary Hermitian perturbations. This allows us to rederive previous separately obtained results in a simpler and unified manner, as is shown explicitly for the case of a global perturbation. The general expression is also used to derive a so far unpublished exact analytical formula for the case of a moving S-wave scatterer. In the second part of the paper, we extend a relation between fidelity decay and parametric level correlations from the universal case of global perturbations to an arbitrary combination of global and local perturbations. Thereby, the relation becomes a versatile tool for the analysis of unknown perturbations.

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I. INTRODUCTION

During the last decade, approximately, considerable efforts have been dedicated to the quantitative prediction of the fidelity decay in chaotic and diffusive quantum systems and classical wave systems [1–4] (see also Ref. [5] and references therein). A very successful approach has been based on random matrix theory, adopting the so-called Bohigas-Giannoni-Schmit conjecture [6]. Applied to the current setting, it suggests that quantum systems with chaotic classical counterpart ("chaotic quantum systems" for short) as well as diffusive wave systems show a universal response to perturbations which can be calculated within an appropriate random matrix model [7]. The first exact analytical results in this respect have been obtained by Stöckmann and Schäfer [8,9] using supersymmetry techniques similar to Ref. [10], for the calculation of correlation functions between scattering matrix elements. More recently, exact analytical results have also been found for scattering systems, where the fidelity amplitude, an expectation value, is replaced by the "scattering fidelity," which is a product of two transition amplitudes [11,12]. These results, published in Ref. [13], have been obtained by a simple but powerful modification of the Verbaarschot-Weidenmüller-Zirnbauer (VWZ) formula from Ref. [10]. We will call this new approach the "scattering approach to fidelity."

As shown first by Kohler *et al.* [14], for a global perturbation of a completely diffusive system, the fidelity amplitude can also be calculated from the parametric level correlations. Subsequent generalizations have been discussed in Refs. [15–17]. Originally, parametric level correlations have been introduced in the area of disordered systems with diffusive dynamics [18,19]. At that moment, they have been considered a universal signature of chaotic and diffusive dynamics, with a functional form, independent of the particular perturbation applied. However, in Ref. [20], it was shown that certain types of perturbations lead to very different behaviors, and it became clear that the "universal" prediction of Ref. [19] applies only to global perturbations, not to local perturbations, characterized by only a few eigenstates with nonzero eigenvalues. The perturbation due to the displacement of a small scatterer discussed in [20] is precisely of that latter type.

The purpose of the present paper is twofold. First, we use the scattering fidelity approach from Ref. [13] to derive exact analytical expressions for the fidelity decay of chaotic and diffusive wave systems in the presence of completely general Hermitian perturbations. This allows us to rederive the known result for the decay of the fidelity amplitude due to a global perturbation [8,9]. More importantly, we then use the expression to describe the decay of the fidelity amplitude due to the displacement of a single S-wave scatterer (local perturbation). We compare our result to experimental data published in Ref. [21], based on earlier measurements, described in Ref. [20].

Second, we generalize the relation between the fidelity amplitude and the parametric level correlations ("FA-PLC relation" for short) from Ref. [14] to arbitrary perturbations. To do so, we compare the analytical expression for the parametric level correlations [22] and its analog for the fidelity amplitude for general perturbations. Our result shows that the fidelity amplitude depends only on spectral data and is therefore a basis independent quantity: a surprising fact, taking into account that the perturbation may be completely arbitrary. Our generalization constitutes an important result, since it turns the FA-PLC relation into an analytical tool, useful in an inverse problem setting. Previously, the FA-PLC relation could be applied only if the perturbation was known to be global (implying a universal form of fidelity amplitude and parametric correlations). Now we have shown that it can also be applied in situations where we would like to obtain information about an unknown perturbation by analyzing the fidelity amplitude or the parametric level correlations.

The present paper is organized as follows: In the next section, we follow Ref. [13] to describe the connection between scattering fidelity [11] and scattering matrix correlation functions as considered in Ref. [10]. We then use this connection to derive an exact analytical expression for the fidelity amplitude valid for arbitrary perturbations. In Sec. III we discuss the differences between local and global perturbations, and we use our general formula to rederive the known result for a global perturbation. In the main part of that section, we calculate the fidelity amplitude in the case of a moving scatterer and compare the resulting theoretical prediction to experimental data from Ref. [21]. In Sec. IV we evaluate the general integral expression for the fidelity amplitude in the perturbative (long time) limit. In Sec. V we generalize the relation between parametric level correlations and the fidelity amplitude to arbitrary perturbations. Conclusions are presented in Sec. VI.

II. SCATTERING APPROACH TO FIDELITY

In this section we introduce the central quantity of this work, the fidelity amplitude of a closed quantum or classical wave system, with quantum chaotic or diffusive dynamics. We assume that random matrix theory can be used to describe the fidelity decay. While the first part contains some general statements about fidelity and the random matrix models used, the second part describes the description of the algebraic scattering model to which the fidelity problem is mapped. This mapping, introduced in Ref. [13] provides an exact analytical description of the fidelity decay.

A. Fidelity

The fidelity and the fidelity amplitude for a Hamiltonian $H_\alpha = H_0 + W_\alpha$ with perturbation W_α are defined as

$$F(t) = |f(t)|^2, \quad f(t) = \langle a|\, e^{2\pi i H_\alpha t}\, e^{-2\pi i H_\beta t}\, |a\rangle, \tag{1}$$

where $|a\rangle$ is the initial state and W_α is the perturbation depending on an external parameter α. We assume that the energy is measured in units of the mean level spacing d_0 in the spectrum of H_0, and time in units of the Heisenberg time $t_{\rm H} = 2\pi\hbar/d_0$. As a result, the variable t in Eq. (1) becomes dimensionless.

For our purpose it will prove convenient to write the perturbation in terms of a normalized eigenbasis:

$$W_\alpha = \sum_j w_j(\alpha)\, |v_j\rangle\langle v_j|, \tag{2}$$

where the orthonormal eigenstates $\{\,|v_j\rangle\,\}$ are assumed to be independent of α. This is normally well fulfilled in the case of global perturbations, and also in the case of many types of local perturbations, such as pointlike scatterers. A more detailed discussion of this assumption is given in Sec. III. In other words, Eq. (2) implies that we assume $[W_\alpha, W_\beta] = 0$ for any α,β in the allowed range. Note that it is often possible to consider $H_0 + W_\alpha$ as the unperturbed Hamiltonian H_0'. Then $W_\alpha' = 0$ and $W_\beta' = W_\beta - W_\alpha$ such that $[W_\alpha', W_\beta'] = 0$, trivially.

Returning to our original setup, we choose H_0 from one of the invariant ensembles, the Gaussian orthogonal ensemble (GOE) or the Gaussian unitary ensemble (GUE) [23]. Correspondingly, we assume that the perturbation W_α can be diagonalized either by an orthogonal (GOE case) or unitary (GUE case) transformation. In either case, we arrive at

$$H_\alpha = H_0 + \sum_j w_j(\alpha)\, |j\rangle\langle j|, \tag{3}$$

without changes in the random matrix ensemble for H_0. Here the states $|j\rangle$ simply are the elements of the canonical basis of a complex vector space $\mathbb{C}^N$, where N may be assumed arbitrarily large but finite. In this situation, one may use the results of Ref. [13] to calculate the fidelity amplitude averaged over H_0 as the average of a certain scattering matrix correlation function within the framework of statistical scattering [10]. In what follows, we concentrate on the GOE case. The GUE case (which turns out to be even simpler) may be treated along similar lines, using Ref. [24].

