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Phase transitions of the *p*-spin model on pure Husimi lattices

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We consider the p-spin model with spin $1/2$ on all pure Husimi lattices. Using an effective representation of the recursion relations, the phase transitions of the model on all pure Husimi lattices are investigated. First, the nonexistence of the second order phase transitions in the model on all pure Husimi lattices is proven exactly. Then the existence and properties of the first order phase transitions in a zero external magnetic field are studied in detail. An implicit polynomial equation for determining temperatures below which the paramagnetic and ferromagnetic phases coexist in the model on all pure Husimi lattices is found. In addition, an implicit equation for exactly determining the transition temperatures of the first order phase transitions in a zero external magnetic field on all pure Husimi lattices is derived and discussed.

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I. INTRODUCTION

During the last several decades the Husimi trees and lattices [1–3] have played a significant role in describing many interesting physical systems and phenomena, especially in situations where multisite interactions play a relevant role, e.g., in binary alloys, rare gases, or lipid bilayer problems [4]. At the same time, it was also shown (see, e.g., Refs. [5–7]) that for such systems one can obtain better results by using the Husimi lattices than using the mean-field theory approximation on real lattices [8,9]. Therefore, although the Husimi lattices (as all the so-called recursive lattices) behave effectively as infinite dimensional systems, nevertheless various Husimi lattices were also widely used for describing properties of many other physical systems such as amorphous solids [10], spin liquids [11], Ising spin glasses [12–18], various polymer models [19–27], Abelian sandpiles [28,29], lattice gasses [30], ^{3}He systems [31,32], and random optimization problems [33]. In addition, during last two decades many theoretical studies have been carried out to analyze mathematical properties of the Ising and Ising-like models on the Husimi lattices (see, e.g. Refs. [34–43] as well as references cited therein).

However, the common property of the classical Ising and Ising-like models on various recursive lattices, namely, that they can be studied with arbitrary precision in the form of recursion relations (see, e.g., Ref. [44]), gives us a rather unique opportunity in statistical mechanics for studying global statistical properties of the models with different values of the spin variable and/or with different values of the parameters that characterize the corresponding recursive lattices (e.g., the coordination number) on the same footing and to analyze also possible nontrivial relations among them. It can be said without doubt that apart from purely theoretical and mathematical interest in this analysis such investigations are also certainly important for deeper understanding of already existing results as well as the fact that they have large predictive power when the corresponding models are applied in new physical situations. Therefore, it is rather surprising that despite these facts only a few studies exist in the literature devoted to such general analysis. In this respect, for example, in Ref. [45] general investigation of the $1/2 - s$ mixed spin Ising model on the Bethe lattices was done. On the other hand, in Ref. [46] it was shown that there must exist a strong relation between the critical temperatures of the Ising model with arbitrary values of spin on a given Bethe lattice. In addition, quite recently [47] it was shown that there is also the strong relation between critical temperatures of spin $1/2$ Ising model on all pure Husimi lattices with arbitrary coordination numbers (see the next section for precise definition of pure Husimi lattices). Strictly speaking, in Ref. [47] an effective general representation of the recursion relations for the spin $1/2$ Ising model on arbitrary pure Husimi lattices was found and was used to derive a polynomial equation which determines exactly the position of the critical temperatures of the model on all pure Husimi lattices. In addition, it was also shown that the existence of this effective representation of the recursion relations allows one to write common expressions for all important quantities of the model for all pure Husimi lattices; i.e., it was shown that the spin $1/2$ Ising model on all pure Husimi lattices can be studied simultaneously. This is a nontrivial result which, on one hand, from the mathematical point of view, sheds light on the internal structure of the solutions of the model on various Husimi lattices and, on the other hand, can be immediately applied in many physical problems. Without doubt, it would be also interesting to see if the corresponding general analysis can be carried out with other models usually studied on various Husimi lattices.

In this respect, in the present paper we shall concentrate on the so-called p-spin model on pure Husimi lattices, i.e., on the model with the multisite interaction among all sites of elementary polygons out of which the corresponding pure Husimi lattice is built up. An effective representation of the recursion relations will be found which will allow us to perform the global analysis of the phase transitions of the model simultaneously for all pure Husimi lattices.

The existence of the aforementioned representation of the recursion relations will allow us to prove exactly that the phase transitions of the second order are not possible in this model regardless of the form of the pure Husimi lattice. On the other hand, the phase transitions of the first order in a zero external magnetic field between paramagnetic and ferromagnetic phases are studied in detail by using numerical as well as analytical means. As one of the main results of the present paper, an implicit polynomial-like equation is derived which determines exact positions of the transition temperatures of the first order phase transitions of the model in a zero external magnetic field simultaneously on all pure Husimi lattices. In addition, an explicit expression for exact determining temperatures below which the paramagnetic and ferromagnetic phases coexist is derived.

The paper is organized as follows. In Sec. II, the model is briefly defined. In Sec. III, an appropriate representation of the recursion relations is found and discussed. In Sec. IV, it is shown exactly that the phase transitions of the second order are not possible in the present model. In Sec. V, the phase transitions of the first order in a zero external magnetic field are studied in detail. In Sec. VI, the main results of the paper are reviewed.

II. THE MODEL

Let us consider the so-called p-spin model (also known as the multisite interaction model) with spin $s = 1/2$ on an arbitrary pure Husimi lattice defined as follows [1,2]: By definition a *Husimi tree* is a connected graph in which no line lies on more than one cycle. A *pure Husimi tree* is a special type of Husimi tree which consists of only one type of figure (lines, triangles, etc.) out of which it is built up. For example, the well-known Cayley tree which consists only of lines, i.e., it has no cycles, is the simplest case of the pure Husimi tree. The pure Husimi lattice is now obtained in the same way as the Bethe lattice is defined from the corresponding Cayley tree (see, e.g., Ref. [44]); i.e., we suppose that we are located deep inside of the graph, where all sites are equivalent to each other. Thus, each pure Husimi lattice is exactly defined by two numbers, namely, by the number of sites of the single polygon $p \geqslant 2$ and by the coordination number $z = 2q$ (for $p > 2$), where q denotes the number of p polygons which meet at each site (see Fig. 1). In addition, in the special case with $p = 2$ one comes to the Bethe lattice with the coordination number $z = q$.

The Hamiltonian of the model has the following form:

$$\mathcal{H} = -J \sum_{P} \prod_{s_i \in P} s_i - H \sum_{i} s_i, \qquad (1)$$

where each variable s_i can acquire one of the two possible values, namely, $1/2$ and $-1/2$, J is the multisite interaction among the sites of each polygon P, and H represents the external magnetic field. In Hamiltonian (1), the first sum runs over all polygons, and the second sum runs over all spin

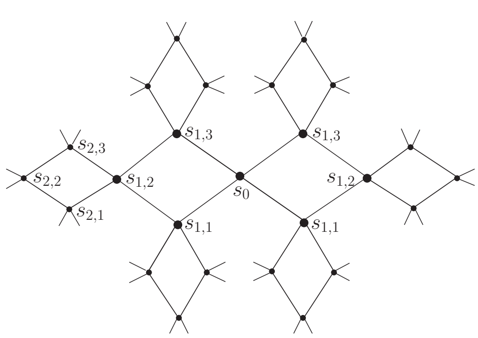

FIG. 1. The pure Husimi lattice with $p = 4$ and $q = z/2 = 2$. The site denoted as s_0 is taken to be the central site. The sites denoted as $s_{1,i}, i = 1, 2, 3$ form the first layer of the Husimi lattice, etc.

sites. In the special case $p = 2$, Hamiltonian (1) turns to the Hamiltonian of the well-known Ising spin $1/2$ model, i.e., to the model with the nearest-neighbor interaction, on the Bethe lattices; therefore we shall not consider it here. Thus, in what follows, we always suppose that $p \geqslant 3$.

The partition function of the model (1) is

$$Z \equiv \sum_{s} e^{-\beta\mathcal{H}} = \sum_{s} e^{K \sum_P \prod_{s_i \in P} s_i + h \sum_i s_i}, \qquad (2)$$

where $\beta = 1/(k_B T)$, T is the temperature, k_B is the Boltzmann constant, $K = \beta J$, and $h = \beta H$. The sum over s in Eq. (2) means the summation over all possible spin configurations on the considered lattice.

