Rogue wave solutions to the generalized nonlinear Schrödinger equation with variable coefficients

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A similarity transformation is utilized to reduce the generalized nonlinear Schrödinger (NLS) equation with variable coefficients to the standard NLS equation with constant coefficients, whose rogue wave solutions are then transformed back into the solutions of the original equation. In this way, Ma breathers, the first- and second-order rogue wave solutions of the generalized equation, are constructed. Properties of a few specific solutions and controllability of their characteristics are discussed. The results obtained may raise the possibility of performing relevant experiments and achieving potential applications.

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I. INTRODUCTION

The study of nonlinear (NL) optical waves has received much attention in recent years. Often, they are described by various nonlinear Schrödinger (NLS) equations, which appear in many branches of physics such as NL optics [1], photonics [2], and Bose-Einstein condensates [3]. Many authors have studied different types of NLS equations and discussed propagation properties of solutions using a variety of methods [4], including Hirota's method [5], Darboux transformation [6], similarity transformation [7], and others.

Rogue waves are giant single waves that may suddenly appear in oceans [8]. Their appearance can be quite unexpected and mysterious in origin. Rogue waves have been recently identified in other fields—in NL optics [9,10], atmosphere [11], plasmas, etc. [12–14]. Although solitary by nature, rogue waves are different from the usual solitons, in that they are rare, short-lived, and unstable. They can emerge from a turbulent state of random fields, while ordinary solitons are stable waves with characteristic collision properties, commonly appearing in deterministic settings of NL $(1+1)$-dimensional $[(1+1)\text{D}]$ evolution partial differential equations (PDEs). Still, rogue waves are related to solitons; one of the mechanisms for the generation of optical rogue waves appears to be the oblique collision of bright solitons in fibers [14].

The first breather-type rogue solution of the NLS was found by Ma in 1979 [12]. This rogue wave solution breathes temporally and is spatially localized. By extending the temporal period of this solution to infinity, Peregrine in 1983 obtained a soliton solution localized in both space and time [13]. Recent studies in NL optics demonstrated that rogue waves can be obtained as analytical solutions of some integrable NLS equations, in the form of Akhmediev breathers [14] or as their limiting cases when the spatial and temporal periods are taken to infinity. These solutions are known as the rational soliton (RS) solutions [15–17]. The Peregrine soliton is also a limiting case of the Akhmediev breather when the spatial period is taken to be infinite. RS solutions come in the form of ratios of two polynomial functions of space and time. A hierarchy of RSs has been identified in the framework of integrable one-dimensional (1D) NLS equations [14]. Thus, the Peregrine solution can be considered as the first-order RS. Because of their feature of being localized in both space and time, the RSs may be viewed as prototypes of rogue waves [18].

We search for exact rogue wave solutions of a generalized 1D NLS equation with variable coefficients. With the help of exact solutions, the phenomena modeled by this equation can be better understood. In particular, the possibility of controlling the propagation of optical rogue waves opens vast new opportunities. One of the important classes of exact solutions is the so-called managed rogue wave solution. This solution maintains its overall shape but allows for changes in the width and amplitude, according to the management of the system's parameters; these may include diffraction, nonlinearity, and gain or loss [19]. Management opens a venue into experimental manipulation of rogue waves, which has needed more exploration.

In recent years, many works have been devoted to the construction of analytical solutions of the NLS equation with variable coefficients, such as the pioneering work of Serkin *et al.* [20]. Senthilnathan *et al.* [21] have investigated the evolution of optical pulses in NL media. Tang and Shukla have presented some analytical soliton solutions of the spatially inhomogeneous NLS equation with an external potential [22]. These authors have constructed solutions by reducing the NLS equation with variable coefficients into the stationary NLS equation. Following a similar procedure, we employ an analytical method to obtain rogue wave solutions by reducing the generalized NLS equation with variable coefficients into the standard NLS equation and connecting the solutions of the two.