B. Scattering matrix correlation functions

According to Ref. [10], the scattering matrix may be written as

$$S_{ab}(E) = \delta_{ab} - 2i\pi\ V^\dagger\, \frac{1}{E - H_{\rm eff}}\, V, \tag{4}$$

where $H_{\rm eff} = H_0 - i\pi\ V\,V^\dagger$ with H_0 from the GOE and V a fixed coupling matrix whose matrix elements V_{ja} may be interpreted as transition amplitudes from an internal state $|j\rangle$ to a decay channel a. Typically, it is assumed that there are much less scattering channels $a = 1,\ldots,M$ than internal states, such that $M \ll N$. The Kronecker delta δ_{ab} implies the any direct processes are absent or negligible. Applying the singular value decomposition to V, demonstrates that there exists and external basis for the scattering states as well as an internal basis in the Hilbert space of H_0, such that the matrix $V\,V^\dagger$ becomes diagonal with an $N-M$-fold degenerate eigenvalue zero, and M real and positive eigenvalues $\{\gamma_a\}_{a=1,\ldots,M}$. In what follows, we use these special basis sets, which allows us to write

$$H_{\rm eff} = H_0 - i\pi\ \sum_a \gamma_a\ |a\rangle\langle a|, \tag{5}$$

where the vectors $\{|a\rangle\}_{a=1,\ldots,M}$ simply denote the first M canonical basis vectors in the Hilbert space $\mathbb{R}^N$. According to Ref. [10], the average scattering matrix (averaged over the GOE for H_0), is then given as

$$\mathbb{E}(S_{ab}) = \frac{1-\kappa_a}{1+\kappa_a}, \quad \kappa_a = \frac{\pi^2\,\gamma_a}{N}, \tag{6}$$

where we have assumed that the average level spacing for H_0 is equal to one. Here we introduced the somewhat unusual notation $\mathbb{E}(\ldots)$ for the ensemble average over the GOE, to avoid possible conflicts with the Dirac notation used below. The main result of Ref. [10] consists in a triple integral which gives the spectral correlation function between different S-matrix elements,

$$\begin{aligned} C[S_{ab}^*, S_{cd}](w) = {}& \mathbb{E}[\, S_{ab}(E)^*\, S_{cd}(E+w)] \\ & - \mathbb{E}[S_{ab}(E)^*]\ \mathbb{E}[S_{cd}(E+w)], \end{aligned} \tag{7}$$

depending on the transmission coefficients $T_a = 4\kappa_a\,(1+\kappa_a)^{-2}$ only. Due to the convolution theorem, the Fourier transform of these correlation functions yields an average over different amplitudes of the evolution operator for the effective Hamiltonian $H_{\rm eff}$. Namely, for $t > 0$:

$$\begin{aligned} \hat C[S_{ab}^*, S_{cd}](t) &\propto \mathbb{E}[\, \hat S_{ab}(t)^*\, \hat S_{cd}(t)\,] \\ &= \mathbb{E}\big[\, \langle b|{\rm e}^{2\pi i H_{\rm eff}^\dagger t}|a\rangle\langle c|{\rm e}^{-2\pi i H_{\rm eff} t}|d\rangle\,\big]. \end{aligned} \tag{8}$$

Here the states $|a\rangle$, $|b\rangle$, $|c\rangle$, and $|d\rangle$ have to be chosen from the first M elements of the canonical basis in $\mathbb{R}^N$, as introduced above. In what follows, we will only be concerned with the case $c = a$, $d = b$. That allows to write for the correlation function in Eq. (8):

$$\hat C[S_{ab}^*, S_{ab}](t) = \delta_{ab}\, T_a^2\, \big\langle Z\, J_a^2 \big\rangle_I + (1+\delta_{ab})\, T_a T_b\, \langle Z\, P_{ab}\rangle_I\,, \tag{9}$$

where Z, J_a, and P_{ab} are abbreviations for more complicated expressions given below, while the angular brackets $\langle\cdots\rangle_I$ denote the following weighted double integral:

$$\langle\cdots\rangle_I = \int_{\max(0,t-1)}^{t} dr \int_0^r du\, \frac{(t-r)(r+1-t)(\cdots)}{(2u+1)(t^2-r^2+x^2)^2}. \tag{10}$$

The above-mentioned expressions are as follows:

$$Z = \prod_{j=1} \frac{1-T_j(t-r)}{\sqrt{1+2T_j\, r + T_j^2\, x^2}}, \quad x^2 = u^2\,\frac{2r+1}{2u+1}, \tag{11}$$

$$J_a = 2\sqrt{1-T_a}\left[\frac{r+T_a x^2}{1+2T_a r+T_a^2 x^2} + \frac{t-r}{1-T_a(t-r)}\right],$$

$$P_{ab} = 2\left\{\frac{T_a T_b x^4 + d_{ab}x^2 + (2r+1)r}{\left[1+2T_a r+T_a^2 x^2\right]\left[1+2T_b r+T_b^2 x^2\right]} + \frac{(t-r)(r+1-t)}{[1-T_a(t-r)]\,[1-T_b(t-r)]}\right\}, \tag{12}$$

with $d_{ab} = T_a T_b + (T_a+T_b)(r+a) - 1$. Note that changing the integration variable from u to x yields

$$2x\,dx = (2r+1)\left[\frac{2u}{2u+1} - \frac{2u^2}{(2u+1)^2}\right]du$$
$$\Rightarrow \quad \frac{du}{2u+1} = \frac{dx}{\sqrt{x^2+2r+1}}, \tag{13}$$

such that Eq. (10) may be written equivalently as

$$\langle\cdots\rangle_I = \int_{\max(0,t-1)}^{t} dr \int_0^r \frac{dx(t-r)(r+1-t)(\cdots)}{\sqrt{x^2+2r+1}\,(t^2-r^2+x^2)^2}\,. \tag{14}$$

That expression allows easier comparisons with the results in Refs. [8,9].

C. Connection to fidelity

Starting from Eq. (1), we insert the projection onto a random state $|b\rangle\langle b|$ into the definition of the fidelity amplitude:

$$f(t) \to f_{ab}(t) \propto \mathbb{E}(\,\langle a|e^{2\pi i H_\beta t}\,|b\rangle\langle b|\,e^{-2\pi i H_\alpha t}\,|a\rangle\,), \tag{15}$$

where the average over the random state $|b\rangle\langle b|$ simply yields the identity times a normalization constant equal to the inverse Hilbert space dimension. As a result, we obtain the product of two transition amplitudes which we propose to call "scattering fidelity" even if the system is closed. The reason is that in the light of what we call the scattering approach to fidelity, the more fundamental distinction between fidelity and scattering fidelity as defined in Ref. [11] is in the different algebraic structure and not in the question whether one has a true scattering system or not. According to Ref. [11], the scattering fidelity should be normalized in such a way that $f_{ab}(t) = 1$ whenever $H_\alpha = H_\beta$. In that case the right-hand side of Eq. (15) becomes and autocorrelation function. Thus the normalization factor will typically contain autocorrelation functions in the denominator, as can be seen in the following section.

Comparing the effective Hamiltonian in Eq. (5) with the perturbed Hamiltonian for a closed system as given in Eq. (3), we find that they share the same structure, and that we need only to allow the coupling parameters γ_a to become complex to unify both descriptions. In Ref. [13] it has then been shown that the analytical result for the correlation functions in Eq. (9) can be generalized to different effective Hamiltonians $H_{\rm eff}$ and $H'_{\rm eff}$, which differ only in the eigenvalues of the coupling term γ_a and γ'_a. In that case, one just needs to calculate the effective transmission coefficients

$$T_j = \frac{2\,(\kappa'_j + \kappa^*_j)}{(1+\kappa'_j)(1+\kappa^*_j)} \tag{16}$$

from the coupling parameters κ_a (corresponding to $H_{\rm eff}$) and κ'_a (corresponding to $H'_{\rm eff}$), as defined in Eq. (6). Then the double integral in Eq. (10) yields the scattering fidelity, when replacing the transmission coefficients in the term Z by the effective transmission coefficients defined here.