Standardly, the model (1) on an arbitrary pure Husimi lattice can be studied by using the method of recursion relations in a similar way as it is usually performed on the Bethe lattices (see, e.g., Ref. [44]). However, using the formalism introduced in Ref. [47] a simple representation for the recursion relations can be found which allows one to study the properties of the model on different pure Husimi lattices on the same basis; e.g., it will allow us to make a general analytic analysis of the properties of the phase transitions of the model.

III. GENERAL SOLUTION OF THE MODEL

A. A suitable representation of recursion relations

If a pure Husimi recursive lattice is cut in the site 0 (see Fig. 1), then the corresponding graph splits into $q = z/2$ (for $p \geqslant 3$) disconnected identical pieces (subgraphs) and the partition function (2) can be rewritten as follows:

$$Z = \sum_{s_0} \exp\left(h s_0\right) [g_n(s_0)]^q, \qquad (3)$$

where functions $g_n(s_0)$ for $s_0 = 1/2$ and $-1/2$ can be expressed in the form of recursion relations, namely,

$$g_n(s_0) = \sum_{s_{1,1},\ldots,s_{1,p-1}} \exp\left\{K s_0 \prod_{l=1}^{p-1} s_{1,l} + h \sum_{l=1}^{p-1} s_{1,l}\right\} \times \prod_{l=1}^{p-1} [g_{n-1}(s_{1,l})]^{q-1}, \qquad (4)$$

where we suppose that the pure Husimi lattice has n layers and $s_{1,l}$, $l = 1,\ldots,p-1$ represent the spin variables of sites which lie on shell 1 (see Fig. 1).

However, it is usually appropriate to rewrite the recursion relations (4) for $g_n(s_0)$ into the form of recursion relations for their ratios:

$$x_n(i,t) = g_n\left(s_0^i\right)/g_n\left(s_0^t\right), \quad i = 0,1, \qquad (5)$$

where $s_0^i = 1/2 - i$ is introduced for possible values of the spin variable and the integer t can be chosen arbitrary from two possible values 0 and 1. The choice is completely free. But once one of the two possibilities is chosen, then it must be used in all calculations.

The explicit form of recursion relations $x_n(i,t), i = 0,1$ in Eq. (5) for the model (1) is given as follows:

$$x_n(i,t)=\frac{\sum_{j_1=0}^{1}\cdots\sum_{j_{p-1}=0}^{1} A_{i;j_1,\ldots,j_{p-1}} \prod_{l=1}^{p-1}[x_{n-1}(j_l,t)]^{q-1}}{\sum_{j_1=0}^{1}\cdots\sum_{j_{p-1}=0}^{1} A_{t;j_1,\ldots,j_{p-1}} \prod_{l=1}^{p-1}[x_{n-1}(j_l,t)]^{q-1}}, \tag{6}$$

where coefficients $A_{a;j_1,\ldots,j_{p-1}}$ for $a = i,t$ are

$$A_{a;j_1,\ldots,j_{p-1}} = e^{K(1/2-a)\prod_{l=1}^{p-1}(1/2-j_l)+h\sum_{l=1}^{p-1}(1/2-j_l)}. \tag{7}$$

Although, recursion relations (5) play the central role in the numerical analysis of the present model for given values of the parameters, nevertheless their representation given in Eq. (6) is not suitable for a general analysis of the model simultaneously on all pure Husimi lattices simply because representation (6) contains a different number of summations for different values of parameter p. However, as was shown in Ref. [47], a more appropriate and effective representation of the recursion relations (6) exists. This representation can be further simplified when only pure multisite interaction within single polygons is considered. To this end, first, it is necessary to fix the value of t in Eq. (5). Without loss of generality, let us put $t = 0$. In this case, $x_n(0,0) = 1$, and one has only one independent recursion relation for the quantity $x_n \equiv x_n(1,0)$ in Eq. (6), which can be rewritten in the following convenient form:

$$x_n = \frac{Y_1(x_{n-1})}{Y_0(x_{n-1})}, \tag{8}$$

where now

$$Y_0(x) = \sum_{j=0}^{p-1}\binom{p-1}{j} x^{j(q-1)} e^{\frac{h(p-2j-1)}{2}} \times \left[\sigma_j\, e^{-\frac{K}{2^p}} + (1-\sigma_j)e^{\frac{K}{2^p}}\right] \tag{9}$$

and

$$Y_1(x) = \sum_{j=0}^{p-1}\binom{p-1}{j} x^{j(q-1)} e^{\frac{h(p-2j-1)}{2}} \times \left[\sigma_j\, e^{\frac{K}{2^p}} + (1-\sigma_j)e^{-\frac{K}{2^p}}\right], \tag{10}$$

where

$$\sigma_j \equiv j (\mathrm{mod}\, 2), \tag{11}$$

and $\binom{a}{b} = a!/[b!(a-b)!]$ is the binomial coefficient.

Derivation of representation (8)–(10) for recursion relations (6) with (7), which plays the central role in what follows, is based on relatively simple straightforward combinatory analysis. The main idea is the following. All exponents in Eq. (7) have the form $e^{\alpha_1 K+\alpha_2 h}$, where α_1 can obtain only two possible values, namely, $-1/2^p$ and $1/2^p$, depending on the spin configuration in the polygon. On the other hand, α_2 runs from $-(p-1)/2$ to $(p-1)/2$ with increment 1. Thus, the main task is to find the number of all spin configurations in the polygon that simultaneously lead to given values of α_1 and α_2. These numbers are exactly given by the corresponding binomial coefficients together with factors σ_j and $1-\sigma_j$ as they are shown explicitly in Eqs. (9) and (10).

For convenience it is also appropriate to introduce the following notation:

$$z_j = j(q-1), \tag{12}$$

$$B_j^{\pm} = \pm\frac{1}{2^p} + \frac{h}{K}\frac{(p-2j-1)}{2}, \tag{13}$$

$$C_j = \binom{p-1}{j}. \tag{14}$$

This notation allows us to write functions Y_0 and Y_1 in Eqs. (9) and (10) in the following compact symmetric form appropriate for further analysis:

$$Y_0(x) = \sum_{j=0}^{p-1} C_j\, x^{z_j}\left[\sigma_j\, e^{KB_j^-} + (1-\sigma_j)e^{KB_j^+}\right], \tag{15}$$

$$Y_1(x) = \sum_{j=0}^{p-1} C_j\, x^{z_j}\left[\sigma_j\, e^{KB_j^+} + (1-\sigma_j)e^{KB_j^-}\right]. \tag{16}$$

B. Magnetization and spontaneous magnetization

For the model under consideration the standard definition of the magnetization per site m gives

$$m \equiv \langle s_0\rangle = Z^{-1}\sum_s s_0 \exp\left(K\sum_P\prod_{s_i\in P} s_i + h\sum_i s_i\right), \tag{17}$$

and by using the ratios defined in Eq. (5) the expression for magnetization (17) can be written in the following form:

$$m = \frac{1}{Z'}\sum_{i=0}^{1}(1/2-i)\exp[h(1/2-i)]\,[x_n(i,t)]^q, \tag{18}$$

where the explicit expression for $Z' \equiv Z/[g_n(s_0^t)]^q$ is

$$Z' = \sum_{i=0}^{1}\exp\,[h(1/2-i)]\;[x_n(i,t)]^q. \tag{19}$$

Finally, using the representation of the recursion relation (8) with (15) and (16) in the limit $n\to\infty$, when x_n obtains the fixed point value $x \equiv \lim_{n\to\infty} x_n$, one has

$$m = \frac{1}{2}\frac{e^{h/2} - e^{-h/2}x^q}{e^{h/2} + e^{-h/2}x^q}, \tag{20}$$

and for the spontaneous magnetization in the zero external magnetic field one finally obtains

$$m = \frac{1}{2}\frac{1-x^q}{1+x^q}. \tag{21}$$

Thus, solving the recursion relation (8) with (15) and (16) for given reduced temperature $K^{-1} = k_BT/J$ and for a given value of the magnetic field h the behavior of the magnetization of the system can be studied for an arbitrary pure Husimi lattice with given p and q. We shall return to the analysis of the behavior of the spontaneous magnetization (21) in Sec. V, where the first order phase transitions of the model will be discussed.