The present Brief Report is organized as follows. In Sec. II, we extend the similarity method given in Refs. [21,22] to Eq. (1), to reduce the generalized NLS equation with variable coefficients to the standard NLS equation and present exact rogue wave solutions. In Sec. III, we investigate some characteristics of the rogue waves, by selecting parameters of the original equation. In Sec. IV, conclusions are outlined briefly.

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II. SIMILARITY TRANSFORMATION OF NLS EQUATION WITH VARIABLE COEFFICIENTS

The optical pulse propagation in Kerr media can be investigated with the help of the generalized NLS equation with variable coefficients, in the form

$$i\frac{\partial u}{\partial z}+\frac{1}{2}\beta(z)\frac{\partial^2 u}{\partial x^2}+\chi(z)|u|^2u=i\gamma(z)u, \tag{1}$$

where $u(z,x)$ is the complex envelope of the optical field; here the dimensionless propagation distance z and the transverse coordinate x are measured in some convenient units. When the propagation of pulses in fibers is considered, the variable x is interpreted as local time. The function $\beta(z)$ represents the diffraction coefficient, $\chi(z)$ the nonlinearity coefficient, and $\gamma(z)$ the gain ($\gamma>0$) or the loss ($\gamma<0$) coefficient. Inspired by our previous work [23,24], we search for a similarity transformation that would reduce Eq. (1) to the standard NLS equation,

$$i\frac{\partial V}{\partial T}+\frac{1}{2}\frac{\partial^2 V}{\partial X^2}+|V|^2V=0, \tag{2}$$

where the complex field $V(T,X)$, the effective dimensionless propagation distance $T(z)$, and the similarity variable $X(z,x)$ are functions to be determined. To connect the rogue wave solution of Eq. (1) with those of Eq. (2) we use the similarity transformation

$$u(z,x)=A(z)V(T,X)e^{iB(z,x)}, \tag{3}$$

where $A(z)$ is the amplitude and $B(z,x)$ is the phase of the wave, assumed to be real functions. Substituting Eq. (3) into Eq. (1) will lead to Eq. (2), provided a system of relations and PDEs for T, X, A, and B is satisfied:

$$w^2(X_x)^2=1, \tag{4a}$$

$$T_z=\frac{\beta}{w(z)^2}, \tag{4b}$$

$$\chi w^2A^2=\beta, \tag{4c}$$

$$-\frac{\partial B}{\partial z}-\frac{1}{2}\beta\left(\frac{\partial B}{\partial x}\right)^2=0, \tag{4d}$$

$$X_z+\beta X_xB_x=0, \tag{4e}$$

$$2A_z+\beta AB_{xx}=2\gamma A. \tag{4f}$$

The subscripts mean the partial derivatives with respect to z or x, respectively.

From Eq. (4a), the similarity variable can be assumed as

$$X(z,x)=\frac{x}{w(z)}, \tag{5a}$$

where $w(z)$ is the width of the rogue wave; in general, it may depend on the propagation distance z. The phase can be expressed in the form [25] $B(z,x)=a(z)x^2+b(z)x+c(z)$, where $a(z)$ is the chirp function. With the help of symbolic computations, we obtain the solutions of Eqs. (4):

$$T=\int_0^z\frac{\beta}{w^2(z)}dz, \tag{5b}$$

$$w(z)=\frac{\omega_0\beta}{\chi}e^{-2\int_0^z\gamma(z)dz}, \tag{5c}$$

$$A=\frac{1}{w(z)}\sqrt{\frac{\beta}{\chi}}, \tag{5d}$$

$$B(z,x)=\frac{1}{2\beta}\frac{1}{w}\frac{dw}{dz}x^2+c_0, \tag{5e}$$

where w_0 is the initial width and c_0 the initial phase shift. In the examples below, we choose $w_0=1$, $c_0=0$, without loss of generality. In addition, the following condition must be imposed on the beam width and the diffraction coefficient:

$$\beta w_{zz}=\beta_z w_z. \tag{6}$$

Hence, the solutions found can exist only under certain conditions and the system coefficients $\chi(z)$, $\gamma(z)$, and $\beta(z)$ cannot be all chosen independently. Two coefficients can be chosen independently; the third one has to be consistent with Eq. (6).

