

# Potential condensed-matter realization of space-fractional quantum mechanics: The one-dimensional Lévy crystal

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We introduce and discuss the one-dimensional Lévy crystal as a probable candidate for an experimentally accessible realization of space-fractional quantum mechanics (SFQM) in a condensed-matter environment. The discretization of the space-fractional Schrödinger equation with the help of shifted Grünwald-Letnikov derivatives delivers a straightforward route to define the Lévy crystal of order  $\alpha \in (1,2]$ . As key ingredients for its experimental identification we study the dispersion relation as well as the density of states for arbitrary  $\alpha \in (1,2]$ . It is demonstrated that in the limit of small wave numbers all interesting properties of continuous-space SFQM are recovered, while for  $\alpha \rightarrow 2$  the well-established nearest-neighbor one-dimensional tight-binding chain arises.

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## I. INTRODUCTION

Space-fractional quantum mechanics (SFQM), as introduced by Laskin [1–3], is a natural generalization of standard quantum mechanics that arises when the Brownian trajectories in Feynman path integrals are replaced by Lévy flights. The classical Lévy flight is a stochastic process that, in one dimension, is described by a jump length probability density function (PDF) of the form [4,5]

$$p_\alpha(x) \propto \frac{1}{|x|^{\alpha+1}} \quad \text{for } |x| \rightarrow \infty, \quad (1)$$

where  $\alpha \in (0,2]$  is referred to as the Lévy index. Although there are numerous applications of classical Lévy flights [4] such as the description of particle trajectories in a rotating flow [6] or the traveling behavior of humans [7], reports of experimental realization or observation of SFQM are lacking. While the current literature seems to concentrate on more mathematical aspects of the theory [8,9], it is still an open question for which systems manifestations of SFQM are to be expected. It is the aim of this paper to address this shortcoming and to point the route towards an experimental realization of SFQM.

In order to position the current work in an appropriate context, let us briefly survey the basic notions of SFQM. The one-dimensional (1D) space-fractional Schrödinger equation reads [1,2]

$$D_\alpha \hat{P}^\alpha |\psi\rangle + \hat{V} |\psi\rangle = E |\psi\rangle, \quad (2)$$

where  $D_\alpha \in \mathbb{R}$  ( $D_2 = 1/2m$ ) is a constant,  $\hat{P}^\alpha$  is the  $\alpha$ th power of the momentum operator,  $\hat{V}$  is the potential operator, and  $|\psi\rangle$  is the eigenstate pertaining to eigenenergy  $E$ . The requirement of the first moments of the Lévy process with PDF (1) to exist constrains  $\alpha$  to  $\alpha \in (1,2]$  [3]. The position space representation of the  $\alpha$ th power of the momentum operator is given by [3]

$$\langle x | \hat{P}^\alpha | \psi \rangle = -\hbar^\alpha \mathcal{D}_{|x|}^\alpha \psi(x), \quad (3)$$

where  $\mathcal{D}_{|x|}^\alpha$  is the Riesz fractional derivative operator of order  $\alpha$  [10] and  $\hbar$  is the reduced Planck constant. For  $\alpha = 2$  the

Riesz fractional derivative is equivalent to the standard second-order derivative [10]. The free-particle solution of the space-fractional Schrödinger equation (2) is easily determined to be of the form [1,2]

$$\psi(x) = \exp(ikx) \quad \text{with } E = D_\alpha \hbar^\alpha |k|^\alpha, \quad (4)$$

i.e., the dispersion is proportional to  $|k|^\alpha$ . Hence, in the limit  $\alpha \rightarrow 2$  we obtain a parabolic dispersion as, for instance, for conduction electrons near the band minimum [11] and for  $\alpha \rightarrow 1$ , the dispersion shows the behavior  $|k|$  as, for instance, a 1D acoustic phonon band for  $k \rightarrow 0$  [12]. The solution of the fractional Schrödinger equation in other, more complex situations appears to be rather difficult due to the nonlocality of Eq. (2) [8]. It is an important property of the space-fractional Schrödinger equation that it is not possible to solve the equation locally and to obtain a global solution from matching conditions as it is usual in standard quantum mechanics. Numeric solutions for the 1D particle in a box, for instance, have been presented in [13].

These open questions apparently make it rather difficult to search for possible realizations of SFQM. For instance, Lenzi *et al.* [14] determined the specific heat of noncrystalline solids for low temperatures with the help of a fractional generalization of the thermodynamic Green's function; however, their fits to experimental data resulted in values  $\alpha > 2$ . In contrast, in Ref. [15] the semiclassical evolution equation resulting from Eq. (2) was investigated. It was left as an open problem which particular form of the Wigner transform has to be employed and therefore which semiclassical behavior of the free-flight term is to be expected.

