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Turbulent Prandtl number in a model of passively advected vector field: Two-loop renormalization group result

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The turbulent Prandtl number in the model of a passive vector field advected by the turbulent environment driven by the stochastic Navier-Stokes equation is studied by using the field theoretic renormalization group technique in the two-loop approximation. It is shown that unlike the turbulent Prandtl number in the model of passively advected scalar field, as well as the turbulent magnetic Prandtl number of passively advected magnetic field in the framework of the kinematic magnetohydrodynamic turbulence, where the two-loop corrections to the corresponding Prandtl numbers are very small (less than 2% of their one-loop values), the two-loop correction to the turbulent Prandtl number of passively advected vector field is considerably larger; namely, it is 27% of its one-loop value. At the same time, the calculated two-loop value of the turbulent vector Prandtl number, $\mathrm{Pr}_{v,t} = 0.7307$, is surprisingly very close to the two-loop value of the turbulent Prandtl number of passively advected scalar field, $\mathrm{Pr}_t = 0.7040$.

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Without doubt, one of the most important characteristics of the diffusion processes in fluids are the so-called Prandtl numbers, the dimensionless ratios of the coefficient of kinematic viscosity to the corresponding diffusion coefficients (e.g., to the coefficient of thermal diffusivity in the temperature diffusion problem, to the coefficient of molecular diffusivity in the impurity concentration problem, or to the coefficient of magnetic diffusivity (resistivity) in the diffusion problem of magnetic field in a conductive medium) [1–4]. On the other hand, it is well known that in the case when the fluid is in the state of fully developed turbulence, then the corresponding diffusion processes are rapidly accelerated. This fact is expressed in the appearance of effective values of the diffusion coefficients, the so-called turbulent diffusion coefficients. The ratios of the turbulent viscosity to the various coefficients of turbulent diffusivity are known as the turbulent Prandtl numbers, e.g., turbulent Prandtl number Pr_t in the temperature diffusion problem or turbulent magnetic Prandtl number $\mathrm{Pr}_{m,t}$ in the magnetohydrodynamic (MHD) turbulence.

Recently, the turbulent Prandtl numbers have been investigated by using the field theoretic renormalization group (RG) method [5–7], which represents an effective technique for investigating the universal properties of processes in fully developed turbulent systems [8,9]. In Ref. [5], the two-loop scheme independent formula for the inverse turbulent Prandtl number in the model of passively advected scalar quantity was derived and it was shown that the two-loop correction to the turbulent Prandtl number is less than 2% of its one-loop value [5,6]. Thus, it seems that the turbulent Prandtl number demonstrates awfully strong stability with respect to the corresponding perturbation expansion. On the other hand, quite recently the turbulent magnetic Prandtl number in the framework of the kinematic MHD turbulence was also investigated within two-loop RG approximation [7], and it was shown that the value of the turbulent magnetic Prandtl number in the model of kinematic MHD turbulence, where the magnetic field behaves as a kind of passively advected vector quantity, is completely identical with the corresponding turbulent Prandtl number of passively advected scalar field studied in Refs. [5,6].

However, there exists another model of a passively advected vector field, namely, the $A = 0$ model in which the so-called "stretching term," which is present in the kinematic MHD, is omitted (see, e.g., Ref. [10] for details). This model is completely analogical to the model of passively advected scalar quantity and the main reason for its investigation is related to the fact that (in some important features) the problem of anomalous scaling in the framework of $A = 0$ vector models resembles the problem of the anomalous scaling in genuine Navier-Stokes turbulence. In this respect, the $A = 0$ model of the passive vector advection was investigated in various Gaussian turbulent velocity fields [10–16] and quite recently also in the non-Gaussian turbulent velocity field governed by the Navier-Stokes equation [17].

However, when investigating the problem of a passive vector field advected by the Navier-Stokes velocity field in the framework of the $A = 0$ model, the question of the value of the corresponding turbulent Prandtl number immediately arises. In this respect, in the present paper we shall concentrate on the calculation of this turbulent "vector" Prandtl number using the field theoretic RG technique in the two-loop approximation. We shall find the corresponding explicit two-loop RG expression for the inverse turbulent vector Prandtl number and the obtained value of the turbulent vector Prandtl number will be compared to the turbulent Prandtl number of passively advected scalar field [5,6], which is the same as the turbulent magnetic Prandtl number of the kinematic MHD turbulence [7]. We shall show that, at least at the two-loop level of approximation, the values of these turbulent Prandtl numbers are surprisingly very close to each other. This nontrivial and rather unexpected fact will allow us to make some interesting conclusions.