C. Free energy

One of the most important quantity, especially for analysis of the first order phase transitions, is the free energy standardly given by the partition function Z as follows:

$$\beta F = -\ln Z. \tag{22}$$

However, when one works with recursive lattices the surface sites of the corresponding trees play important role even in the limit $n \to \infty$, and the free energy defined by Eq. (22) strongly depends on surface conditions (see, e.g., Refs. [7,39] and references cited therein). In this situation, the free energy defined directly through the entire partition function Z can lead to strange results that are usually not in agreement with results obtained on regular lattices [48–50]. Therefore, although the free energy defined by Eq. (22) is exact, it represents the free energy of the recursive trees (e.g., Caley trees or pure Husimi trees) and cannot be considered as an appropriate definition of the "interior" free energy of the corresponding recursive lattices that should be independent of the surface properties.

The general analysis of the free energy of the Ising model with arbitrary value of spin on arbitrary pure Husimi lattices was performed in Ref. [39] (see also discussion in Ref. [7]). Thus, using the results of Ref. [39] one can write the interior free energy per site of the present model on pure Husimi lattices as follows:

$$\beta F = \frac{(p-1)(q-1)-1}{p}\ln[Y_2(x)] - \frac{q}{p}\ln[Y_0(x)], \tag{23}$$

where $Y_0(x)$ is given in Eq. (15) and

$$Y_2(x) = e^{h/2} + e^{-h/2}x^q. \tag{24}$$

Let us note that the free energy (23) for $p = 2$, i.e., for the Bethe lattices, is equal to the free energy per site given by Baxter [44] and is $2/q$ times the free energy derived by Gujrati [7]. In Sec. V, the free energy defined by Eq. (23) will be used for analysis of the first order phase transitions in the present model. It should be also noted that the free energy obtained by using the procedure shown, e.g., in Ref. [47], which is in fact the free energy per site for infinite pure Husimi trees, cannot be used for this purpose.

IV. NONEXISTENCE OF THE SECOND ORDER PHASE TRANSITIONS

One of the basic characteristics of the lattice statistical models is the existence or nonexistence of the critical temperature T_c, i.e., the temperature at which the second order phase transition occurs in zero external magnetic field $h = 0$. We shall show that the existence of the phase transition of the second order is not possible in the pure multisite model on arbitrary pure Husimi lattices.

As was already mentioned in the previous section, in the limit $n \to \infty$, i.e., when we are deep inside of the Husimi lattice, the recursion relation $x_n \equiv x_n(1,0)$ in Eq. (8) obtains a fixed point value x. In principle, there can exist more than one stable fixed point. Each of them has its own region of attractiveness, i.e., the value of the fixed point obtained by the recursion relation can depend on the initial conditions. However, all possible fixed points of the recursion relation (8) must be also solutions of the following implicit equation for x, namely,

$$x - Y_1(x)/Y_0(x) = 0, \tag{25}$$

where Y_0 and Y_1 are defined in Eqs. (9) and (10) or in Eqs. (15) and (16). Because Y_0 is always positive, i.e., $Y_0(x) \neq 0$, condition (25) can be rewritten into more suitable form, namely,

$$xY_0(x) - Y_1(x) = 0. \tag{26}$$

Further, let us suppose that there exists the critical temperature T_c, i.e., the temperature in zero magnetic field at which the second order phase transition occurs. Under this assumption and from the fact that for $T \geqslant T_c$ the spontaneous magnetization vanishes, it follows that $x = 1$ must be the solution of Eq. (26) taken at $h = 0$. It means that, at least for $T \geqslant T_c$,

$$\mathcal{Y}_0(1) = \mathcal{Y}_1(1), \tag{27}$$

where functions $\mathcal{Y}_0(x)$ and $\mathcal{Y}_1(x)$ are functions $Y_0(x)$ and $Y_1(x)$ defined in Eqs. (9) and (10), respectively, taken at zero external magnetic field, i.e.,

$$\mathcal{Y}_0(x) = \sum_{j=0}^{p-1} C_j\, x^{z_j}\left[\sigma_j\, e^{-\frac{K}{2^p}} + (1-\sigma_j)e^{\frac{K}{2^p}}\right], \tag{28}$$

$$\mathcal{Y}_1(x) = \sum_{j=0}^{p-1} C_j\, x^{z_j}\left[\sigma_j\, e^{\frac{K}{2^p}} + (1-\sigma_j)e^{-\frac{K}{2^p}}\right]. \tag{29}$$

Now, it is easy to see that the necessary condition (27) for the existence of the second order phase transition is satisfied for arbitrary temperature $T > 0$, as well as for arbitrary possible values of parameters p and q.

At the same time, if the critical temperature exists then for $T < T_c$ in addition to solution $x = 1$ two other solutions, namely, $x' < 1$ and $x'' > 1$, must exist which correspond to the two values of the nonzero total spontaneous magnetization $\pm|m|$. Therefore, directly at the critical point ($T = T_c$) the function $x - \mathcal{Y}_1(x)/\mathcal{Y}_0(x)$ or $x\mathcal{Y}_0(x) - \mathcal{Y}_1(x)$ has the inflection point at $x = 1$; i.e., the first and the second derivatives of this function with respect to x are simultaneously equal to zero at this point.

Thus, the necessary conditions for the existence of the second order phase transition which must be fulfilled simultaneously directly at the critical point are

$$f(x)|_{T=T_c,x=1} = 0, \tag{30}$$

$$\left(\frac{\partial f(x)}{\partial x}\right)_{T=T_c,x=1} = 0, \tag{31}$$

$$\left(\frac{\partial^2 f(x)}{\partial x^2}\right)_{T=T_c,x=1} = 0, \tag{32}$$

where we have defined

$$f(x) = x\mathcal{Y}_0(x) - \mathcal{Y}_1(x), \tag{33}$$

with $\mathcal{Y}_0(x)$ and $\mathcal{Y}_1(x)$ given in Eqs. (28) and (29).

As was discussed above, the first necessary condition for the existence of the second order phase transition, namely, Eq. (30) is satisfied for arbitrary temperature and arbitrary values of the parameters p and q.

Let us have a look at the second necessary condition given in Eq. (31), which can be written as follows:

$$\{\partial/\partial x \mathcal{Y}_1(x) - x\partial/\partial x \mathcal{Y}_0(x) - \mathcal{Y}_0(x)\}_{T=T_c,x=1} = 0. \quad (34)$$

Its explicit form is

$$y^{-1/2}\sum_{j=0}^{p-1} C_j\{(1-y)[\sigma_j(2z_j+1)-z_j]+y\}=0, \quad (35)$$

where σ_j, z_j, and C_j are given in Eqs. (11), (12), and (14) and

$$y = e^{\frac{K}{2^{p-1}}}. \quad (36)$$

After simple algebraic manipulations, Eq. (35) can be rewritten into the following form:

$$(1+y)y^{-1/2}\sum_{j=0}^{p-2}\binom{p-2}{j} = 0. \quad (37)$$

Now, using the fact that $\sum_{j=0}^{p-2}\binom{p-2}{j} = 2^{p-2}$ as well as the definition of y in Eq. (36) one obtains the final explicit form of the second necessary condition for the existence of the second order phase transition on arbitrary pure Husimi lattices, namely,

$$2^{p-1}\cosh(K/2^p) = 0, \quad (38)$$

from which it is immediately evident that this condition, which is independent of the coordination number q, cannot be satisfied for any value of p, as well as for any value of the temperature. Because the second necessary condition for the existence of the second order phase transitions is not satisfied, therefore, regardless of the validity of the third necessary condition given in Eq. (32), we can conclude that the second order phase transitions do not exist in the spin 1/2 p-spin (multisite) model on arbitrary pure Husimi lattices with arbitrary values of parameters p and q.