Within this paper we suggest a different route toward an experimental identification of SFQM. In particular, we discuss a probable candidate for the realization of SFQM in a condensed-matter environment by introducing a 1D infinite-range tight-binding chain, which we shall refer to as the 1D Lévy crystal of order  $\alpha \in (1,2]$  for reasons of convenience. This model emerges from the mapping of the Hamiltonian (2) onto a 1D lattice and thus represents a natural generalization of the well established 1D nearest-neighbor tight-binding chain ( $\alpha = 2$ ). It has to be emphasized that such a model can only be regarded as an idealization and that, from this point of

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view, SFQM has to be understood as an effective theory that may be employed under certain conditions. Furthermore, the investigation of the Lévy crystal allows us to resolve the dilemma of ambiguous Wigner transforms raised in [15], at least within this context. Please note that an example for a solid state system that induces classical Lévy flights is the Lévy-Lorentz gas [16].

This paper is structured as follows. In Sec. II we define the Lévy crystal and in Sec. III we determine its dispersion relation as well as its density of states (DOS). Moreover, we discuss an interesting interpretation of this model and briefly investigate the dilemma raised in [15]. Finally, in Sec. IV the work is summarized.

## II. THE 1D LÉVY CRYSTAL

We start with a closer inspection of the space-fractional Schrödinger equation (2) by expressing the Riesz fractional derivative operator  $\mathcal{D}_{|x|}^\alpha$  as [17]

$$\mathcal{D}_{|x|}^\alpha = -\frac{1}{2 \cos\left(\frac{\alpha\pi}{2}\right)} (I_+^{-\alpha} + I_-^{-\alpha}), \quad (5)$$

where  $I_\pm^{-\alpha}$  can be written with the help of shifted Grünwald-Letnikov derivatives [17] as

$$I_\pm^{-\alpha} \psi(x) = \lim_{a \rightarrow 0} \frac{1}{a^\alpha} \sum_{n=0}^{\infty} (-1)^n \binom{\alpha}{n} \psi[x \mp (n-1)a] \quad (6)$$

for sufficiently well behaved functions  $\psi(x)$ . Here  $\binom{\alpha}{n}$  is the generalized binomial coefficient defined by

$$\binom{\alpha}{n} = \frac{\Gamma(\alpha+1)}{\Gamma(n+1)\Gamma(\alpha-n+1)}, \quad (7)$$

with  $\Gamma(\cdot)$  the Gamma function. In order to arrive at a tight-binding model we follow the procedure outlined by Datta [18] and replace the exact momentum operator (5) by its discretized version. We regard equally spaced grid points  $x_\ell = \ell a$ , where  $a > 0$  is the lattice constant and  $\ell \in \mathbb{Z}$ . The position space element for a particular grid point  $x_\ell$  of the kinetic term of Eq. (2) acting on  $|\psi\rangle$  is therefore replaced by

$$\begin{aligned} \langle x_\ell | -D_\alpha \hat{P}^\alpha | \psi \rangle \\ \rightarrow \langle x_\ell | -D_\alpha \hat{P}_a^\alpha | \psi \rangle := \frac{t_0}{2} \sum_{n=0}^{\infty} (-1)^n \binom{\alpha}{n} \\ \times \{ \psi[x_\ell + (n-1)a] + \psi[x_\ell - (n-1)a] \}, \end{aligned} \quad (8)$$

where we defined the hopping amplitude  $t_0$ :

$$t_0 = \frac{D_\alpha \hbar^\alpha}{a^\alpha \cos\left(\frac{\alpha\pi}{2}\right)}. \quad (9)$$

In a first step we rewrite Eq. (8):

$$\begin{aligned} \langle x_\ell | -D_\alpha \hat{P}_a^\alpha | \psi \rangle \\ = \frac{t_0}{2} \sum_{n \neq 0} (-1)^{n+1} \binom{\alpha}{n+1} \psi(x_\ell + na) \\ + \frac{t_0}{2} [\psi(x_\ell - a) + \psi(x_\ell + a)] - \alpha t_0 \psi(x_\ell). \end{aligned} \quad (10)$$

Then we assume that the potential operator  $\hat{V}$  is diagonal in position space and periodic with periodicity  $a$ , i.e.,  $V(x_\ell) \equiv U$