Restricting ourselves to closed systems with Hermitian perturbations, we obtain from the comparison of Eq. (3) with Eq. (5) that $-i\pi\gamma_j = w_j(\alpha)$, such that according to Eq. (6)

$$\kappa_j = \frac{\pi^2\gamma_j}{N} = \frac{i\,\pi\,w_j(\alpha)}{N}, \quad \kappa'_j = \frac{i\,\pi\,w_j(\beta)}{N}. \tag{17}$$

Finally, to make sure that we really have a closed system, we need the transmission coefficients T_a and T_b to be negligibly small. That means that the dynamics of the system is probed from the outside via scattering channels which are so weakly coupled to the system that their effect on the dynamics is negligible. In that limit, the expressions to be integrated in Eq. (9) become

$$J_a \to 2t,$$
$$P_{ab} \to P_0 = 2\,[\,r^2 + (2r+1)\,t - t^2 - x^2\,]. \tag{18}$$

Thereby, we obtain for the scattering fidelity

$$f_{ab}(\{\kappa_j\},\{\kappa'_j\}\,;\,t) \propto \delta_{ab}\,T_a^2\,4t^2\,\langle Z\rangle_I + (1+\delta_{ab})\,T_a\,T_b\,\langle Z\,P_0\rangle_I, \tag{19}$$

where its dependence on the coupling parameters $\{\kappa_j\}$ and $\{\kappa'_j\}$ is denoted explicitly, as their value will become important below, where we discuss normalization.

In the calculation of the scattering fidelity $f_{ab}(t)$, there is still the problem of normalization to be solved. This is because $f_{ab}(t)$ becomes an autocorrelation function for $H_{\rm eff} = H'_{\rm eff}$, which may still decay to zero in time. In Ref. [11], this problem has been solved by dividing the scattering fidelity through the geometric mean of the auto correlation functions of $H_{\rm eff}$ and $H'_{\rm eff}$. Below we will see that this normalization procedure is somewhat simpler in the case of closed systems.

As mentioned earlier, when one is really interested in fidelity and $|b\rangle\langle b|$ has been inserted just for convenience as described in Eq. (15), one can normally assume that $a \neq b$. In addition, the case $a \neq b$ arises in the case of an explicit scattering fidelity experiment, where in- and outgoing channels are chosen to be different (transmission measurement). Then the formula for $f_{ab}(t)$ simplifies to

$$f_{ab}(t) \propto T_a T_b\,\langle P_0\,Z\rangle_I. \tag{20}$$

In order to apply the normalization scheme from Ref. [11], we note that for the autocorrelation functions:

$$f_{ab}(\{\kappa_j\},\{\kappa_j\}\,;\,t) = f_{ab}(\{\kappa'_j\},\{\kappa'_j\}\,;\,t) \propto T_a T_b\,\langle P_0\rangle_I. \tag{21}$$

This follows from the fact that $\kappa_j + \kappa_j^* = \kappa_j' + \kappa_j'^* = 0$ since the coupling parameters are purely imaginary in both cases. That implies that the effective transmission coefficients are zero, so that Z becomes equal to one. Since the autocorrelation functions are the same, the geometric mean is also the same. Therefore, we obtain for $f(t)$ in Eq. (1) and $f_{ab}(t)$ in Eq. (15)

$$f(t) = f_{ab}(t) = \frac{T_a T_b \, \langle P_0 \; Z \rangle_I}{T_a T_b \, \langle P_0 \rangle_I} = \frac{\langle P_0 \; Z \rangle_I}{\langle P_0 \rangle_I}. \qquad (22)$$

Now, it has been shown in Ref. [8] that for $Z = 1$, the resulting double integral yields

$$\langle P_0 \rangle_I = \int_{\max(0,t-1)}^{t} dr \int_0^r \frac{dx (t-r)(r+1-t)\, P_0}{\sqrt{x^2+2r+1}\,(t^2-r^2+x^2)^2} = 1 \qquad (23)$$

for any $t > 0$, so that for $a \neq b$

$$f(t) = f_{ab}(t) = \langle P_0 \; Z \rangle_I. \qquad (24)$$

This formula constitutes the first important result of our work, since it gives an exact analytical expression for the fidelity amplitude of a chaotic and diffusive wave system for an arbitrary perturbation.

In the special case when the scattering fidelity is measured from a reflection amplitude ($a = b$), we find

$$f_{aa}(\{\kappa_j\},\{\kappa_j'\}\,;\, t) = \frac{T_a^2 [\, 4t^2 \, \langle Z \rangle_I + 2 \, \langle Z \; P_0 \rangle_I \,]}{N(t)}, \qquad (25)$$

where the geometric mean of the autocorrelation functions, denoted by $N(t)$, turns out to be time dependent. While the effective transmission coefficients are still zero and $Z = 1$, the autocorrelation functions now read:

$$f_{aa}(\{\kappa_j\},\{\kappa_j\}\,;\, t) = f_{aa}(\{\kappa_j'\},\{\kappa_j'\}\,;\, t) = N(t) = T_a^2 \, [\, 4t^2 \, \langle 1 \rangle_I + 2 \, \langle P_0 \rangle_I \,]. \qquad (26)$$

The integral $\langle 1 \rangle_I$ has been calculated in Refs. [25,26], with the result $4t^2 \, \langle 1 \rangle_I = 1 - b_2(t)$, where $b_2(t)$ is the two-point spectral form factor of the Gaussian orthogonal ensemble [23]. Hence, we obtain

$$f_{aa}(t) = \frac{4t^2 \, \langle Z \rangle_I + 2 \, \langle Z \; P_0 \rangle_I}{3 - b_2(t)}. \qquad (27)$$

This result will be used in Sec. III B, where we discuss experimental results for perturbations due to the displacement of an S-wave scatterer.

III. LOCAL VERSUS GLOBAL PERTURBATIONS

A detailed discussion of the differences between local and global perturbations can be found in Ref. [22]. Consider Eq. (2), where a perturbation results in the change of several eigenvalues of the perturbation operator W. In order to affect the dynamics of the system (leading to the decay of the fidelity amplitude), one may change either only a few eigenvalues by a large amount (local perturbation) or very many eigenvalues by a small amount (global perturbation). Both cases are considered in this section.

A. Global perturbation

This was the first problem solved analytically in the context of fidelity decay of quantum-chaotic systems [8,9,12]. Experimentally, the perturbation was realized in a chaotic microwave billiard by displacing one of the straight billiard boundaries. If described by Eq. (1) and Eq. (2), W_α may represent absolute displacements with respect to some initial position. Then its eigenvector representation in Eq. (2) runs over a large number N of states. According to Sec. II C, and in particular Eq. (16) and Eq. (17), the effective transmission coefficients become

$$T_j = \frac{2\pi}{N} \, \frac{i \, w_j(\beta) - i \, w_j(\alpha)}{[1 + i\pi \, w_j(\beta)/N]\,[1 - i\pi \, w_j(\alpha)/N]}$$
$$= 2\delta_j \, (1 - i\pi \, \delta_j) + O\left\{ \left[\frac{w_j(\beta)}{N}\right]^3 , \left[\frac{w_j(\alpha)}{N}\right]^3 \right\}, \qquad (28)$$

where $\delta_j = [w_j(\beta) - w_j(\alpha)]/N$. In this setting, global perturbations are characterized by the fact that the contribution of each individual term is negligible, while the perturbation becomes noticeable only because it is the sum of very many such contributions. This allows us to perform a Taylor expansion of $\ln Z$ with respect to the coupling parameters δ_j. Starting from the Taylor expansion of Z_j with respect to the transmission coefficients

$$Z_j = [1 - T_j \, (t-r)] \left[1 + 2T_j \, r + T_j^2 \, x^2\right]^{-1/2}$$
$$= [1 - T_j(t-r)] \left[1 - rT_j - (x^2 - 3r^2)\, T_j^2/2\right] + O\left(T_j^3\right)$$
$$= 1 - t \; T_j + [rt + r^2/2 - x^2/2] \; T_j^2 + O\left(T_j^3\right). \qquad (29)$$