V. THE FIRST ORDER PHASE TRANSITIONS IN A ZERO EXTERNAL MAGNETIC FIELD

A. General discussion

As was shown in previous section, $x = 1$ is always the solution of Eq. (25) in a zero external magnetic field, therefore the paramagnetic solution with $m = 0$ still exists. Moreover, because the model does not exhibit the second order phase transitions, i.e., there does not exist critical temperature T_c for which the function (33) has inflection point at $x = 1$, then solution $x = 1$ is always the stable fixed point of recursion relation (8). It means that if, for given temperature $T > 0$, there exists another stable fixed point $x > 0$, which corresponds to the ferromagnetic phase with nonzero spontaneous magnetization, then only one of the two phases, the paramagnetic or the ferromagnetic one, is physically (thermodynamically) stable. The answer to the question which of them is stable is given by the values of the free energy for these two phases. The stable phase has a smaller value for the free energy, and the second phase with the larger value of the free energy is the so-called metastable phase. It should be emphasized that, in principle, there can exist more than one stable fixed point $x > 0 \wedge x \neq 1$ which correspond to different ferromagnetic phases. In this case, a physically stable phase will be again given by the fixed point with the smallest value of the free energy.

Thus, the coexistence of the paramagnetic and the ferromagnetic phases is given by the specific behavior of function $f(x)$ defined in Eq. (33), and, strictly speaking, it is related to the existence of more than one stable fixed point of the recursion relation (8) with nonsingular regions of attractiveness. The borders of these temperature regions, where the number of the stable fixed points is changed, are accompanied by a jump of the value of the order parameter (the spontaneous magnetization). Let us denote these temperatures as T_{ms} [13]. However, in what follows, we shall always have at most one independent ferromagnetic stable fixed point; therefore, in our case, temperature T_{ms} will always represent the border such that for $T < T_{ms}$ the ferromagnetic and paramagnetic phases coexist, and for $T > T_{ms}$ only the paramagnetic phase exists. As we shall see, the existence of the representation (8)–(10) of the recursion relations (6) allows us to determine T_{ms} exactly for arbitrary pure Husimi lattice.

Thus, the paramagnetic and ferromagnetic phases coexist for $T < T_{ms}$. Which of them is thermodynamically stable is determined by the corresponding values of the free energy. Here, at least, three typical possibilities can exist. Two of them are trivial, namely, either the free energy for the paramagnetic phase is smaller than the ferromagnetic one for all temperatures $0 < T < T_{ms}$, i.e., the paramagnetic phase is stable and the ferromagnetic phase is always metastable, or, vice versa, the free energy for the ferromagnetic phase is smaller than the paramagnetic one throughout interval $0 < T < T_{ms}$, i.e., now the ferromagnetic phase is stable and the paramagnetic phase is metastable. However, the most interesting is the third possibility, namely, when temperature T_t ($0 < T_t < T_{ms}$) exists for which the free energies for the paramagnetic and ferromagnetic phases are equal to each other. In this case, for temperatures $T < T_t$, depending on the behavior of the corresponding free energies, one of the two phases is thermodynamically stable and the second one is metastable, and for $T > T_t$ the situation is the opposite. Directly at transition temperature T_t the first order phase transition occurs. As we shall see, the existence of the first order phase transitions in the framework of the present model strongly depends on the form of the pure Husimi lattice, i.e., it depends on the values of parameters p and q.

B. Behavior of function $f(x)$ and stable fixed points of recursion relation

As was already mentioned, the main role in the analysis of the first order phase transitions of the model is played by the behavior of function $f(x)$ defined in Eq. (33) together with properties of the free energy given in Eq. (23). First, let us look at function $f(x)$. Detailed analysis of $f(x)$ as a function of x as well as of reduced temperature $K^{-1} = k_BT/|J|$ shows that its behavior is different for models with odd and even numbers of sites in the polygons ($p \geqslant 3$), as well as for models with positive and negative values of J. In Figs. 2–5, a typical behavior of $f(x)$ for $x > 0$ is shown for odd p, $J > 0$, and $q \geqslant 3$ (Fig. 2); for odd p, $J < 0$, and $q \geqslant 3$ (Fig. 3); for even p, $J > 0$, and $q \geqslant 3$ (Fig. 4); and for even p, $J < 0$, and $q \geqslant 2$ (Fig. 5). As for all cases with $q = 2$, independently of

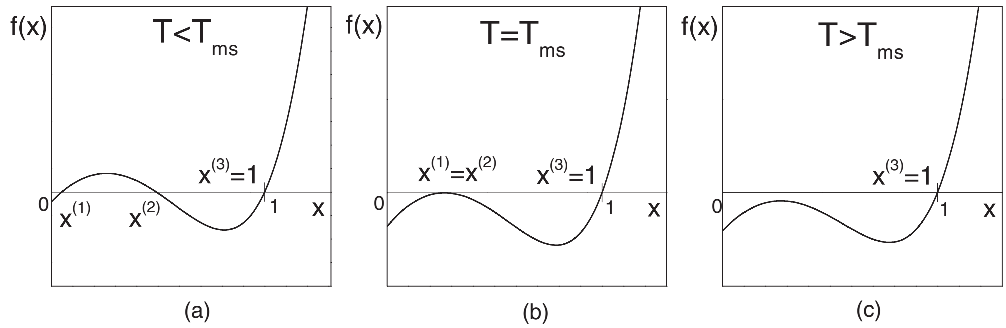

FIG. 2. The typical behavior of function $f(x)$ in Eq. (33) for odd values of p, $J > 0$, and $q \geqslant 3$ below temperature T_{ms} (a), at T_{ms} (b), and above T_{ms} (c).

the value of p as well as of the sign of J, the typical behavior of $f(x)$ is similar with the behavior shown in Fig. 5 for even p and $J < 0$.

First, as is shown in Fig. 2, below temperature T_{ms} three real positive solutions of equation $f(x) = 0$ for odd p, positive J, and $q \geqslant 3$ exist. However, only two of them are also stable fixed points of recursion relation (8) with nonsingular regions of attractiveness, namely, $x^{(1)}$ and $x^{(3)} = 1$. The region of attractiveness of $x^{(1)}$ is interval $0 \leqslant x_0 < x^{(2)}$ and of $x^{(3)} = 1$ it is interval $x^{(2)} < x_0 < \infty$. The third real positive solution $x^{(2)}$ is an unstable fixed point of recursion relation (8) with a singular region of attractiveness that consists of only one point, namely, $x_0 = x^{(2)}$. The fixed point $x^{(1)}$ corresponds to the nonzero positive value of the spontaneous magnetization m (the ferromagnetic phase), and the fixed point $x^{(3)} = 1$ corresponds to $m = 0$ (the paramagnetic phase). Depending on the initial conditions for recursion relation (8) both regimes can be achieved; however, only one of them can be thermodynamically stable for given value of the temperature (see below).

When the temperature increases, but is still below T_{ms}, solutions $x^{(1)}$ and $x^{(2)}$ get near each other, and directly at temperature T_{ms} they merge [see Fig. 2(b)], the ferromagnetic phase ceases to exist, and if the ferromagnetic phase for temperatures slightly below T_{ms} is thermodynamically stable, then the finite jump of the spontaneous magnetization into the paramagnetic phase with zero spontaneous magnetization occurs. Above T_{ms} the only solution of equation $f(x) = 0$ exists, namely, $x^{(3)} = 1$ [see Fig. 2(c)], which is also the only stable fixed point of recursion relation (8) with region of attractiveness $0 \leqslant x_0 < \infty$.