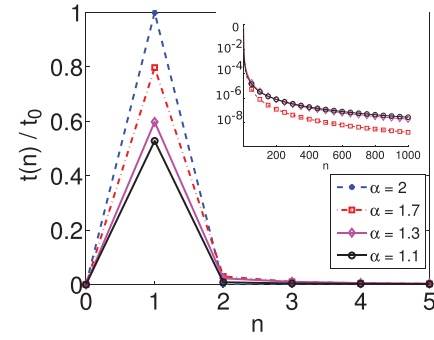


FIG. 1. (Color online) Normalized hopping parameters  $t(n)/t_0$  as a function of the distance  $n$  vs  $n$  for different values of  $\alpha \in (1,2]$ .

for all  $\ell \in \mathbb{Z}$ . This suggests the replacement

$$\langle x_\ell | -D_\alpha \hat{P}^\alpha + \hat{V} | \psi \rangle \rightarrow \varepsilon \psi(x_\ell) + \sum_{n \neq 0} t(n) \psi(x_\ell + na), \quad (11)$$

where we defined the on-site energy

$$\varepsilon = U - \alpha t_0 \quad (12)$$

together with hopping parameters

$$t(n) = \frac{t_0}{2} \left[ (-1)^{|n|+1} \binom{\alpha}{|n|+1} + \delta_{|n|,1} \right], \quad (13)$$

where  $\alpha \in (1,2]$ ,  $n \neq 0$ , and  $\delta_{nm}$  is the Kronecker delta. The hopping parameters (13) are illustrated in Fig. 1 for different values of  $\alpha \in (1,2]$ . Let us briefly discuss this particular form of the hopping parameters  $t(n)$ . First of all, we note that the constraint  $\alpha \in (1,2]$  ensures that  $t(n)/t_0 \geq 0$  for all  $n \in \mathbb{N}$ . Furthermore, for  $\alpha = 2$ , the hopping parameters (13) reduce to

$$t(n) = t_0 \delta_{|n|,1}, \quad (14)$$

i.e., they account only for nearest-neighbor interaction and therefore give rise to a 1D nearest-neighbor tight-binding chain. This is entirely consistent with the above discretization since the replacement (8) is equivalent to the finite-difference approximation of the second-order derivative for  $\alpha = 2$  [18]. Moreover, we note that the hopping parameters obey

$$\frac{1}{t_0} \sum_{n=-\infty}^{\infty} t(n) = \alpha, \quad (15)$$

where we employed that [17]

$$\sum_{n=0}^{\infty} (-1)^n \binom{\alpha}{n} z^n = (1-z)^\alpha \quad (16)$$

for  $|z| \leq 1$  and we defined  $t(0) \equiv 0$ .

In a final step we identify the replacement (11) with the definition of a tight-binding Hamiltonian

$$H = \varepsilon \sum_n |n\rangle \langle n| + \sum_{nm} t(m-n) |n\rangle \langle m|, \quad (17)$$

where  $|n\rangle$  are the basis kets, which are centered at position  $x_n$  and form an orthonormal basis, i.e.,  $\langle n|m\rangle = \delta_{nm}$ . The tight-binding Hamiltonian (17) defines the Lévy crystal for  $\alpha \in (1,2]$  with on-site energy  $\varepsilon$  and hopping amplitude  $t_0$  if  $t(n)$

is given by Eq. (13). We are now in the position to study some of the properties of this system. It is important to remark that an alternative discretization of the integrals (6) would have resulted in an alternative tight-binding Hamiltonian (17). However, the main aspects of the following discussion remain unchanged if an alternative discretization would have been chosen. Moreover, we chose the particular version (6) because it is a straightforward generalization of the 1D monatomic nearest-neighbor tight-binding model and thus the electronic structure will have only one maximum or minimum in the first Brillouin zone.

### III. DISPERSION RELATION AND DENSITY OF STATES

The electronic structure as well as the DOS of the Lévy crystal might be of central significance for an experimental identification of SFQM. In a first step, we solve the stationary Schrödinger equation for the Hamiltonian (17) following the standard procedure [19]. We express the wave function  $|\psi\rangle$  as a linear combination of the localized orbitals  $|\ell\rangle$ , i.e.,

$$|\psi\rangle = \sum_{\ell} c_{\ell}(k)|\ell\rangle; \quad (18)$$

then we employ Bloch's theorem

$$c_{\ell}(k) = \exp[ika(\ell - \ell')]c_{\ell'}(k), \quad (19)$$

where we introduce the wave number  $k \in [-\frac{\pi}{a}, \frac{\pi}{a}]$  and insert Eq. (18) into Eq. (17). Together with Eq. (19) we obtain