Insertion of Eq. (28) into the above Taylor expansion yields

$$\ln Z = -2\pi \sum_j \left[\, it \, \delta_j + \pi(\, r^2 + (2r+1)\, t - t^2 - x^2 \,)\, \delta_j^2\right] + O\left\{ \left[\frac{w_j(\beta)}{N}\right]^3 , \left[\frac{w_j(\alpha)}{N}\right]^3 \right\}. \qquad (30)$$

In order to obtain a well-defined function $Z(t,r,x)$ in the limit of $N \to \infty$, and vanishing perturbation $\delta_j \to 0$, the parameters δ_j must scale with an appropriate negative power of N:

(1) For $\delta_j \sim N^{-1}$, $\ln Z$ would converge to the finite value $-2\pi i \sum_j \delta_j \, t$.

(2) For $\delta_j \sim N^{-1/2}$, the sum $\sum_j \delta_j$ could still converge, if the δ_j had different signs. In addition, the sum $\sum_j {\delta_j}^2$ would always converge, while any higher order terms would vanish.

(3) For powers larger than $-1/2$, the sum $\sum_j {\delta_j}^2$ would always diverge, and the function $Z(t,r,x)$ would not be well defined.

Hence, the cases (1) and (2) are the only viable options, where

$$\sum_j \delta_j = \delta_s, \quad \sum_j {\delta_j}^2 = \lambda^2 \qquad (31)$$

converge to finite values. This finally leads to

$$\lim_{N\to\infty} Z = \exp(-2\pi i \, \delta_s \, t - \pi^2 \lambda^2 \, P_0), \qquad (32)$$

with P_0 given in Eq. (18). Note that by considering the fidelity $F(t) = |f(t)|^2$, instead of the fidelity amplitude, the dependence on δ_s disappears and with it any possible problems with the convergence of this term.

To conclude this section about global perturbations, let us discuss the random matrix model for fidelity decay, as has first been introduced in Ref. [7]. This model may be written as

$$H_\alpha = H_0 + \alpha\, V, \tag{33}$$

where the matrices H_0 and V are independent GOE matrices, normalized in such a way that the mean level spacing in the center of the spectrum of H_0 is $d_0 = 1$, while for the perturbation matrix it holds that

$$\langle V_{ij}\, V_{kl}\rangle = \delta_{jk}\delta_{il} + \delta_{ik}\delta_{jl}. \tag{34}$$

Now, representing H_α in the eigenbasis of V, the perturbation becomes diagonal with eigenvalues $w_j(\alpha)$ showing a semicircle distribution between $-2\alpha\sqrt{N}$ and $2\alpha\sqrt{N}$. Then, according to Eq. (2):

$$\kappa_j = 0, \quad \kappa_j' = \frac{i\pi}{N}\, w_j(\alpha) \quad \Rightarrow \quad \delta_j = \frac{w_j(\alpha)}{N}, \tag{35}$$

which is of order $N^{-1/2}$, as stipulated above. From the semicircle distribution of the eigenvalues $w_j(\alpha)$ it follows that

$$\sum_j \delta_j = \frac{1}{N}\sum_j w_j(\alpha) = 0, \tag{36}$$

$$\sum_j {\delta_j}^2 = \frac{1}{N^2}\sum_j w_j(\alpha)^2 = \alpha^2. \tag{37}$$

This shows that Eq. (32) applies for this case if we set $\delta_s = 0$ and $\lambda = \alpha$. From Eq. (24) it then follows that

$$f(t) = \left\langle P_0\, \mathrm{e}^{-\pi^2\lambda^2 P_0}\right\rangle_I, \tag{38}$$

in agreement with Ref. [8].

B. Local perturbations

In the case of local perturbations, W_α and W_β differ strongly in a finite-dimensional, typically rather small subspace. A Taylor series expansion as in the previous case is therefore not useful, and we may normally not assume that $[W_\alpha, W_\beta] = 0$. Instead, we redefine the perturbation by considering H_α as the unperturbed system and $W_\beta - W_\alpha$ as the perturbation. In doing so, we assume that including W_α into H_0 does not change its statistical (i.e. random matrix) properties. That is justified by the quantum chaos conjecture as long as the perturbation is classically small. We may then choose a basis in which $W_\beta - W_\alpha$ is diagonal, without affecting the the random matrix ensemble for H_0.

In contrast to the previous case, instead of Eq. (3), we now have

$$H_\beta = H_0' + \sum_{j=1}^{M} w_j'(\alpha,\beta)\, |v_j\rangle\langle v_j| \quad \text{where} \quad w_j'(\alpha,\alpha) = 0, \tag{39}$$

with $1 \leqslant M \ll N$. Then the Eqs. (24) and (27) give the exact analytical expressions for the scattering fidelity in the case of a transmission experiment $f_{ab}(t)$ $(a \neq b)$, or a reflection experiment $f_{aa}(t)$, respectively. We only have to express the effective transmission coefficients in Eq. (16) and the coupling parameters in Eq. (17) in terms of the new eigenvalues:

$$T_j = \frac{2\kappa_j'}{1+\kappa_j'}, \quad \kappa_j' = i\pi\, \frac{w_j'(\alpha,\beta)}{N}, \tag{40}$$

since $\kappa_j = 0$. Here it can be seen in the case of a finite M, the eigenvalues $w_j'(\alpha,\beta)$ must scale with N in order to make the fidelity or scattering fidelity decay on finite times (in units of the Heisenberg time). Note however that in order to derive the Eqs. (24) and (27) we assumed that the measurement channels a and b are coupled very weakly to the system, such that the corresponding transmission coefficients are negligibly small. This implies that the measurement channels must have negligible overlap with the eigenvectors $\{|v_j\rangle\}_{j=1,\ldots M}$ defined above. In the case of fidelity, as defined in Eq. (1), it is enough that the initial state $|a\rangle$ has negligible overlap, since for sufficiently large N, the overlap of a random state $|b\rangle$ as required in Eq. (15) with a finite-dimensional subspace is always negligible.

1. Moving scatterer

This case refers to the displacement of a small scatterer from a position $\vec{r}_1$ to another position $\vec{r}_2$. This may be modeled by the unperturbed Hamiltonian consisting of the system with scatterer at position $\vec{r}_1$, while the perturbation consists in removing the scatterer from position $\vec{r}_1$ and placing it at position $\vec{r}_2$. For a pointlike scatterer, the effect of the scatterer can be described by one single state (the perturbation operator corresponding to that scatterer has only one nonzero eigenvalue). Therefore,

$$H_\beta = H_0 + w(\beta)\,(\,|v_2\rangle\langle v_2| - |v_1\rangle\langle v_1|\,). \tag{41}$$

For the scattering approach to fidelity, this means that the perturbation must be described by two effective transmission coefficients:

$$T_1 = \frac{-2\pi i\, \delta_1}{1 - i\pi\, \delta_1}, \quad T_2 = \frac{2\pi i\, \delta_1}{1 + i\pi\, \delta_1}. \tag{42}$$

If $\delta_1 = w(\beta)/N$ becomes very large, T_1 as a function of δ_1 traces a path in the complex plane which starts at $T_1 = 0$ and ends at $T_1 = 2$, while $T_2 = T_1^*$. Then

$$Z = \prod_{j=1}^{M} \frac{1 - T_j(t-r)}{\sqrt{1 + 2T_j r + T_j^2 x^2}} = \frac{|1 - T_1\,(t-r)|^2}{|1 + 2T_1\, r + T_1^2 x^2|} \tag{43}$$

in Eqs. (24) and (27).