Thus, let us consider a solenoidal vector field $\mathbf{w} \equiv \mathbf{w}(x)$ $[x \equiv (t,\mathbf{x})]$ passively advected by the fully symmetric isotropic turbulent environment in the framework of the $A = 0$ model, which is described by the following system of stochastic equations [20]

$$\partial_t \mathbf{w} = \nu_0 u_0 \triangle \mathbf{w} - (\mathbf{v}\cdot\partial)\mathbf{w} - \partial Q + \mathbf{f}^{\mathbf{w}}, \tag{1}$$

$$\partial_t \mathbf{v} = \nu_0 \triangle \mathbf{v} - (\mathbf{v}\cdot\partial)\mathbf{v} - \partial P + \mathbf{f}^{\mathbf{v}}. \tag{2}$$

Here, the standard notation is used: $\partial_t \equiv \partial/\partial t$, $\partial_i \equiv \partial/\partial x_i$, $\triangle \equiv \partial^2$ is the Laplace operator, ν_0 is the viscosity coefficient, u_0 is the reciprocal "vector" Prandtl number, $\mathbf{v} \equiv \mathbf{v}(x)$ is the incompressible velocity field, and $P \equiv P(x)$ and $Q \equiv Q(x)$ are the corresponding pressures. Due to the assumption of incompressibility, the velocity field $\mathbf{v} \equiv \mathbf{v}(x)$ is also solenoidal. Thus, both $\mathbf{v}$ and $\mathbf{w}$ are divergence-free vector fields: $\partial \cdot \mathbf{v} = \partial \cdot \mathbf{w} = 0$.

In Eq. (1), the transverse random noise $\mathbf{f}^{\mathbf{w}} = \mathbf{f}^{\mathbf{w}}(x)$ is taken in the form of a Gaussian distribution with correlator

$$D^w_{ij}(x;0) \equiv \langle f^{\mathbf{w}}_i(x) f^{\mathbf{w}}_j(0)\rangle = \delta(t) C_{ij}(|\mathbf{x}|/L). \tag{3}$$

It represents the source of the fluctuations of the vector field, where L is an integral scale related to the corresponding stirring and C_{ij} is a function finite in the limit $L \to \infty$. Its detailed form is not important in what follows because it does not enter into the calculations. The only condition that must be satisfied is that C_{ij} decreases rapidly for $|\mathbf{x}| \gg L$. On the other hand, the explicit form of the transverse random force per unit mass $\mathbf{f}^{\mathbf{v}}$ is essential. We shall assume that it also obeys a Gaussian distribution with zero mean and correlator

$$D^v_{ij}(x;x') = \langle f^{\mathbf{v}}_i(x) f^{\mathbf{v}}_j(x')\rangle = \delta(t-t') D_0 \int \frac{d^d\mathbf{k}}{(2\pi)^d} P_{ij}(\mathbf{k}) k^{4-d-2\varepsilon} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}, \tag{4}$$

where d is the space dimension and $P_{ij}(\mathbf{k})$ is a transverse projector which, in general, describes geometric properties of the random force. In the simplest isotropic case, which is considered in what follows, it reads $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2$. In Eq. (4), $D_0 > 0$ is a positive amplitude and the most realistic value of the exponent $0 < \varepsilon \leqslant 2$ is $\varepsilon = 2$ (see, e.g., Refs. [8,9] for details). The correlator Eq. (4) is written in a form that realizes a realistic, i.e., infrared, introduction of the energy (by large-scale eddies) into the system and, at the same time, it has the power-law asymptotic form at large k. The last condition is necessary for application of the field theoretic RG technique. In Eq. (4), the needed infrared regularization is given by a restriction of the integration from below, namely, $k \geqslant m$, where m corresponds to another integral scale. In what follows, we shall suppose that $L \gg 1/m$. It is also useful to introduce new bare coupling constant g_0 by relation $D_0 \equiv g_0 \nu_0^3$. Then, g_0 is a formal small parameter of the ordinary perturbation theory related to the characteristic ultraviolet (UV) momentum scale Λ (or inner length $l \sim \Lambda^{-1}$) by the relation $g_0 \simeq \Lambda^{2\varepsilon}$ (see, e.g., Refs. [8,9] for details).