$$\begin{aligned} H|\psi\rangle &= \varepsilon|\psi\rangle + \sum_{n\ell} t(\ell - n)c_{\ell}|n\rangle \\ &= \left\{ \varepsilon + 2 \operatorname{Re} \left[ \sum_{\ell \geq 1} t(\ell) \exp(ika\ell) \right] \right\} |\psi\rangle, \end{aligned} \quad (20)$$

where  $\operatorname{Re}(\cdot)$  denotes the real part. It is important to note that Bloch's theorem (19) is valid because the Lévy crystal is infinitely extended. It is therefore comparable to the plane-wave solution of the one-dimensional space-fractional Schrödinger equation (2) [1,2].

With the help of Eq. (16) we can express the sum for  $\alpha \in (1,2]$  as

$$\begin{aligned} \sum_{\ell \geq 1} t(\ell) \exp(ika\ell) &= \frac{t_0}{2} \exp(-ika) [1 - \exp(ika)]^{\alpha} \\ &\quad + it_0 \sin(ka) + \frac{\alpha t_0}{2}. \end{aligned} \quad (21)$$

Hence the dispersion relation of the 1D Lévy crystal is described by

$$\begin{aligned} E_{\alpha}(k) &= \varepsilon + \alpha t_0 + t_0 \operatorname{Re}\{\exp(-ika) \\ &\quad \times [1 - \exp(ika)]^{\alpha}\} \quad \text{for } \alpha \in (1,2]. \end{aligned} \quad (22)$$

This is illustrated in Fig. 2. In the particular case that  $\alpha = 2$  the dispersion reduces to

$$E_2(k) = \varepsilon + 2t_0 \cos(ka), \quad (23)$$

which is well known from the 1D nearest-neighbor tight-binding chain [18,19]. Next we transform the dispersion

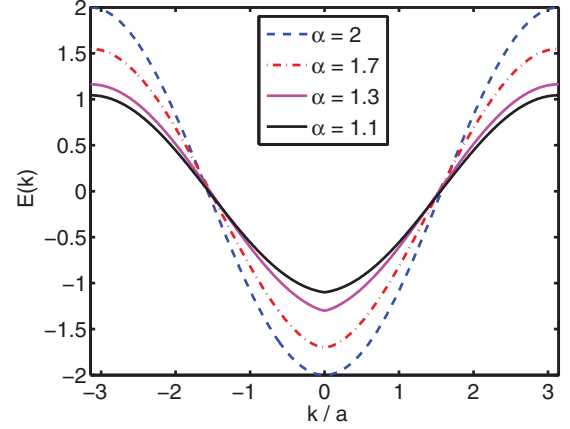


FIG. 2. (Color online) Dispersion relation  $E_{\alpha}(k)$  vs  $k$  for different values of  $\alpha \in (1,2]$  and the particular choice  $\varepsilon = 0$  and  $t_0 = -1$ .

relation (22) into a more convenient form. We note that

$$\begin{aligned} |\exp(-ika)[1 - \exp(ika)]^{\alpha}| &= 2^{\alpha/2} [1 - \cos(ka)]^{\alpha/2} \\ &= 2^{\alpha} \left| \sin\left(\frac{ka}{2}\right) \right|^{\alpha}. \end{aligned} \quad (24)$$

Moreover, we rewrite

$$\begin{aligned} \arg[1 - \exp(ika)] &= -\arctan\left(\frac{\sin(ka)}{1 - \cos(ka)}\right) \\ &= -\arctan\left[\cot\left(\frac{ka}{2}\right)\right] = \frac{ka}{2} - \frac{\pi}{2} \operatorname{sgn}(k) \end{aligned} \quad (25)$$

and hence

$$\arg\{\exp(-ika)[1 - \exp(ika)]^{\alpha}\} = -ka + \frac{\alpha}{2}[ka - \pi \operatorname{sgn}(k)]. \quad (26)$$

Combining Eqs. (24) and (26) finally allows us to rewrite the dispersion (22) as

$$\begin{aligned} E_{\alpha}(k) &= \varepsilon + \alpha t_0 + 2^{\alpha} t_0 \left| \sin\left(\frac{ka}{2}\right) \right|^{\alpha} \\ &\quad \times \cos\left[ka\left(1 - \frac{\alpha}{2}\right) + \frac{\alpha\pi}{2} \operatorname{sgn}(k)\right]. \end{aligned} \quad (27)$$