2. Comparison with experiment

The perturbation we just described applies precisely to an experiment published in Refs. [20,21]. There a small disk of diameter 4.6 mm has been moved in steps of $|\Delta r| = 1$ mm through a rectangular two-dimensional microwave billiard with 19 additional random scatterers. Then the reflection spectrum has been measured for 300 different positions of the moving disk in a frequency range from 3.5 to 6 GHz. In this frequency range, it was still possible to extract resonance positions and amplitudes by Lorentzian fits. The statistical properties of the spectrum as well as the wave functions were in agreement with the random matrix expectation for quantum chaotic or weakly disordered systems.

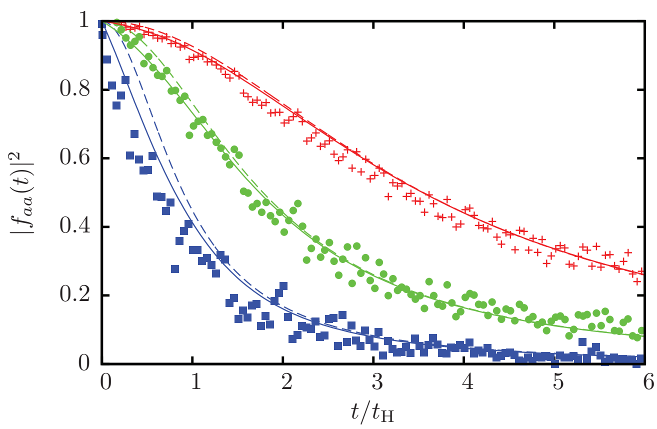

FIG. 1. (Color online) Experimental data for the fidelity decay due to a moving scatterer from Ref. [21], compared to the approximate (dashed lines) and to the exact theory (solid lines). The different colors, red (slowest decay), green (intermediate decay), and blue (fastest decay), correspond to different displacements $\Delta r = 1$ mm ($\delta_1 \approx 0.07$), 2 mm ($\delta_1 \approx 0.14$), and 4 mm ($\delta_1 \approx 0.28$), respectively. The corresponding experimental data (in the same order) are shown by red crosses, green circles, and blue squares.

From Berry's model of the random superposition of plane waves [27], it is possible to obtain a connection between the displacement $|\Delta r|$ of the movable disk and the parameter δ_1 which measures the perturbation strength. Translating the corresponding equation from Ref. [21] to our system of units and parameters, we obtain

$$\delta_1 = \frac{\alpha}{4}\,\sqrt{1 - J_0(k|\Delta r|)^2}, \tag{44}$$

where α is a dimensionless factor related to the electromagnetic properties of the movable disk, which can be determined independently (e.g., from the variance of the level velocities). For the wave number k, we choose a value which corresponds to the frequency in the center of the range considered, which yields

$$k = \frac{2\pi\, f}{c} = 0.996\ \text{cm}^{-1}. \tag{45}$$

For the displacements considered in Ref. [21], the relation between δ_1 and $|\Delta r|$ is still approximately linear, as can be seen from the fact that $k\,|\Delta r|$ is small as compared to one in all cases (see the captions of Fig. 1). Finally, we find that $\alpha = 1$ provides the best agreement between the theory and experiment. Using time-independent perturbation theory, the authors of Ref. [21] obtained the following asymptotic result for the decay of the fidelity amplitude:

$$f(t) = \frac{1}{\sqrt{1 + (4\delta_1 t)^2}}, \tag{46}$$

valid for finite $\delta_1\, t$, in the case where $\delta_1 \to 0$ and $t \to \infty$, and in agreement with our asymptotic result derived below in Sec. IV D. The most notable difference to our exact result in Eq. (27) is the quadratic decay at small times, where the exact result shows a linear decay. Note that we use an energy scale, where the mean level spacing is equal to one [see the text below Eq. (6)], such that the spectral span of the Hamiltonian H_0 goes to infinity in the semiclassical limit. That implies that the Zeno time scale, below which the fidelity decay must be quadratic (as discussed in Ref. [5]), goes to zero.

In Fig. 1 we show the experimental data for the decay of the absolute value squared $|f_{aa}(t)|^2$ as obtained in Ref. [21]. In the experiment, this quantity is obtained from ensemble averages of the respective correlation functions, introduced in Eq. (15). The results are compared to the theoretical predictions based on the perturbative result, Eq. (46), and on our exact analytical expression for a reflection measurement, Eq. (27). We note that both theoretical descriptions are quite close to the experimental data. However, focusing on the behavior of the scattering fidelity at small times, where the asymptotic formula is expected to be less accurate, we find indeed significant deviations for the cases $\delta_1 \approx 0.14$ (green points vs dashed green line) and $\delta_1 \approx 0.28$ (blue points vs dashed blue line). For these cases, the experimental fidelity decay has a notable linear component at small times, which cannot be reproduced by the perturbative formula, Eq. (46). By contrast, our exact analytical result (solid lines) contains that linear component and therefore agrees better with the experimental results. Still, in particular for $\delta_1 \approx 0.28$, some differences remain. We believe that these are rather due to problems on the experimental side. One error source consists in the wide frequency range used, which, according to Eq. (44) leads to a considerable variation in the perturbation strength. Another problem is related to the upper end of the frequency range, which implies rather small wavelengths, for which the scatterer to be moved may no longer be considered as pointlike. For a more significant test, one would need a different experimental design, allowing higher accuracies at strong perturbations in the vicinity of the Heisenberg time.

In the perturbative result, Eq. (46), it makes no difference whether the measurement is performed as a transmission measurement or as a reflection measurement with only one measurement antenna. As we have seen from Eqs. (24) and (27), for the exact analytical result this is no longer true. In Fig. 2 we compare both cases for four different perturbation strengths. The results show that the difference $|f_{aa}(t)|^2 - |f_{ab}(t)|^2$ is relatively small, but clearly noticeable. However, for sufficiently large ($\delta_1 = 0.56$, narrow peak at

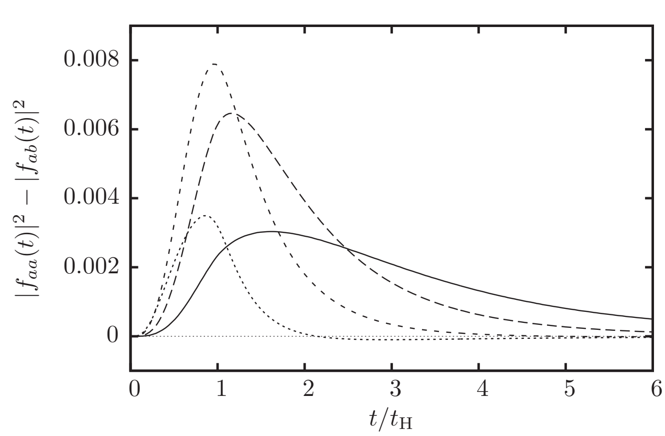

FIG. 2. Comparison between fidelity decay, measured in reflection $|f_{aa}(t)|^2$ and in transmission $|f_{ab}(t)|^2$. Different colors correspond to different perturbation strengths, $\delta_1 = 0.07$ (solid line), 0.14, 0.28, and 0.56 (dashed lines with decreasing dash lengths).

$t \lesssim t_H$) or sufficiently small perturbation strengths, it seems that the difference tends to disappear. This is consistent with our treatment of the perturbative case in the next section.