Now, let us briefly discuss the model with odd p but $J < 0$ for $q \geqslant 3$. A typical behavior of function $f(x)$ is shown in Fig. 3. The behavior is quite similar to the case with positive values of J discussed above. However, now all possible real positive solutions of equation $f(x) = 0$ are larger or equal to one. As it can be seen in Fig. 3, below temperature T_{ms}, there again exist three real positive solutions of equation $f(x) = 0$, and, at the same time, two of them are also stable fixed points of recursion relation (8) with nonsingular regions of attractiveness, namely, $x^{(3)} = 1$ and $x^{(5)}$. The region of attractiveness for $x^{(3)} = 1$ is interval $0 \leqslant x_0 < x^{(4)}$, and for $x^{(5)}$ it is interval $x^{(4)} < x_0 < \infty$. The role of the border between the regions of attractiveness of these two stable fixed points is now played by the third real positive solution $x^{(4)}$, which is the unstable fixed point of recursion relation (8) with a singular region of attractiveness that consists of only one point, namely, $x_0 = x^{(4)}$. The fixed point $x^{(5)}$ corresponds to the nonzero negative spontaneous magnetization $m = -|m|$ (the ferromagnetic phase), and its absolute value (for the same value of the temperature) is the same as the corresponding spontaneous magnetization related to the fixed point $x^{(2)}$ in the case with positive value of J. It means that

$$m(x^{(5)}, -|J|, p, q) = -m(x^{(1)}, |J|, p, q), \quad (39)$$

where p is odd and $q \geqslant 3$. On the other hand, as in the case with $J > 0$, the second stable fixed point $x^{(3)} = 1$ corresponds to the paramagnetic phase ($m = 0$); and depending on the initial conditions both regimes can be achieved; however, again only one of them can be thermodynamically stable for given value of the temperature (depending on the values of the free energies).

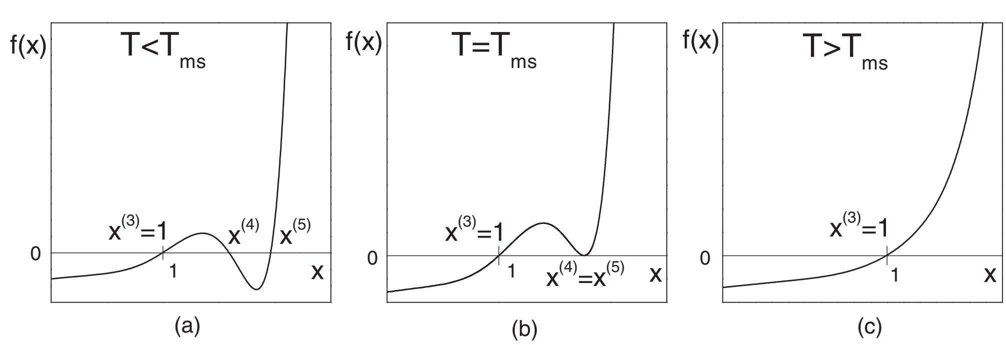

FIG. 3. The typical behavior of function $f(x)$ in Eq. (33) for odd values of p, $J < 0$, and $q \geqslant 3$ below temperature T_{ms} (a), at T_{ms} (b), and above T_{ms} (c).

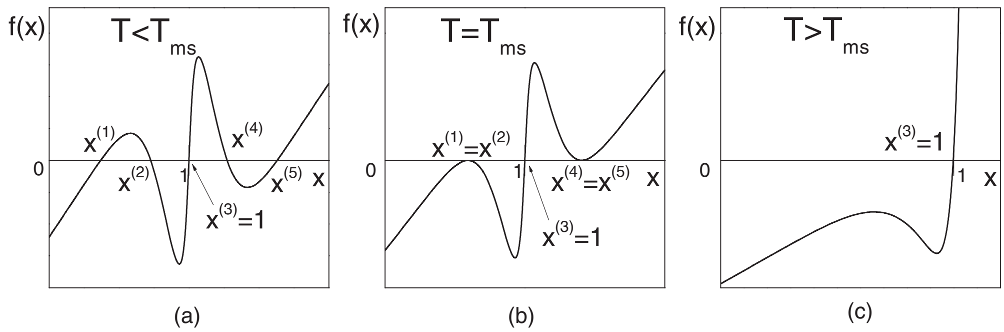

FIG. 4. The typical behavior of function $f(x)$ in Eq. (33) for even values of p, $J > 0$, and $q \geqslant 3$ below temperature T_{ms} (a), at T_{ms} (b), and above T_{ms} (c).

At temperature T_{ms} solutions $x^{(4)}$ and $x^{(5)}$ merge [see Fig. 3(b)] and the ferromagnetic phase disappears, and, again, if the ferromagnetic phase for temperatures slightly below T_{ms} is thermodynamically stable, then the finite jump of the spontaneous magnetization into the paramagnetic phase with zero spontaneous magnetization occurs. For temperatures $T > T_{ms}$ the value $x^{(3)} = 1$ becomes the only solution of equation $f(x) = 0$ [see Fig. 3(c)] which is also the only stable fixed point of recursion relation (8) with region of attractiveness $0 \leqslant x_0 < \infty$. It is also important to notice that for given values of parameters p and q, as well as for given absolute value of interaction parameter J, the values of temperature T_{ms} are the same for the models with positive and negative values of J, i.e.,

$$T_{ms}(-|J|,p,q) = T_{ms}(|J|,p,q). \quad (40)$$

Thus, the properties of the model for odd values of p and $q \geqslant 3$ are completely the same for positive and negative values of the interaction parameter J. On the other hand, this is not true for the model with even values of p. Let us start the corresponding analysis with the positive values of J. In this case, the typical behavior of function $f(x)$ is shown in Fig. 4. As is evident from Fig. 4, below temperature T_{ms} five real positive solutions of equation $f(x) = 0$ exist for even p, positive J, and $q \geqslant 3$, and three of them are also stable fixed points of recursion relation (8) with nonsingular regions of attractiveness, namely, $x^{(1)}$, $x^{(3)} = 1$, and $x^{(5)}$. The region of attractiveness of fixed point $x^{(1)}$ is interval $0 \leqslant x_0 < x^{(2)}$, of fixed point $x^{(3)} = 1$ it is interval $x^{(2)} < x_0 < x^{(4)}$, and of fixed point $x^{(5)}$ it is interval $x^{(2)} < x_0 < \infty$. Two other solutions, namely, $x^{(2)}$ and $x^{(4)}$, are unstable fixed points of recursion relation (8) with singular regions of attractiveness which consist of only one point $x_0 = x^{(2)}$ and $x_0 = x^{(4)}$, respectively. Stable fixed points $x^{(1)}$ and $x^{(5)}$ correspond to the nonzero positive (fixed point $x^{(1)}$) and negative (fixed point $x^{(5)}$) values of the spontaneous magnetization which has the same absolute value $|m|$; i.e., they correspond to the two equivalent ferromagnetic phases. On the other hand, fixed point $x^{(3)} = 1$ corresponds to $m = 0$ (the paramagnetic phase). As was already mentioned, all three fixed points ($x^{(1)}$, $x^{(3)}$, and $x^{(5)}$) have their own regions of attractiveness and depending on the chosen initial condition x_0 in recursion relation (8) all corresponding regimes can be reached; however, again, for given value of the temperature, either the paramagnetic phase or the ferromagnetic one is thermodynamically stable (depending on the values of the corresponding free energies). Further, when temperature ($T < T_{ms}$) increases, then the solutions $x^{(1)}$ and $x^{(2)}$, as well as $x^{(4)}$ and $x^{(5)}$, get near each other, and directly at temperature T_{ms} they merge, i.e., $x^{(1)} = x^{(2)}$ and $x^{(4)} = x^{(5)}$ [see Fig. 4(b)], and both ferromagnetic phases with positive and negative spontaneous magnetization [$m(x^{(1)}) = -m(x^{(5)})$] disappear, and if the ferromagnetic phases for temperatures slightly below T_{ms} are thermodynamically stable, then the finite jump of the spontaneous magnetization into the paramagnetic phase with zero spontaneous magnetization will appear. Above T_{ms} the only solution of equation $f(x) = 0$ exists, namely, $x^{(3)} = 1$ [see Fig. 4(c), which becomes also the only stable fixed point of recursion relation (8) with region of attractiveness $0 \leqslant x_0 < \infty$

Now, let us consider $J < 0$ for even value of p and arbitrary $q \geqslant 2$. In this case a typical behavior of function $f(x)$ is shown in Fig. 5. It means that regardless of the value of temperature $T > 0$, the equation $f(x) = 0$ has only one real positive solution $x^{(3)} = 1$, which is also the only stable fixed point of recursion relation (8) and corresponds to the zero spontaneous magnetization $m = 0$. Thus, in this case only the paramagnetic phase exists for all $T > 0$.