The stochastic problem, Eqs. (1)–(4), can be rewritten into a field theoretic model of the double set of fields $\Phi = \{\mathbf{v},\mathbf{w},\mathbf{v}',\mathbf{w}'\}$ (see, e.g., Ref. [9] and references cited therein) with the action functional given as follows:

$$S(\Phi) = \frac{1}{2}\int dt_1\, d^d\mathbf{x}_1\, dt_2\, d^d\mathbf{x}_2 \big[v'_i(x_1) D^v_{ij}(x_1;x_2) v'_j(x_2) + w'_i(x_1) D^w_{ij}(x_1;x_2) w'_j(x_2)\big] + \int dt\, d^d\mathbf{x} \{\mathbf{v}'[-\partial_t + \nu_0\triangle - (\mathbf{v}\cdot\partial)]\mathbf{v} + \mathbf{w}'[-\partial_t \mathbf{w} + \nu_0 u_0 \triangle \mathbf{w} - (\mathbf{v}\cdot\partial)\mathbf{w}]\}, \tag{5}$$

where $x_l = (t_l,\mathbf{x}_l)$, $l = 1,2$, $\mathbf{v}'(x)$ and $\mathbf{w}'(x)$ are auxiliary transverse fields with the same tensor properties as fields $\mathbf{v}(x)$ and $\mathbf{w}(x)$, D^w_{ij} and D^v_{ij} are given in Eqs. (3) and (4), respectively, and all required summations over dummy indices are assumed.

The advantage of the formulation of the stochastic problem given by Eqs. (1)–(4) through action functional Eq. (5) is that it makes it possible to apply the well-defined field theoretic means, e.g., the RG technique, to analyze the problem [8,9].

Now, using standard dimensional analysis of the canonical dimensions [8,9], it can be shown that the model Eq. (5) is logarithmic (i.e., the coupling constant g_0 is dimensionless) at $\varepsilon = 0$. In addition, for $d > 2$ the superficial UV divergences are present only in the 1-irreducible Green's functions $\langle v'_i v_j\rangle_{1-ir}$ and $\langle w'_i w_j\rangle_{1-ir}$ and can be removed by using the multiplicative renormalization of bare parameters g_0, u_0, and ν_0 in the following form [21]:

$$\nu_0 = \nu Z_1, \quad g_0 = g\mu^{2\varepsilon} Z_1^{-3}, \quad u_0 = u Z_2 Z_1^{-1}, \tag{6}$$

where the dimensionless parameters g,u, and ν are the renormalized counterparts of the bare parameters, μ is the renormalization mass (a scale-setting parameter), an artefact of the dimensional regularization, and $Z_i = Z_i(g,u;d;\varepsilon), i = 1,2$ are two independent renormalization constants. Their general explicit form in the minimal subtraction (MS) scheme is given as follows:

$$Z_i(g,\varepsilon) = 1 + \sum_{n=1}^{\infty} g^n \sum_{j=1}^{n} \frac{z^{(i)}_{nj}}{\varepsilon^j}, \quad i = 1,2, \tag{7}$$

where coefficients $z^{(i)}_{nj}, i = 1,2$ are independent of g and ε. Their explicit form is determined by the requirement that the 1-irreducible Green's functions $\langle v'_i v_j\rangle_{1-ir}$ and $\langle w'_i w_j\rangle_{1-ir}$ are UV finite when are written in the renormalized variables, i.e., they must be free of poles in ε.

The expansion of the renormalization constant Z_1 to the second order in g (two-loop approximation) has been known for a long time [18]. On the other hand, one-loop contribution $z^{(2)}_{11}$ to the renormalization constant Z_2 is

$$z^{(2)}_{11} = -\frac{S_d}{(2\pi)^d}\frac{(d^2-3)}{4u(1+u)d(d+2)}, \tag{8}$$

where $S_d = 2\pi^{d/2}/\Gamma(d/2)$ denotes the surface area of the d-dimensional unit sphere and $\Gamma(x)$ is Euler's gamma function. On the other hand, the two-loop contribution $z^{(2)}_{21}$ in general d-dimensional space is complicated expression and will be discussed elsewhere. However, for the most important three-dimensional case at the fixed point (see below) one obtains

$$z^{(2)}_{21} = -1.01914 \times 10^{-5}, \tag{9}$$

where the one-loop fixed-point value for parameter u, namely, $u^{(1)}_* = 1$, is already used (see below). Finally, the quantity $z^{(2)}_{22}$ is not important in what follows, therefore, we shall not analyze it at all.