IV. PERTURBATIVE REGIME

In the perturbative regime, the fidelity amplitude of Eq. (1) can be calculated using first order time-independent perturbation theory [2]. For $\{|j\rangle\}$ denoting the eigenbasis of H_β and $V = H_\alpha - H_\beta$, we then find

$$f(t) \approx \sum_j \langle\alpha|j\rangle\, e^{2\pi i E_j(\beta)t}\, e^{-2\pi i[E_j(\beta)+\langle j|V|j\rangle]t}\, \langle j|\alpha\rangle$$
$$= \sum_j |\langle j|\alpha\rangle|^2\, e^{-2\pi i\langle j|V|j\rangle t}, \quad (47)$$

where $|\alpha\rangle$ denotes some initial state, and $\{E_j(\beta)\}$ denote the eigenvalues of H_β. This expression shows that in the perturbative regime the fidelity decay depends on the product between time and perturbation strength. The perturbative result becomes exact only in the limit of vanishing perturbation strength. To yield a finite value for the fidelity amplitude, time must then tend to infinity such that the product between perturbation strength and time remains constant. We therefore define the perturbative regime as the limit

$$t \to \infty, \quad \forall j\ :\ T_j \to 0, \quad (48)$$

such that $t\sum T_j$ and $t^2 \sum_j T_j^2$ remain both finite. In what follows, we calculate $f_{ab}(t)$ and $f_{aa}(t)$ in that limit, starting from the exact analytical expressions in Eqs. (24) and (27), via an asymptotic expansion of the respective integrals. This is done in two steps. Note that including the summations over the transmission coefficients, our treatment is valid for local and global perturbations.

A. Step one

Here we will demonstrate that $\langle\cdots\rangle_I \sim \langle\cdots\rangle_A$, with

$$\langle\cdots\rangle_A = \int_{t-1}^{t} dr \int_0^{\sqrt{r}} du\, \frac{(t-r)(r+1-t)(\cdots)}{(2u+1)(t^2-r^2+x^2)^2}, \quad (49)$$

where the ellipsis above may be replaced by either $Z\,P_0$ or $4t^2\,Z$. Here and in what follows, the symbol $\sim$ denotes the perturbative limit we are interested in.

For the first case, our claim follows from

$$\int_{t-1}^{t} dr \int_{\sqrt{r}}^{r} du\, \frac{(t-r)(r+1-t)\,P_0\,Z}{(2u+1)(t^2-r^2+x^2)^2} \sim 0. \quad (50)$$

Since $P_0 > 0$ and $0 < Z < 1$ in the whole region of integration, it is sufficient to show that Eq. (50) holds for $Z = 1$. Furthermore, since we can maximize $(t-r)(r+1-t)$ in the interval $t-1 < r < t$ by 1/4, it is sufficient to prove that

$$\max_{t-1<r<t} \int_{\sqrt{r}}^{r} du\, \frac{r^2+(2r+1)t-t^2-x^2}{(2u+1)(t^2-r^2+x^2)^2} \sim 0, \quad (51)$$

where we have used that $P_0 = 2\,(r^2+(2r+1)t-t^2-x^2)$. Denoting this integral with J, we realize that

$$J < \int_{\sqrt{r}}^{r} du\, \frac{r^2+(2r+1)t-t^2}{(2u+1)(t^2-r^2+x^2)^2}. \quad (52)$$

Then, because $r^2+(2r+1)t-t^2 = t+2r^2-(r^2-2rt+t^2)$,

$$J < \int_{\sqrt{r}}^{r} du \frac{2t^2+t}{(2u+1)x^4} = (2t^2+t)\int_{\sqrt{r}}^{r} du \frac{2u+1}{(2r+1)^2\,u^4}. \quad (53)$$

Evaluating the last integral we finally obtain

$$J < \frac{t\,(2t+1)}{(2r+1)^2}\left(\frac{3\sqrt{r}+1}{3r^{3/2}} - \frac{3r+1}{3r^3}\right) \sim 0, \quad (54)$$

which completes the proof. For the second case, we replace P_0 with $2t^2$ and arrive at the same result, which may be seen from Eq. (53).

B. Step two

According to Eq. (49) the perturbative limit only requires integration of u up to $u = \sqrt{r}$. This implies that

$$Z = \prod_j \frac{1-T_j(t-r)}{\sqrt{1+2T_j\,r+T_j^2\,x^2}} \sim \prod_j \frac{1}{\sqrt{1+2T_j\,t}}, \quad (55)$$

since $(t-r)$ is of order one, $t-1 < r < t$, and

$$T_j^2\,x^2 = T_j^2\,u^2\,\frac{2r+1}{2u+1} < T_j^2\,r\,\frac{2r+1}{2\sqrt{r}+1} \sim 0. \quad (56)$$

Therefore, we obtain for $f_{ab}(t) = \langle Z\,P_0\rangle_I$:

$$f_{ab}(t) \sim \prod_j \frac{1}{\sqrt{1+2T_j\,t}}\langle P_0\rangle_A \sim \prod_j \frac{1}{\sqrt{1+2T_j\,t}}. \quad (57)$$

This simply follows from the fact that $1 = \langle P_0\rangle_I \sim \langle P_0\rangle_A$. For the scattering fidelity in a reflection measurement, we obtain

$$f_{aa}(t) \sim \frac{4t^2\,\langle Z\rangle_I + 2\,\langle Z\,P_0\rangle_I}{3}, \quad (58)$$

since $b_2(t)$ in Eq. (27) tends to zero for large times. Here it only remains to treat the first term in the nominator:

$$4t^2\,\langle Z\rangle_I \sim \prod_j \frac{1}{\sqrt{1+2T_j\,t}}\,4t^2\,\langle 1\rangle_A \sim \prod_j \frac{1}{\sqrt{1+2T_j\,t}}, \quad (59)$$

since $1 = 4t^2\,\langle 1\rangle_I \sim 4t^2\,\langle 1\rangle_A$. Thus, in the perturbative regime, we obtain the same result no matter whether we perform a transmission or a reflection measurement:

$$f_{aa}(t) \sim f_{ab}(t) \sim f_{\text{pert}}(t) = \prod_j \frac{1}{\sqrt{1+2T_j\,t}}. \quad (60)$$

C. Global perturbation

Global perturbations are discussed in detail in Sec. III A, where Eq. (28) relates the effective transmission coefficients with the perturbation strengths δ_j. Taking also Eq. (31) into account, we may write for $f(t)$ up to second order in the perturbation strength:

$$\ln\, f_{\text{pert}}(t) \sim -\frac{1}{2}\sum_j \ln[\,1+4\pi i\,\delta_j\,(1-i\pi\,\delta_j)\,t\,]$$
$$\sim -\sum_j \left[2\pi i\,\delta_j\,t + 2\pi^2\delta_j^2\,(2t^2+t)\right]. \quad (61)$$

Since we are working in the perturbative regime, where t goes as fast to infinity as the δ_j go to zero, terms containing $\delta_j^2\, t$ can be neglected. In addition, the summation contains a number of terms which are of the order of N, taken to infinity in our analytical approach. In this limit, δ_j^2 scales with N^{-1}, such that we obtain

$$f_{\rm pert}(t) = \exp(-2\pi i\, \delta_{\rm s}\, t - 4\pi^2\lambda^2\, t^2). \tag{62}$$

D. Moving scatterer

Inserting the effective transmission coefficients from Eq. (42) describing a moving scatterer into Eq. (60), we find

$$f_{\rm pert}(t) = \left| 1 - \frac{4i\, \delta_1\, t}{1 - i\, \delta_1} \right|^{-1}. \tag{63}$$

In the perturbative limit considered this implies that $T_1 \to 0$, $t \to \infty$ such that $T_1\, t$ remains constant. Consequently, also $\delta_1 \to 0$ such that $\delta_1\, t$ remains constant. Therefore $f_{\rm pert}(t) = |1 - 4i\, \delta_1\, t|^{-1}$ in agreement with Eq. (46), from Ref. [21].