In the limit  $a \rightarrow 0$  or  $k \rightarrow 0$  we expect, due to our definition of the Lévy crystal, to obtain a dispersion that resembles the free-particle space-fractional Schrödinger equation (4). Indeed, we obtain from Eq. (27)

$$E_{\alpha}(k) \approx \varepsilon + \alpha t_0 + t_0 (a|k|)^{\alpha} \cos\left(\frac{\alpha\pi}{2}\right) \quad (28)$$

for small  $|k|$ . Hence, as expected, the dispersion (22) of the Lévy crystal is nonanalytic in  $k$  at  $k = 0$ , indicating an infinite-band velocity. The question arises whether such a model can be of any interest for real-world applications. Obviously, the particular form of Eq. (28) is the result of the infinite sum (21) evaluated with the help of Eq. (16). More precisely, it is a result of the asymptotic behavior of the hopping parameters (13). Let us briefly investigate how the dispersion (21) is changed if the hoppings are truncated to finite-range interactions, i.e., if we

set  $t(n) = 0$  for all  $n > M$ , where  $M \in \mathbb{N}$  is left arbitrary. Then the dispersion (22) takes on the form

$$\begin{aligned} E_\alpha(k) &= \varepsilon + 2 \operatorname{Re} \left[ \sum_{\ell=1}^M t(\ell) \exp(ika\ell) \right] \\ &= \varepsilon + 2 \sum_{\ell=1}^M t(\ell) \cos(ka\ell) \\ &\approx \varepsilon + 2 \sum_{\ell=1}^M t(\ell) \left[ 1 - \frac{(ka\ell)^2}{2} \right], \end{aligned} \quad (29)$$

i.e., the dispersion is analytic and parabolic for  $k \rightarrow 0$  for all  $M < \infty$ . Hence the Lévy crystal can be regarded as an effective model for a crystal that possesses a dispersion proportional to  $|k|^\alpha$  in the vicinity of  $k = 0$ .

Let us turn our attention to some further properties of the dispersion relation of the Lévy crystal. We restrict our discussion to the case  $t_0 < 0$  for reasons of simplicity. From the dispersion relation (22) we observe that the band minimum is located at  $k = 0$  and takes on the value

$$E_\alpha(k = 0) = \varepsilon + \alpha t_0, \quad (30)$$

while the band maximum at  $k = \pm \frac{\pi}{a}$  is given by

$$E_\alpha \left( k = \pm \frac{\pi}{a} \right) = \varepsilon - (2^\alpha - \alpha) t_0. \quad (31)$$

Please note that  $2^\alpha - \alpha > 0$  for all  $\alpha \in (1, 2]$ . Hence the total bandwidth  $\delta_\alpha$  is given by

$$\begin{aligned} \delta_\alpha &:= \left| E_\alpha(k = 0) - E_\alpha \left( k = \pm \frac{\pi}{a} \right) \right| \\ &= 2^\alpha |t_0|, \end{aligned} \quad (32)$$

i.e. it increases with increasing  $\alpha$ . Interestingly, we have

$$\begin{aligned} E_\alpha \left( k = \pm \frac{\pi}{2a} \right) &= \varepsilon + \alpha t_0 + t_0 \operatorname{Re}[i(1+i)^\alpha] \\ &= \varepsilon - t_0 \left[ 2^{\alpha/2} \sin \left( \frac{\alpha\pi}{4} \right) - \alpha \right] \end{aligned} \quad (33)$$

and we note that

$$\left[ 2^{\alpha/2} \sin \left( \frac{\alpha\pi}{4} \right) - \alpha \right] \geq 0, \quad (34)$$

where the equal sign applies to  $\alpha = 2$ . Hence the point  $k = \pm \frac{\pi}{2a}$  does not coincide with the inflection point of the dispersion for  $\alpha < 2$ . This already indicates that the DOS cannot be symmetric in  $E$ .

Of course, we may also regard a Lévy crystal of finite length, i.e., in the Hamiltonian (17)  $n$  is restricted to  $n = -N, \dots, N$  with  $N \in \mathbb{N}$ . In this case, Bloch's theorem (19) cannot be applied and an analytic solution of the problem is not straightforward. In fact, this situation is comparable to the solution of the space-fractional Schrödinger equation (2) on a finite domain, as in the case of the one-dimensional particle in a box [8,13]. In Figs. 3(a)–3(d) we illustrate the numerically obtained energy levels of the Lévy crystal of finite length for  $\alpha = 2, 1.7, 1.3$ , and  $1.1$ , respectively, and for different values of  $N$  together with the characteristic points of the dispersion, Eqs. (30), (31) and (33). From these graphs we observe that the DOS increases with decreasing  $\alpha$  due to the reduced bandwidth. Moreover, as stated above, the DOS is larger for  $E > E(\frac{\pi}{2a})$  than for  $E < E(\frac{\pi}{2a})$  since  $E(\frac{\pi}{2a})$  is slightly shifted to positive energies according to Eq. (33) and there has to be the same number of states above this energy and below.