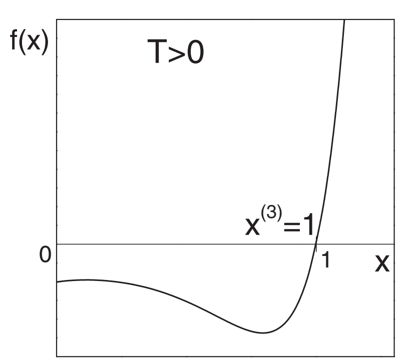

FIG. 5. The typical behavior of function $f(x)$ in Eq. (33) for even values of p, $J < 0$, and $q \geqslant 2$ for arbitrary value of the temperature $T > 0$, as well as for $q = 2$ independently of the value of p and of the sign of J.

The same behavior as shown in Fig. 5 also holds for all models with $q = 2$, regardless of the value of p, as well as regardless of the sign of J, i.e., the equation $f(x) = 0$ has only one real positive solution $x^{(3)} = 1$ with $m = 0$. Therefore, in all these cases only the paramagnetic phase exists. In the next subsection, this fact will be proven exactly. In addition, we shall also find an implicit equation for exactly determining the temperatures T_{ms} for all specific cases discussed in this subsection.

C. Exact determination of temperature T_{ms}

As discussed above, the necessary and sufficient condition for the existence of temperature T_{ms}, below which the thermodynamically stable and metastable phases coexist, is the existence of point $x > 0$ ($x \neq 1$) for which the following conditions are satisfied:

$$f(x)|_{x>0\wedge x\neq 1} = 0, \tag{41}$$

$$\left(\frac{\partial f(x)}{\partial x}\right)_{x>0\wedge x\neq 1} = 0, \tag{42}$$

where $f(x)$ is defined in Eq. (33). Using the effective representation of functions $\mathcal{Y}_0(x)$ and $\mathcal{Y}_1(x)$ given in Eqs. (28) and (29) it is possible to find a single implicit equation for exactly determining positions of these temperatures, as well as to give an exact proof for the nonexistence of these temperatures in situations when the metastable phases cannot exist at all.

Conditions (41) and (42) have the explicit form

$$\sum_{j=0}^{p-1} C_j x^{z_j}[\sigma_j(x-y) + (1-\sigma_j)(xy-1)] = 0, \tag{43}$$

$$\sum_{j=0}^{p-1} C_j x^{z_j-1}\{z_j[\sigma_j(x-y) + (1-\sigma_j)(xy-1)] + x[\sigma_j + (1-\sigma_j)y]\} = 0, \tag{44}$$

where σ_j, z_j, and C_j are given in Eqs. (11), (12), and (14), respectively, and y is defined in Eq. (36). A solution of these equations for given values of p and q determines temperature T_{ms} if $x > 0$ and $x \neq 1$.

However, using straightforward algebraic manipulations with Eqs. (43) and (44), the following explicit expression for reduced temperature $k_B T_{ms}/J$ can be obtained, namely,:

$$\frac{k_B T_{ms}}{J} = \frac{1}{2^{p-1}\ln\left(\frac{V}{W}\right)}, \tag{45}$$

where

$$V = \sum_{j=0}^{p-1} C_j x^{z_j}[1 - \sigma_j(x+1)], \tag{46}$$

$$W = \sum_{j=0}^{p-1} C_j x^{z_j}[x - \sigma_j(x+1)], \tag{47}$$

and where x is given by the solution of the following simple implicit polynomial equation:

$$x^{2q} - x^2 + (p-1)(q-1)(1-x^2)x^q = 0. \tag{48}$$

Thus, it is evident that the necessary and sufficient condition for the very existence of the metastable phases in the present model on the corresponding pure Husimi lattice defined by the concrete values of parameters p and q is reduced to the existence of real positive solutions ($x > 0 \wedge x \neq 1$) of Eq. (48). If such a solution exists, then the value of the corresponding reduced temperature $K_{ms}^{-1} = k_B T_{ms}/J$ is given by Eq. (45). It is important also to point out that Eqs. (45) and (48) determine values of temperatures T_{ms} for $J > 0$ as well as for $J < 0$. If for the given positive solution of Eq. (48) one has a positive value of K_{ms}^{-1} in Eq. (45), then this solution corresponds to a positive value of parameter J. On the other hand, if one obtains a negative value of reduced temperature K_{ms}^{-1}, then it corresponds to the model with negative value of J.

In addition, Eq. (48) immediately confirms our conclusion made in the previous subsection, namely, that the coexistence of two phases (the paramagnetic one and the ferromagnetic one) is not found in the present model on all pure Husimi lattices with $q = 2$ (in a zero external magnetic field). For $q = 2$, Eq. (48) is reduced to the following one:

$$(1 - x^2)x^2(p-2) = 0, \tag{49}$$

where it is evident that for $p \geqslant 3$ the only positive real solution of Eq. (49) is $x = 1$, which, as was discussed above, is not relevant here. Thus, for $q = 2$, regardless of the value of p and of the sign of interaction parameter J, only the paramagnetic phase exists, at least, when zero value of the external magnetic field is supposed.

Alternatively, it can be also shown directly by analyzing equation $f(x) = 0$ with $f(x)$ defined in Eq. (33). For $q = 2$ it can be written in the following form:

$$(x^2 - 1)\sum_{j=0}^{p-2}\binom{p-2}{j} x^j y^{\sigma_j} = 0. \tag{50}$$

It is evident again that, regardless of the value of p as well as regardless of the sign of J, the only positive solution of this equation exists, namely, $x = 1$, which, as was shown in previous section, is always the solution of Eq. (33).

On the other hand, Eq. (48) can be also solved exactly for other small values of parameter q, namely, for $q = 3$ and $q = 4$. The corresponding two real positive solutions ($x \neq 1$) for $q = 3$ are

$$x_{\pm} = p - 1 \pm \sqrt{p(p-2)}, \tag{51}$$

and for $q = 4$ one has

$$x_{\pm} = \sqrt{\frac{3p - 4 \pm \sqrt{3(3p^2 - 8p + 4)}}{2}}. \tag{52}$$

One of the two solutions for given q, namely, $x_- < 1$, corresponds to the positive value of the magnetization directly at temperature T_{ms}, i.e., $x_- \equiv x^{(1)} = x^{(2)}$ (see the previous subsection for details), and the second one, namely, $x_+ > 1$, corresponds to the negative value of the corresponding magnetization, i.e., $x_+ \equiv x^{(4)} = x^{(5)}$ (again see the previous subsection). However, the absolute values of both magnetizations, i.e., for x_- and x_+, are the same. At the same time, it is evident that the existence of the explicit expressions for x for $q = 3$ and $q = 4$ and for arbitrary value of p [see Eqs. (51) and (52)] means that in these cases the positions of reduced temperatures $k_B T_{ms}/J$ given by Eq. (45) are known in fully explicit form.

In Table I reduced temperatures $k_B T_{ms}/|J|$ of the model are shown for various values of parameters $p \geqslant 3$ and $q \geqslant 3$. Let us remind readers that, as discussed in detail in the previous

TABLE I. Reduced temperatures $k_B T_{ms}/|J|$ of the spin 1/2 multisite (p-spin) model on pure Husimi lattices for various values of $p \geqslant 3$ and $q \geqslant 3$. For even values of p the temperatures T_{ms} exist only for $J > 0$.