The infrared (IR) asymptotic scaling behavior, i.e., the scaling behavior deep inside the inertial range, of the correlation functions of the model is driven by the IR stable fixed point of the RG equations [8,9]. The coordinates of the fixed point are given by the requirement of the vanishing of the so-called β RG functions of the model, namely,

$$\beta_g(g_*) = 0, \quad \beta_u(g_*,u_*) = 0, \tag{10}$$

where fixed point values of all quantities are denoted by stars and

$$\beta_g = g(-2\varepsilon + 3\gamma_1), \quad \beta_u = u(\gamma_2 - \gamma_1),$$
$$\gamma_i \equiv \mu\partial_\mu \ln Z_i, \quad i = 1,2. \tag{11}$$

Finally, the coordinates of the IR stable fixed point in the two-loop approximation for $d = 3$ are

$$g_* = \frac{40\pi^2}{3}\varepsilon(1 - 1.0994\varepsilon), \quad u_* = 1 - 0.0270\varepsilon. \tag{12}$$

It is IR stable for $\varepsilon > 0$.

In Ref. [5], the two-loop scheme independent RG expression was derived for the inverse turbulent (effective) Prandtl number in the model of passively advected scalar field. The corresponding expression for the two-loop inverse turbulent Prandtl number in the vector model under consideration can be derived and written in the similar form ($d > 2$), namely,

$$u_{\text{eff}} = u_*^{(1)}\left(1 + \varepsilon\left\{\frac{1 + u_*^{(1)}}{1 + 2u_*^{(1)}}\left[\lambda - \frac{128(d+2)^2}{3(d-1)^2}B(u_*^{(1)})\right] + \frac{(2\pi)^d}{S_d}\frac{8(d+2)}{3(d-1)}(a_v - a_w)\right\}\right), \tag{13}$$

where $u_*^{(1)}$ is the one-loop fixed point value of the parameter u (it is also the one-loop value for the inverse turbulent vector Prandtl number). The general $d > 2$ case is given by the following simple equation:

$$u_*^{(1)}[1 + u_*^{(1)}] = 2(d^2 - 3)/[d(d-1)], \tag{14}$$

and for $d = 3$, one has $u_*^{(1)} = 1$. The quantity λ in Eq. (13) is related to the coefficient $z_{21}^{(1)}$ in Eq. (7); i.e., it is given by the two-loop RG analysis of the model of pure fully developed turbulence driven by the stochastic Navier-Stokes equation. It was analyzed in Ref. [18], and for $d = 3$, one has $\lambda = -1.0994$. The quantities a_v and a_w are given by the corresponding expansions to the leading order in ε of the scaling functions of response functions $\langle vv'\rangle$ and $\langle ww'\rangle$ for the velocity field and the advected vector field, respectively (see Ref. [5] for details). Their values for $d = 3$ are

$$a_v = -0.00241744, \quad a_w = -0.00402280. \tag{15}$$

Finally, the quantity $B(u_*^{(1)})$ is related to the coefficient $z_{21}^{(2)}$ in Eq. (9) by the relation ($d = 3$)

$$B(u_*^{(1)}) = \frac{(2\pi)^6}{S_3^2}z_{21}^{(2)} = -0.00397094. \tag{16}$$

Thus, using all these facts, one comes to the final two-loop value for the inverse turbulent vector Prandtl number in three dimensions, namely,

$$u_{\text{eff}} = 1 + 0.18426\varepsilon + O(\varepsilon^2), \tag{17}$$

and for physical value $\varepsilon = 2$, one finally obtains the two-loop value of the turbulent vector Prandtl number