V. COMPARISON WITH PARAMETRIC LEVEL CORRELATIONS

In this section we will generalize the relation between fidelity amplitude and the parametric level correlations (FA-PLC relation) discovered in Ref. [14] to arbitrary perturbations. For that purpose we use our initial definition of the fidelity amplitude in Eqs. (1) and (2), but choose $H_\alpha = 0$. This typically does not imply any restriction of generality, as explained at the beginning of Sec. III B. To proceed, we separate global and local perturbations explicitly. Hence,

$$H_{\lambda,\beta} = H_0 + W_\alpha + \sum_{k=1}^{M} w'_k(\beta)\, |v'_k\rangle\langle v'_k|, \tag{64}$$

where the unique quantity to characterize the global perturbation W_α is the strength parameter λ, defined in Sec. III A, which fulfills

$$\lambda^2 = \frac{{\rm tr}\big(W_\alpha^2\big)}{N^2} = \frac{1}{N^2} \sum_j w_j(\alpha)^2, \quad \frac{1}{N} \sum_j w_j(\alpha) = 0. \tag{65}$$

These two equations just mean that the eigenvalues $w_j(\alpha)$ are symmetrically distributed around zero, and that their magnitudes scale with $\sqrt{N}$. In order to derive an analytic expression for the fidelity amplitude we need to find a common eigenbasis for both types of perturbations. For that purpose, we replace W_α with its projection $\tilde W_\alpha$ onto the orthogonal complement of the subspace spanned by the eigenvectors $\{|v'_k\rangle\}$ and show that this will not change the result for the fidelity amplitude decay.

Let $\{|v''_l\rangle\}_{l=1,\ldots,N}$ be a orthonormal basis of the full Hilbert space such that $|v''_k\rangle = |v'_k\rangle$ for all $k = 1,\ldots,M$. Then we obtain with $P_\beta = \sum_{k=1}^M |v'_k\rangle\langle v'_k|$

$$\begin{aligned} {\rm tr}\big[\tilde W_\alpha^2\big] &= {\rm tr}[\,(\mathbb{1} - P_\beta)\, W_\alpha\, (\mathbb{1} - P_\beta)\, W_\alpha\,] \\ &= {\rm tr}\big[W_\alpha^2\big] - 2\, {\rm tr}\big[P_\beta\, W_\alpha^2\big] + {\rm tr}[P_\beta W_\alpha P_\beta W_\alpha] \\ &= N^2\lambda^2 - 2 \sum_{k=1}^{M} \langle v'_k|\, W_\alpha^2\, |v'_k\rangle + \sum_{k,l=1}^{M} |\langle v'_k| W_\alpha |v'_l\rangle|^2. \end{aligned} \tag{66}$$

Then the inequality

$$\begin{aligned} \sum_{k,l=1}^{M} |\langle v'_k| W_\alpha |v'_l\rangle|^2 &\leqslant \sum_{k=1}^{M} \sum_{l=1}^{N} \langle v'_k| W_\alpha\, |v''_l\rangle\langle v''_l|\, W_\alpha |v'_k\rangle \\ &= \sum_{k=1}^{M} \langle v'_k|\, W_\alpha^2\, |v'_k\rangle = \sum_{k=1}^{M} \sum_{j=1}^{N} \langle v'_k|v_j\rangle\, w_j(\alpha)^2\, \langle v_j|v'_k\rangle \\ &\sim \sum_{k=1}^{M} \sum_{j=1}^{N} \langle v'_k|v_j\rangle\, \sqrt{N}\, \langle v_j|v'_k\rangle = M\sqrt{N} \end{aligned} \tag{67}$$

implies that in the limit $N \to \infty$,

$$\frac{1}{N^2}\, {\rm tr}\big[\tilde W_\alpha^2\big] - \lambda^2 \sim \frac{M}{\sqrt{N}} \to 0. \tag{68}$$

This proves the validity of the replacement, such that we may finally write for the above Hamiltonian:

$$H_{\lambda,\beta} = H_0 + \sum_{k=1}^{M} w'_k(\beta)\, |v''_k\rangle\langle v''_k| + \sum_{k=M+1}^{N} \tilde w_k(\alpha)\, |v''_k\rangle\langle v''_k|, \tag{69}$$

where the $\tilde w_k(\alpha)$ are the new eigenvalues of $\tilde W_\alpha$. This results in the following expression for the fidelity amplitude:

$$f(\lambda,t) = \big\langle P_0\, {\rm e}^{-\pi^2\lambda^2 P_0}\, Z_{\rm loc}\big\rangle, \tag{70}$$

with $Z_{\rm loc}$ given by the expression for Z in Eq. (11), but considering the eigenvalues of the local perturbation only.

The parametric level correlations $X(\lambda,r)$ describe the probability to find two eigenvalues, one of H_0 and the other one of $H_{\lambda,\beta}$ at a distance r. The quantity to be compared to the fidelity amplitude is the Fourier transform of the parametric level correlations [14]:

$$K(\lambda,t) = \int dr\; {\rm e}^{2\pi i\, rt}[X(\lambda,r) - 1]. \tag{71}$$

Note that for $\lambda \to 0$, this quantity converges to the complement of the two-point form factor: $K(0,t) = 1 - b_2(t)$ [23,28,29]. The FA-PLC relation discussed in Ref. [14] may now be expressed as

$$f(\lambda,t) = \frac{-\beta}{4\pi^2 t^2}\, \frac{\partial}{\partial\, (\lambda^2)}\, K(\lambda,t), \tag{72}$$

with β being the Dyson parameter [30] which is one, as we consider systems with an anti-unitary symmetry and real matrix representations of their Hamiltonians. In Ref. [14] this relation is proven for a purely global perturbation ($Z_{\rm loc} = 1$) only.

For pedagogical reasons, we will prove Eq. (72) for purely global perturbations first (Sec. V A), as the prove for the general case follows the same lines. Without local perturbations, the expressions are less lengthy, and the line of argument is easier to follow. In Sec. V B we consider the general case and then need only to indicate which, and if so how, some of the expressions must be modified.

A. Global perturbation

In that case $Z_{\rm loc} = 1$, and from Ref. [22] we find

$$X(\lambda,r) = 1 + \mathrm{Re} \iint_1^\infty d\lambda_1 d\lambda_2 \int_{-1}^1 d\mu' \times \frac{(\lambda_1\lambda_2 - \mu')^2 (1-\mu'^2)\, e^{i\pi r_+ (\lambda_1\lambda_2-\mu')}\, e^{-\pi^2\lambda^2 \tilde{P}_0}}{(1+2\lambda_1\lambda_2\,\mu' - \lambda_1^2 - \lambda_2^2 - \mu'^2)^2}, \tag{73}$$

where $2\tilde{P}_0 = 1 + 2\lambda_1^2\lambda_2^2 - \lambda_1^2 - \lambda_2^2 - \mu'^2$, and r_+ equals r plus an additional positive imaginary increment.