Using the inverse scattering technique, Ma has found a breather solution of the standard NLS equation (2), which is periodic in the effective propagation distance $T(z)$ [12]. It can be written as

$$V(T,X)=\left[2\frac{\cos(2\sqrt{2}T)+i\sqrt{2}\sin(2\sqrt{2}T)}{\sqrt{2}\cosh(2X)-\cos(2\sqrt{2}T)}-1\right]e^{iT}. \tag{7a}$$

The same technique is also used by Osborne *et al.* to arrive at a similar solution [26]. In general, the Ma breather can be simplified in a limiting case when the period becomes infinite [27] and the periodic solution then becomes a rogue wave solution with the following basic structure:

$$V_n(T,X)=\left[(-1)^n+\frac{G_n(T,X)+iH_n(T,X)}{D_n(T,X)}\right]e^{iT}, \tag{7b}$$

where $n=1,2,\ldots$. The polynomial $D_n(T,X)$ should have no zeros in the region of interest, to ensure that the solution is finite everywhere.

For the first-order ($n=1$) solution, one finds $G_1=4$, $H_1=8X$, and $D_1=1+4T^2+4X^2$; this solution is known as the Peregrine soliton [13]. The second-order ($n=2$) solution has been found by Akhmediev *et al.* [28]. The solution $u_2(T,X)$ can be considered as a superposition of two first-order rogue waves. It has the form of solution (7b) with G_2, H_2, and D_2 given by

$$G_2=\left(X^2+T^2+\tfrac{3}{4}\right)\left(X^2+5T^2+\tfrac{3}{4}\right)-\tfrac{3}{4}, \tag{8a}$$

$$H_2=T\left[T^2-3X^2+2(X^2+T^2)^2-\tfrac{15}{8}\right], \tag{8b}$$

$$D_2=\tfrac{1}{3}(X^2+T^2)^3+\tfrac{1}{4}(X^2-3T^2)^2+\tfrac{3}{64}(12X^2+44T^2+1). \tag{8c}$$

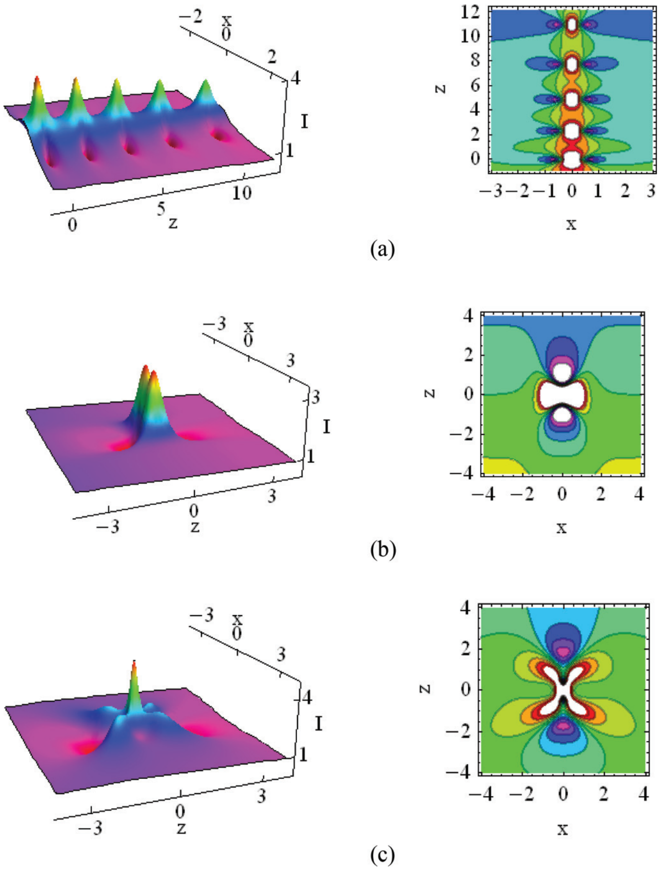

FIG. 1. (Color online) Intensity distributions of rogue waves in a lossy medium. (a) Ma breather; (b) Peregrine first-order wave; (c) Akhmediev second-order wave. The parameters are $w_0 = 1$, $\gamma(z) = -0.01$.