$$\text{Pr}_{v,t} = u_{\text{eff}}^{-1} = 0.7307. \tag{18}$$

Result for the two-loop turbulent vector Prandtl number given in Eq. (18) is instructive at least from two points of view. First, in contrast to the two-loop correction to the turbulent Prandtl number in the model of passively advected scalar field [5,6] (as well as to the turbulent magnetic Prandtl number in the kinematic MHD turbulence [7]), which is only 2% of its one-loop value, the two-loop correction to the turbulent vector Prandtl number in the present model is essentially larger, namely, it is 27% of its one-loop value. However, more interesting is the fact that the rather large relative difference between the one-loop values of the turbulent Prandtl number in the scalar model $\text{Pr}_t^{(1)} = 0.7179$ [5] and the turbulent vector Prandtl number $\text{Pr}_{v,t}^{(1)} = 1$ [see Eq. (14)], namely, $(\text{Pr}_{v,t}^{(1)} - \text{Pr}_t^{(1)})/\text{Pr}_t^{(1)} \simeq 0.39$, i.e., which is 39% in respect to the one-loop turbulent Prandtl number of the model of the passive scalar advection, is radically reduced when two-loop corrections are taken into account, namely, $(\text{Pr}_{v,t} - \text{Pr}_t)/\text{Pr}_t \simeq 0.038$, i.e., the relative difference is only 3.8% in respect to the two-loop value of the turbulent Prandtl number of the model of the passive scalar advection. Here, $\text{Pr}_{v,t} = 0.7307$ as it is given in Eq. (18) and $\text{Pr}_t = 0.7040$ was found in Refs. [6] and [22]. This result is rather surprising and instructive. Namely, it seems that the turbulent environments do not feel essential difference between the internal tensor structure of passively advected scalar and vector fields and, as a result, the properties of the corresponding diffusion processes are very similar.

Here, a few questions for further investigation immediately arises. For example, it would be interesting to analyze in more details the source of rather large two-loop correction to the turbulent vector Prandtl number compared with the small two-loop corrections to the turbulent Prandtl number of passively advected scalar field [5,6] and the turbulent magnetic Prandtl number in the kinematic MHD turbulence [7]. However, to answer this question it is necessary to investigate the problem in general d-dimensional case. Another open question is the behavior of the turbulent vector Prandtl number in turbulent environments with some symmetry breaking (spatial parity violation, anisotropy) compared with the corresponding behaviors of the turbulent Prandtl number and the turbulent magnetic Prandtl number. But maybe the most interesting open question is whether the closeness of the turbulent vector Prandtl number to the turbulent Prandtl number of scalar field, which is seen at two-loop level of approximation studied in this paper, is perturbatively stable. Here, at least, the corresponding three-loop calculations are needed. However, all these questions are open for now.

In conclusion, in this paper we have used the field theoretic RG technique within the two-loop approximation for investigation of the turbulent vector Prandtl number in the model of passive vector quantity advected by the turbulent velocity field driven by the stochastic Navier-Stokes equation. It is shown that the two-loop correction essentially decreases the one-loop value of the turbulent vector Prandtl number. On the other hand, the value of the turbulent vector Prandtl number at the two-loop approximation is very close to the corresponding two-loop values of the turbulent Prandtl number in the model of passively advected scalar field [5,6] and the turbulent magnetic Prandtl number in the framework of the kinematic MHD turbulence [7]. It seems that the properties of diffusion processes in turbulent environments only slightly depend on the internal tensor structure of the advected fields as well as on the form of their interactions with the velocity field, at least, in fully symmetric isotropic turbulent systems. In the end, we can conclude that now the turbulent Prandtl numbers are known at the two-loop RG approximation for all relevant and usually studied models of passive scalar and vector quantities advected by the Navier-Stokes turbulence.

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[20] The name "$A = 0$ model" is given by the following consideration: The general form of the nonlinear part in Eq. (1) can be written as $-(\mathbf{v} \cdot \partial)\mathbf{w} + A(\mathbf{w} \cdot \partial)\mathbf{v}$, where the parameter A in front of the "stretching term" $(\mathbf{w} \cdot \partial)\mathbf{v}$ is not fixed by Galilean symmetry and can be arbitrary [10,17]. If $A = 0$, then one obtains the aforementioned "$A = 0$ model" of passively advected vector impurity.

[21] For $d = 2$, additional divergences appear in the model that cannot be removed by using multiplicative renormalization in the framework of the present formulation. Here, the so-called double expansion approach [19] is needed, which is out of scope of this paper.

[22] Note that this value is a little bit different from the value calculated in Ref. [6], namely 0.7051. It is related to the fact that in the present paper we have used more precise numerical value for the quantity λ, namely, $\lambda = -1.0994$ than in Ref. [6], where less precise value calculated in Ref. [18] was used, namely $\lambda = -1.101$.