	$q=3$	$q=4$	$q=5$	$q=6$	$q=7$	$q=8$	$q=9$	$q=10$
$p=3$	0.122 57	0.203 75	0.280 41	0.355 47	0.429 76	0.503 62	0.577 22	0.650 64
$p=4$	0.050 65	0.085 69	0.118 56	0.150 65	0.182 36	0.213 86	0.245 23	0.276 52
$p=5$	0.022 62	0.038 69	0.053 72	0.068 37	0.082 83	0.097 18	0.111 48	0.125 72
$p=6$	0.010 45	0.018 02	0.025 09	0.031 96	0.038 75	0.045 48	0.052 18	0.058 86
$p=7$	0.004 92	0.008 54	0.011 91	0.015 19	0.018 42	0.021 63	0.024 82	0.028 00
$p=8$	0.002 35	0.004 09	0.005 72	0.007 29	0.008 85	0.010 39	0.011 93	0.013 46
$p=9$	0.001 13	0.001 97	0.002 76	0.003 53	0.004 28	0.005 03	0.005 77	0.006 51
$p=10$	0.000 55	0.000 96	0.001 34	0.001 71	0.002 08	0.002 44	0.002 81	0.003 17
$p=11$	0.000 26	0.000 47	0.000 65	0.000 84	0.001 02	0.001 19	0.001 37	0.001 55

subsection, for odd values of the parameter p, i.e., for odd numbers of sites in the single polygons of the pure Husimi lattices, temperatures T_{ms} exist for models with positive as well as negative values of interaction parameter J, and, at the same time, the corresponding temperatures are the same for the same absolute values of J. On the other hand, for even values of p temperatures T_{ms} exist only in the model with positive values of J. It means that for negative values of J and even values of p only the paramagnetic phase exists. From Table I, it can be seen that temperature T_{ms} for a given value of p increases as a function of q, i.e., it is an increasing function of the coordination number $z = q/2$ of the pure Husimi lattices. On the other hand, this temperature for a given value of q is a decreasing function of the number of sites in the polygons p and for $p \to \infty$ it tends to zero.

D. Exact determination of temperature T_t

By using the explicit expression for the free energy given in Eq. (23), it is easy to check that for temperatures slightly below T_{ms} the free energy of the paramagnetic phase is always lower than the free energy of the ferromagnetic phase (of course, if the ferromagnetic phase exists at all), i.e., $F(x^{(3)} = 1) < F(x^{(1)})$ as well as $F(x^{(3)} = 1) < F(x^{(5)})$, regardless of the possible values of the parameters of the model. It means that directly below T_{ms} the paramagnetic phase is thermodynamically stable and the ferromagnetic phase is metastable. Using this fact, by further decreasing the temperature, at least, two basic scenarios can occur. One of them is trivial, namely, the free energy of the paramagnetic phase remains lower than the corresponding free energy of the ferromagnetic phase for all temperatures $0 < T \leqslant T_{ms}$. In this case, although the ferromagnetic phase exists for $0 \leqslant T \leqslant T_{ms}$, it is thermodynamically metastable for $0 < T \leqslant T_{ms}$ and only the paramagnetic phase is globally realized. Let us also note that for the specific limit case with $T = 0$ the corresponding free energies of the paramagnetic and ferromagnetic phases are equal one to another, i.e., directly at $T = 0$, no of these two phases is thermodynamically more preferable. It is given by the fact that in the limit $T \to 0$ the free energy obtains the following simple form:

$$F_0 \equiv \lim_{T\to 0} F = -\frac{|J|\,q}{2^p\,p}, \tag{53}$$

which is independent of x. It means that in the present model, regardless of the form of the pure Husimi lattice, all possible phases are thermodynamically equally probable at $T = 0$.

More interesting is the second possible basic scenario. By decreasing the temperature towards $T = 0$, a temperature T_t ($0 < T_t < T_{ms}$) can appear at which the free energy of the paramagnetic phase becomes equal to the free energy of the ferromagnetic phase, i.e., $F(x^{(3)} = 1) = F(x^{(1)})$ and/or $F(x^{(3)} = 1) = F(x^{(5)})$. Thus, below T_t ($0 < T < T_t$), the free energy of the ferromagnetic phase becomes smaller than the free energy of the paramagnetic phase, i.e., the ferromagnetic phase is thermodynamically stable and the paramagnetic phase becomes metastable. In this case, directly at T_t the first order phase transition occurs accompanied by a jump of the spontaneous magnetization. However, for $T = 0$, even in this scenario, due to the fact that the free energy of the paramagnetic phase is equal to the free energy of the ferromagnetic phase, again none of these two phases is thermodynamically more preferable.

Moreover, the existence of effective representation (8)–(10) of recursion relations (6) allows one to determine exactly the position of the first order phase transitions of the present model on all pure Husimi lattices. As discussed above, the necessary and sufficient condition for the existence of transition temperature T_t is the existence of a point $x > 0$ ($x \neq 1$) for which the following conditions are satisfied:

$$f(x)|_{x>0\wedge x\neq 1} = 0, \tag{54}$$

$$F(x)|_{x>0\wedge x\neq 1} = F(1), \tag{55}$$

where $f(x)$ is defined in Eq. (33) and free energy $F(x)$ is given in Eq. (23).

The explicit form of condition (54) is given in Eq. (43). On the other hand, condition (55) reads

$$[(p-1)(q-1)-1]\ln(1+x^q) + q\ln(1+y) + (p-q)\ln 2 - q\ln\sum_{j=0}^{p-1} C_j x^{z_j}[\sigma_j + (1-\sigma_j)y] = 0, \tag{56}$$

where z_j and C_j are given in Eqs. (12) and (14), respectively, σ_j is defined in Eq. (11), and y is given in Eq. (36). Solution of Eqs. (43) and (56) for given values of p and q determines transition temperature T_t of the first order phase transition if $x > 0$ and $x \neq 1$. Now, using straightforward algebraic manipulations with the system of Eqs. (43) and (56)

TABLE II. Reduced transition temperatures $k_BT_t/|J|$ of the spin 1/2 multisite (p-spin) model on pure Husimi lattices for various values of $p \geqslant 3$ and $q \geqslant 3$. For even values of p the transition temperatures of the first order phase transitions exist only for $J > 0$. For pure Husimi lattices with $q \leqslant p$ the first order phase transitions does not exist.

	$q = 3$	$q = 4$	$q = 5$	$q = 6$	$q = 7$	$q = 8$	$q = 9$	q=10
$p = 3$	–	0.150 82	0.221 12	0.287 25	0.351 77	0.415 48	0.478 72	0.541 66
$p = 4$	–	–	0.065 60	0.092 84	0.117 90	0.142 08	0.165 80	0.189 25
$p = 5$	–	–	–	0.029 76	0.041 16	0.051 47	0.061 32	0.070 93
$p = 6$	–	–	–	–	0.013 81	0.018 77	0.023 19	0.027 39
$p = 7$	–	–	–	–	–	0.006 50	0.008 72	0.010 67
$p = 8$	–	–	–	–	–	–	0.003 09	0.004 10
$p = 9$	–	–	–	–	–	–	–	0.001 48

one obtains an explicit expression for reduced transition temperature k_BT_t/J in the form

$$\frac{k_BT_t}{J} = \frac{1}{2^{p-1}\ln\left(\frac{V}{W}\right)}, \tag{57}$$

where V and W are given in Eqs. (46) and (47), respectively, and x is given by the solution of the following implicit polynomial-like equation:

$$2^{\frac{p-q}{q}}(V+W)(1+x^q)^{\frac{(p-1)(q-1)-1}{q}} - \sum_{j=0}^{p-1} C_j x^{z_j}[W\sigma_j + (1-\sigma_j)V] = 0. \tag{58}$$

It means that the necessary and sufficient condition for the existence of the first order phase transition is reduced to the existence of real positive solutions ($x > 0 \wedge x \neq 1$) of Eq. (58). If such a solution exists, then the value of the corresponding reduced transition temperature $K_t^{-1} \equiv k_BT_t/J$ is given by Eq. (57). Note that temperatures T_{ms} [see Eq. (45)] and temperatures T_t [see Eq. (57)] are given by the same expression as function of x. However, the corresponding values of x are determined by different implicit equations, namely, by Eq. (48) for temperature T_{ms} and by Eq. (58) for transition temperature T_t. It is also important to note that Eqs. (57) and (58) determine transition temperatures T_t for $J > 0$ as well as for $J < 0$. If for given positive solution of Eq. (58) one has a positive value of reduced transition temperature K_t^{-1} in Eq. (57), then this solution corresponds to the positive value of J. But if one obtains a negative value of reduced transition temperature K_t^{-1}, then it corresponds to the model with a negative value of J.