Let us derive an analytic expression for the DOS of the 1D Lévy crystal. The DOS  $\rho_\alpha(E)$  of the 1D dispersion  $E_\alpha(k)$  reads [11]

$$\rho_\alpha(E) = \frac{1}{\pi} \frac{1}{|\partial_k E_\alpha(k)|} \Big|_{k=k(E)}, \quad (35)$$

where  $k(E)$  is the inverse dispersion and  $\partial_k$  denotes the partial derivative with respect to  $k$ . With the help of Eq. (27) we obtain for  $k \neq 0$

$$\begin{aligned} |\partial_k E_\alpha(k)| &= \left| \alpha \frac{a2^\alpha}{2} t_0 \operatorname{sgn}(k) \left| \sin \left( \frac{ka}{2} \right) \right|^{\alpha-1} \cos \left( \frac{ka}{2} \right) \cos \left[ ka \left( 1 - \frac{\alpha}{2} \right) + \frac{\alpha\pi}{2} \operatorname{sgn}(k) \right] \right. \\ &\quad \left. - t_0 2^\alpha a \left( 1 - \frac{\alpha}{2} \right) \left| \sin \left( \frac{ka}{2} \right) \right|^\alpha \sin \left[ ka \left( 1 - \frac{\alpha}{2} \right) + \frac{\alpha\pi}{2} \operatorname{sgn}(k) \right] \right| \\ &= a2^\alpha \left| t_0 \sin \left( \frac{ka}{2} \right) \right|^{\alpha-1} \left| \frac{\alpha}{2} \cos \left( \frac{ka}{2} \right) \cos \left[ ka \left( 1 - \frac{\alpha}{2} \right) + \frac{\alpha\pi}{2} \operatorname{sgn}(k) \right] \right. \\ &\quad \left. - \left( 1 - \frac{\alpha}{2} \right) \sin \left( \frac{ka}{2} \right) \sin \left[ ka \left( 1 - \frac{\alpha}{2} \right) + \frac{\alpha\pi}{2} \operatorname{sgn}(k) \right] \right|. \end{aligned} \quad (36)$$

For  $k = 0$ , the first derivative of the dispersion, which is essentially the band velocity, diverges. Finally, we have to compute the inverse of the electronic structure (27). First of all, we note the time-reversal invariance of the dispersion  $E_\alpha(k) = E_\alpha(-k)$  and, for  $t_0 < 0$ , that  $\varepsilon - \alpha|t_0| \leq E_\alpha(k) \leq \varepsilon + (2^\alpha - \alpha)|t_0|$ . We rewrite Eq. (27) as

$$\omega := \frac{E - \varepsilon + \alpha|t_0|}{2^\alpha|t_0|} = - \left| \sin \left( \frac{\phi}{2} \right) \right|^\alpha \cos \left[ \phi \left( 1 - \frac{\alpha}{2} \right) + \frac{\alpha\pi}{2} \right], \quad (37)$$