The first substitution, $\mu' \to \mu = (\lambda_1\lambda_2 - \mu')/2$, yields

$$X(\lambda,r) = 1 + 2\,\mathrm{Re} \iint_1^\infty d\lambda_1 d\lambda_2 \int_{(\lambda_1\lambda_2-1)/2}^{(\lambda_1\lambda_2+1)/2} d\mu \times \frac{4\mu^2 (1-\mu'^2)\, e^{2i\pi r_+ \mu}\, e^{-\pi^2\lambda^2 \tilde{P}_0}}{(1+2\lambda_1\lambda_2\,\mu' - \lambda_1^2 - \lambda_2^2 - \mu'^2)^2}. \tag{74}$$

In order to shorten the expressions, we keep writing μ' which must be understood as being a function of μ. Now, we can switch to the Fourier transform, which turns the Fourier factors in delta functions:

$$K(\lambda,t) = \iint_1^\infty d\lambda_1 d\lambda_2 \int_{(\lambda_1\lambda_2-1)/2}^{(\lambda_1\lambda_2+1)/2} d\mu \times \frac{[\,\delta(t+\mu) + \delta(t-\mu)\,]\, 4\mu^2 (1-\mu'^2) e^{-\pi^2\lambda^2 \tilde{P}_0}}{(1+2\lambda_1\lambda_2\,\mu' - \lambda_1^2 - \lambda_2^2 - \mu'^2)^2}. \tag{75}$$

This shows that the function $K(\lambda,t)$ is symmetric in time. In what follows, we thus assume $t > 0$. The remaining delta function already allows us to eliminate the μ integration. However, before actually doing so, we perform a variable transformation on the λ_1,λ_2 double integral:

$$(\lambda_1,\lambda_2) \to (r',x'), \quad r' = \lambda_1\lambda_2, \quad x' = \lambda_2 - \lambda_1. \tag{76}$$

The Jacobian of this transformation is simply $J = (\lambda_1 + \lambda_2)^{-1} = (x'^2 + 4r')^{-1/2}$. Therefore,

$$K(\lambda,t) = \int_1^\infty dr' \int_{1-r'}^{r'-1} \frac{dx'}{\sqrt{x'^2+4r'}} \int_{(r'-1)/2}^{(r'+1)/2} d\mu \times \frac{4\mu^2 (1-r'+2\mu)(1+r'-2\mu)\delta(t-\mu) e^{-\pi^2\lambda^2 \tilde{P}_0}}{[1+2r'(r'-2\mu) - x'^2 - 2r' - (r'-2\mu)^2]^2}. \tag{77}$$

For the μ integral not to yield zero, it must hold that $(r'-1)/2 < t < (r'+1)/2$. This modifies the limits of the r' integral as follows:

$$K(\lambda,t) = 4t^2 \int_{\max(1,2t-1)}^{2t+1} dr' \int_{1-r'}^{r'-1} \frac{dx'}{\sqrt{x'^2+4r'}} \times \frac{(1-r'+2t)(1+r'-2t)\, e^{-\pi^2\lambda^2 \tilde{P}_0}}{[1+2r'(r'-2t) - x'^2 - 2r' - (r'-2t)^2]^2}. \tag{78}$$

Further substitutions $r' = 2r + 1$ and $x' = 2x$ and the fact that the integrand is a symmetric function of x yield

$$\begin{aligned} K(\lambda,t) &= 4t^2 \int_{\max(0,t-1)}^t dr \times \int_0^r dx \frac{(t-r)(r+1-t) e^{-\pi^2\lambda^2 \tilde{P}_0}}{\sqrt{x^2+2r+1}(t^2-r^2+x^2)^2} \\ &= 4\,t^2 \left\langle e^{-\pi^2\lambda^2 \tilde{P}_0} \right\rangle_I. \end{aligned} \tag{79}$$

Performing all variable substitutions to $\tilde{P}_0$ we find that it becomes a function of t, r, and x, which agrees precisely with P_0 defined in Eq. (18). Comparing to Eq. (70) for $Z_{\rm loc} = 1$, it is now easily checked that $K(\lambda,t)$ as defined here fulfills the FA-PLC relation Eq. (72).

B. General perturbation

In Ref. [22] it is shown that parametric level correlations can be calculated for arbitrary perturbations. According to this reference, the term describing the global perturbation $\sigma_{\rm glob} = \pi^2\lambda^2 P_0$ must be replaced by $\sigma = \sigma_{\rm glob} + \sigma_{\rm loc}$, where

$$\sigma_{\rm loc}(\lambda_1,\lambda_2,\mu') = \frac{1}{2}\sum_j \ln\left[\frac{1 + 2i\kappa_j'\lambda_1\lambda_2 - \kappa_j'^2(\lambda_1^2+\lambda_2^2-1)}{(1+i\,\kappa_j'\,\mu')^2}\right]. \tag{80}$$

Note that the discussion in Sec. III A shows that the additional global perturbation could always be incorporated into $\sigma_{\rm loc}$, via a large number of additional channels with infinitesimal perturbations. However, in order to establish the desired relation between fidelity decay and the parametric level correlations, it is important to have the parameter λ describing the global perturbation at hand.

Now, we should go through the calculation of $K(\lambda,t)$ again, replacing $\sigma_{\rm glob}$ in Eq. (74) with the more general expression σ. As a consequence, the integrand is no longer real, which affects Eq. (75). While the delta function $\delta(t-\mu)$ is multiplied with the same term as before, the second delta function $\delta(t+\mu)$ is now multiplied with its complex conjugate. Therefore $K(\lambda,t)$ is no longer symmetric. Instead $K(\lambda,t) = K(\lambda,-t)^*$, which nevertheless allows us to continue the calculation without changes for $t > 0$. Only in Eq. (79) do we need to express $\sigma_{\rm loc}$ in the current integration variables. That results in

$$\exp[-\sigma_{\rm loc}(r,x,t)] = \prod_j \frac{1 - T_j\,(t-r)}{\sqrt{1 + 2T_j\,r + T_j^2\,x^2}}, \tag{81}$$

where $T_j = 2i\kappa_j'/(1+i\,\kappa_j')$, just as in Eq. (28). Inserting this expression into Eq. (79) and comparing to the general result Eq. (24) for the decay of the fidelity amplitude ($a \neq b$) we find that Eq. (72) holds also for this case, where we have an additional term describing local perturbations. This constitutes our second important result. In practice, it means that one can obtain the fidelity amplitude in the presence of arbitrary local and/or global perturbations, by measuring the change of the parametric level correlations under the increment of the global perturbation.

VI. CONCLUSIONS

In this paper we have used the results of Ref. [13] to derive an exact analytical expression for the fidelity decay in a closed chaotic and diffusive wave system, under arbitrary Hermitian perturbations. For illustration, we used that result to rederive the known formula for the fidelity decay in the case of a global perturbation [8,9]. In a second application, we calculated the fidelity amplitude for a moving S-wave scatterer and verified that it describes corresponding experimental results reported in Ref. [21] well. Another analytical result that can be treated along these lines is the case of a symmetry-breaking perturbation [31,32]. Finally, we generalized a relation between the fidelity amplitude and parametric level correlations introduced in Ref. [14] to arbitrary perturbations. The latter is probably the most important result of this work, since it turns the relation into an analytical tool which can be applied to inverse problems, i.e., in cases where we want to deduce properties of the perturbation from measured fidelity or parametric level correlation data.

In the present work, we restricted ourselves to matrix ensembles based on the Gaussian orthogonal ensemble (GOE). For the comparison with experimental data, this is the most important case. However, our results can also be translated to the Gaussian unitary ensemble (GUE), using the analog of the VWZ formula published in Ref. [24]. For the Gaussian symplectic ensemble (GSE), the corresponding analytical expressions for the parametric level correlations and the correlations between scattering matrix elements are unfortunately not yet available, but we would still expect a similar relation to hold.

It would be interesting to perform an experiment similar to the one analyzed in Refs. [20,21] in order to verify our results with higher accuracy and for larger perturbation strengths. Particularly interesting would be the regime where the perturbation strength depends in a nonlinear way on the displacement of the scatterer; see Eq. (44). If the microwave experiment would allow to measure fidelity decay and parametric level correlations at the same time, one could test the applicability of the FA-PLC relation in practice. Finally, one may intend to generalize the FA-PLC relation further to scattering systems and non-Hermitian perturbations (e.g., coupling fidelity).

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