In Table II, the values of reduced transition temperature $k_BT_t/|J|$ of the present model are shown for various values of parameters $p \geqslant 3$ and $q \geqslant 3$. As is evident from the table, the first order phase transitions from the ferromagnetic phase to the paramagnetic phase do not exist for pure Husimi lattices with $q \leqslant p$, at least, up to $p = 9$. Although, we are not able to prove exactly the validity of this property for arbitrary value of p, nevertheless, it seems that it is valid for all p (for a given value of p it can be checked numerically). It means that for such pure Husimi lattices, although the ferromagnetic phase exists, nevertheless it is metastable for all temperatures $0 < T \leqslant T_{ms}$, regardless of the sign of J.

In Figs. 6 and 7, the existence of the first order phase transitions as well as the existence of the metastable ferromagnetic phases are demonstrated on the behavior of the spontaneous magnetization for pure Husimi lattices with odd values of $p \geqslant 3$ (namely, for $p = 3$) for $J > 0$ as well as for $J < 0$ and with even values of $p \geqslant 3$ (namely, for $p = 4$) for $J > 0$. In all cases, we consider $q > p$ as a result of the fact that for $q \leqslant p$ the first order phase transitions do not exist (the ferromagnetic phase is always thermodynamically metastable). These numerical results are in full agreement with our analytical results obtained in this section.

The fact that in a pure multisite model the first order phase transitions exist only on pure Husimi lattices with $q > p$ is maybe the most nontrivial physical result of the present paper. In this respect, let us note that, e.g., in Refs. [12,13] this property was seen on pure Husimi lattices with $p = 3$, namely, it was shown that the first order phase transition exists on a pure Husimi lattice with $p = 3$ and $q = 4$; however, it does not exist on pure Husimi lattice with $p = 3$ and $q = 3$. It is evident that this property is the special case of the general rule shown in the present paper, namely, that the first order phase transitions do not exist on all pure Husimi lattices with $q \leqslant p$.

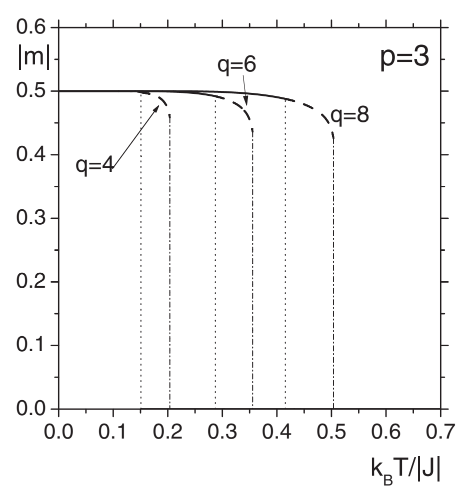

FIG. 6. The behavior of the absolute value of spontaneous magnetization $|m|$ for $p = 3$ and for $q = 4$, 6, and 8. For $J > 0$ the nonzero spontaneous magnetization is positive, i.e., $m = |m|$, and for $J < 0$ the nonzero spontaneous magnetization is negative, i.e., $m = -|m|$. Dashed parts of the magnetization curves correspond to the ferromagnetic metastable phases. Vertical dotted lines determine transition temperatures T_t for $q = 4$, 6, and 8 (from left to right). On the other hand, vertical dashed-dotted lines determine temperatures T_{ms} for $q = 4$,6, and 8 (from left to right).

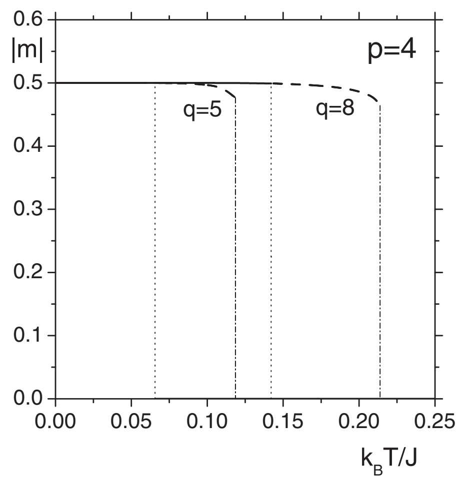

FIG. 7. The behavior of the absolute value of spontaneous magnetization $|m|$ for $p = 4$, $J > 0$, and for $q = 5$ and 8. The nonzero positive spontaneous magnetization $m = |m|$ corresponds to the fixed point $x^{(1)}$. On the other hand, the nonzero negative spontaneous magnetization $m = -|m|$ corresponds to the fixed point $x^{(5)}$. Dashed parts of the magnetization curves correspond to the ferromagnetic metastable phases. Vertical dotted lines determine transition temperatures T_t for $q = 5$ (left line) and 8 (right line). On the other hand, vertical dashed-dotted lines determine temperatures T_{ms} for $q = 5$ (left line) and 8 (right line).

To end, let us also note that temperatures T_{ms} and T_t obtained in Ref. [13] on the Husimi lattice with $p = 3$ and $q = 4$ are the same (up to normalization) as the corresponding temperatures shown in Tables I and II, respectively (in Ref. [13], temperature T_t is denoted as T_{fm}).

VI. CONCLUSION

In conclusion, in the present paper, we have investigated the so-called p-spin model with spin 1/2, on arbitrary pure Husimi lattices. A general formulation of the model is given in the form of recursion relations. A compact and, at the same time, very effective representation of the recursion relations of the model is found that allows us to perform a complete analysis of the phase transitions of the model. On one hand, it is exactly proven that the second order phase transitions do not exist in the model with pure multisite interaction regardless of the values of the parameters that characterize the form of pure Husimi lattices. On the other hand, the complete numerical as well as analytical analysis of simultaneous coexistence of the paramagnetic and ferromagnetic phases below the corresponding temperatures T_{ms} is performed in a zero external magnetic field. The model exhibits a few independent regimes depending on values of the parameters of the model. First, it is proven exactly [see Eq. (48)] that in the present model the ferromagnetic phase does not exist on all pure Husimi lattices with the coordination number $z = 2q = 4$ regardless of the values of the other parameters. It is also shown that for pure Husimi lattices with $q \leqslant p$ ($p > 2$), although the ferromagnetic phase coexists with the paramagnetic phase for $T < T_{ms}$, nevertheless only the paramagnetic phase is thermodynamically stable for all temperatures $T > 0$, i.e., the first order phase transitions do not exist in all these cases. On the other hand, for pure Husimi lattices with $q > p > 2$ the first order phase transitions from the paramagnetic phase to the ferromagnetic phase occur at temperatures T_t ($0 < T_t < T_{ms}$), below which the ferromagnetic phase becomes thermodynamically stable.

By using the effective representation of recursion relations given in Eqs. (8)–(10), expressions for exact determining temperatures T_{ms} [see Eq. (45)] and T_t [see Eq. (57)] of the model on all pure Husimi lattices are derived, where, on one hand, the value of temperature T_{ms}, below which the paramagnetic and ferromagnetic phases coexist, is driven by a simple implicit polynomial equation (48) and, on the other hand, the value of temperature T_t of the first order phase transition is driven by Eq. (58). At the same time, it is shown that for the coordination numbers $z = 2q = 6$ and 8, Eq. (48) has exact explicit solutions, i.e., in these cases temperatures T_{ms} are known in a fully explicit form for all values of p, as well as for negative and positive values of J.

Thus, we can conclude that the structure of the phase transitions of the spin 1/2 model with multisite interaction in zero external magnetic field on all pure Husimi lattices is now completely determined and known. The very existence of Eqs. (45) and (48) as well as Eqs. (57) and (58) for exact determining temperatures T_{ms} and T_t, respectively, demonstrates the existence of strong relations among the properties of the model on all pure Husimi lattices. We suppose that the corresponding relations among properties of the model on various pure Husimi lattices also hold in models with higher values of spin as well as with a different form of the Hamiltonian (see, e.g., Ref. [47]). We intend to return to these open questions in the near future. In addition, in the present paper we have studied the properties of the first order phase transitions only in a zero external magnetic field. In this respect, another interesting question arises, namely, how the general structure of the first order phase transitions obtained in the present paper will be changed when the external magnetic field is switched on. This problem is still open and requires a separate analysis.

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