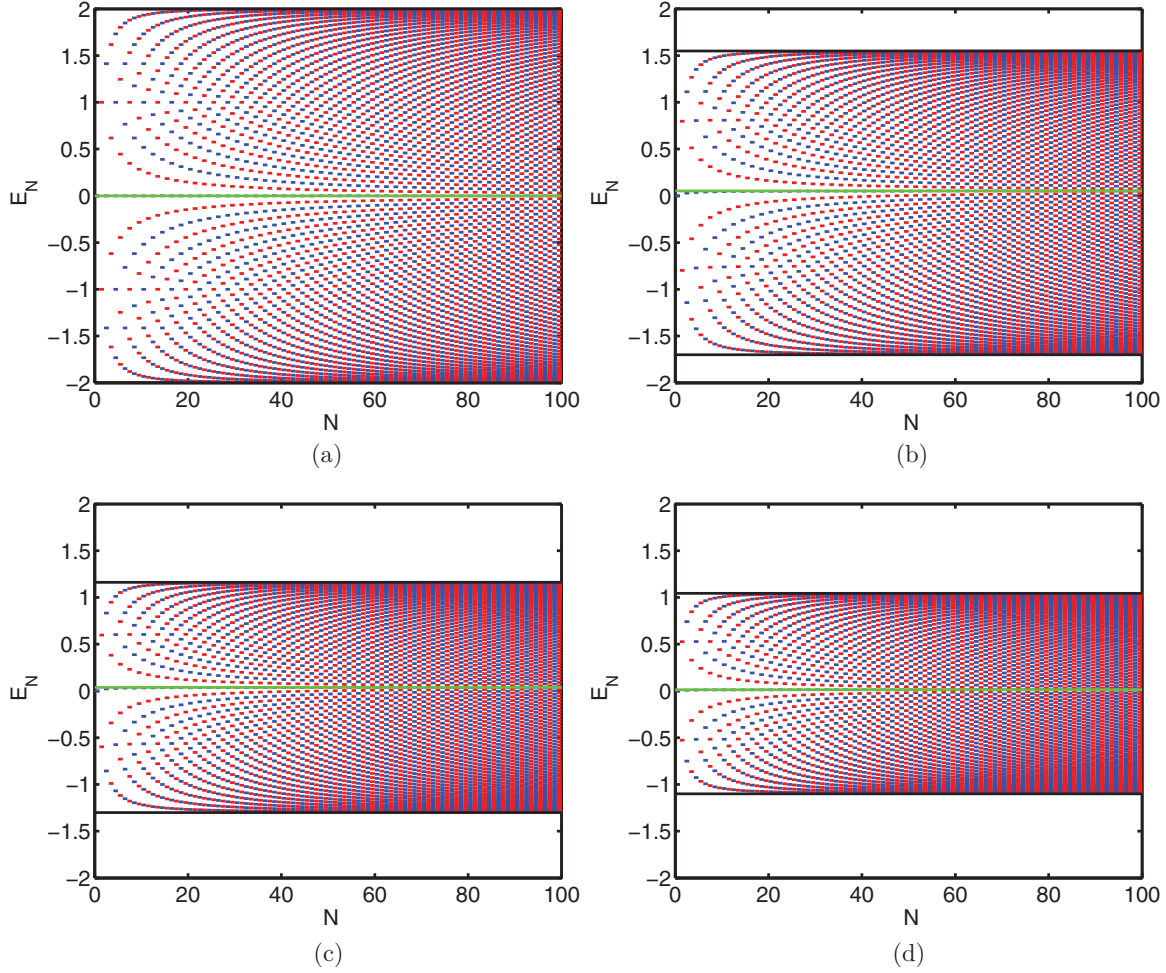


FIG. 3. (Color online) Energy levels  $E_N$  for different  $\alpha \in (1,2]$  vs  $N$  together with the characteristic points for  $N \rightarrow \infty$  at  $k = 0, \pm \frac{\pi}{2a}, \pm \frac{\pi}{a}$ . (a)  $\alpha = 2$ , (b)  $\alpha = 1.7$ , (c)  $\alpha = 1.3$ , and (d)  $\alpha = 1.1$ . Moreover, we chose  $\varepsilon = 0$  and  $t_0 = -1$ .

where  $0 \leq \omega \leq 1$  and  $\phi = ka$ . We restrict the above equation to  $\phi \in [0, \pi]$  due to time-reversal invariance of the band structure. Equation (37) is solved numerically for different  $\omega \in [0, 1]$  in order to obtain  $k(E)$ . In Fig. 4 we present the DOS

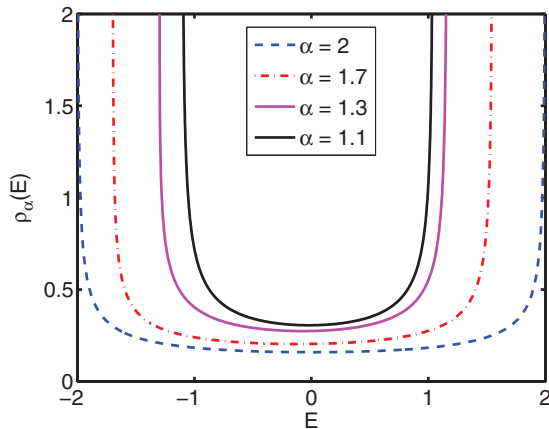


FIG. 4. (Color online) Density of states  $\rho_\alpha(E)$  of the Lévy crystal for different  $\alpha \in (1,2]$  vs  $E$ . Again, we chose  $\varepsilon = 0$  together with  $t_0 = -1$ .

for  $\alpha = 2, 1.7, 1.3$ , and  $1.1$ , respectively. Again, we observe that the DOS at  $E = 0$  increases with decreasing  $\alpha$ . This results from the reduced bandwidth [see Eq. (32) and Figs. 2 and 3]. Moreover, the DOS is not symmetric with respect to  $E = 0$  for  $\alpha \neq 2$ . This phenomenon is due to the shifted center of energy (33) and can already be observed in Fig. 3. The two main characteristics, namely, that the dispersion is proportional to  $|k|^\alpha$  and the asymmetry of the DOS, might be of great interest in identifying the Lévy crystal in a condensed-matter environment.

Before concluding let us briefly focus on another interesting point: The formulation of the space-fractional Schrödinger equation (2) as the low-wave-number limit of a tight-binding Hamiltonian allows for an interesting interpretation. In the general condensed-matter case, the Hamiltonian is of the form  $\hat{H} = \hat{T} + \sum_n \hat{V}_n$ , where  $\hat{T}$  denotes the standard kinetic energy operator and  $\hat{V}_n$  is the potential operator pertaining to unit cell  $n$ . If we employ the ansatz  $|\psi\rangle = \sum_n c_n |n\rangle$  and calculate the matrix elements  $\langle \ell | \hat{H} | \psi \rangle$  we have to cope with terms of the form  $\langle \ell | \hat{V}_m | n \rangle$ . At this point it is common to employ two different simplifications [20]: (a) the three-center approximation states that matrix elements  $\langle \ell | \hat{V}_m | n \rangle$  are only nonzero if at least two of the three indices  $\ell, m, n$  coincide



and (b) the  $M$ th nearest-neighbor approximation states that  $\langle \ell | \hat{V}_n | n \rangle$  is only nonzero if  $|n - \ell| \leq M$ . This offers a very interesting interpretation of the Lévy crystal Hamiltonian (17): Again, we employ the three-center approximation and identify  $\varepsilon = \langle n | \hat{T} | n \rangle + \langle n | \hat{V}_n | n \rangle$  and  $\langle m | \hat{V}_m | n \rangle = t(n - m)$ . If  $\langle m | \hat{V}_m | n \rangle = t(m - n)$  follows the particular behavior (13), the space-fractional Schrödinger equation might be an appropriate model for small values of  $k$ . Moreover, we deduce that the Wigner transform II discussed in [15] might be the adequate one for this model. This follows from the fact that the kinetic term of the semiclassical evolution equation for a solid state system is of the form  $\nabla_k E(k) \cdot \nabla_x w(x, k, t)$ , with  $E(k)$  the dispersion and  $w(x, k, t)$  the semiclassical distribution function [21]. Inserting for  $E(k)$  the dispersion of the Lévy crystal  $E_\alpha(k)$  gives for small  $k$  the equation resulting from the Wigner transform II in Ref. [15].

#### IV. CONCLUSION

We defined and investigated some properties of the 1D Lévy crystal. Its definition was particularly motivated by the quest for a possible realization of SFQM in a solid state environment. Of course, the Lévy crystal has to be regarded as an idealized model comparable to the 1D monatomic nearest-neighbor tight-binding model, which follows in the limit  $\alpha = 2$ . Let us briefly review the main steps of our discussion.

The definition of the Lévy crystal is based on the discretization of the space-fractional Schrödinger equation with Lévy index  $\alpha$  with the help of shifted Grünwald-Letnikov derivatives [17] on an equally spaced grid. In analogy to the introduction of the 1D nearest-neighbor tight-binding chain [18], the grid points are interpreted as lattice points, which finally defines the Lévy crystal of order  $\alpha$ . Hence, in this picture the Lévy crystal may be regarded as an effective model for a 1D crystal if the overlap integrals  $\langle m | \hat{V}_m | n \rangle$  show the characteristic distance dependence of  $t(n - m)$  given by Eq. (13). The asymptotic behavior of the hopping parameters gives rise to a dispersion relation  $E_\alpha(k) \propto |k|^\alpha$  in the limit  $k \rightarrow 0$  or  $a \rightarrow 0$ . Furthermore, for  $\alpha \rightarrow 2$  the well known nearest-neighbor tight-binding model arises [19]. These considerations allow us to identify the Wigner transform II of Ref. [15] as the correct one for this particular case. The DOS  $\rho_\alpha(k)$  of the Lévy crystal for  $\alpha \neq 2$  is no longer symmetric in  $E$ . Interestingly, we obtain a higher density of states for  $E > E_\alpha(\frac{\pi}{2a})$  since the central state located at  $E_\alpha(k = \pm \frac{\pi}{2a})$  is shifted to higher energies, (see Fig. 3). Moreover, for decreasing  $\alpha$  the bandwidth decreases, which forces the overall DOS to increase.

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