Another second-order solution, with two free parameters, has been presented in Ref. [29]. Thus, collecting the partial solutions together, we obtain the exact rogue wave solution of Eq. (1):

$$u(z,x) = \frac{1}{w}\sqrt{\frac{\beta}{\chi}}V(T,X)\,e^{i(T+\frac{1}{2\beta}\frac{1}{w}\frac{dw}{dz}x^2+c_0)}. \quad (9)$$

Equation (9) describes the dynamics of various rogue waves. It is worthwhile noting the importance of the chirp function in our procedure; it is the factor multiplying x^2 in the phase of the wave, Eq. (5e). It influences the phase, but can also influence the amplitude, through the dependence in Eq. (5d). One can see that the chirp vanishes when $dw/dz = 0$. This is expected, as the chirp also represents the wave-front curvature. There exist three parameter functions in the model, $\beta(z)$, $\chi(z)$, and $\gamma(z)$; by selecting two of these functions appropriately, we can manage rogue waves, to obtain desirable physical characteristics.

III. CHARACTERISTIC DISTRIBUTIONS OF ROGUE WAVES

In this section we present some examples, to illustrate the characteristics of the analytic solution (9). To this end, we make some choices for $\beta(z)$, $\chi(z)$, $\gamma(z)$ and present the

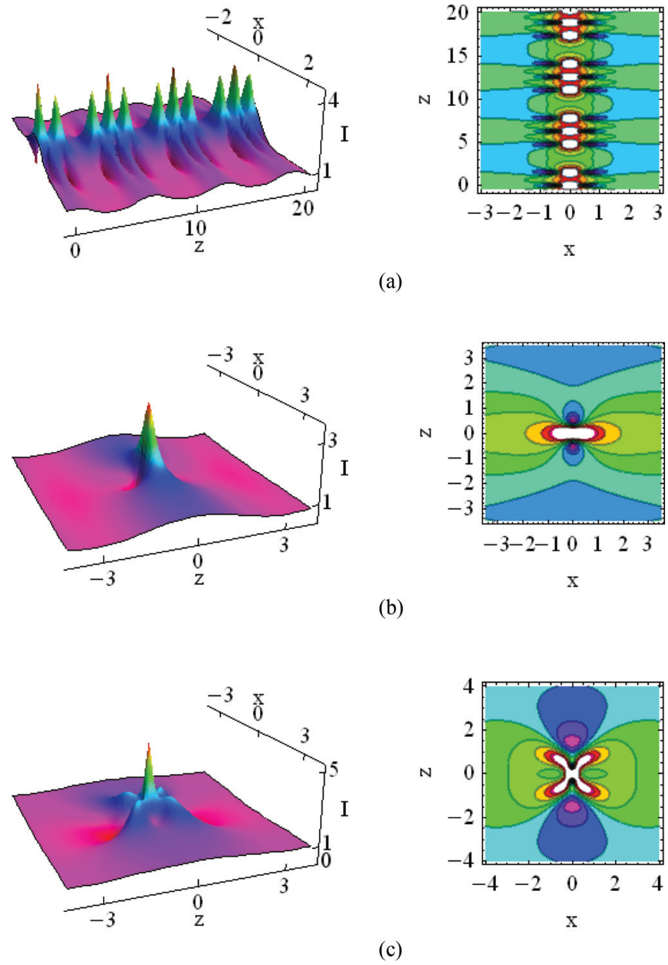

FIG. 2. (Color online) Rogue wave solutions with the periodic modulation of the diffraction and nonlinearity coefficients. Setup is the same as in Fig. 1. The parameters are given in the text.

corresponding system management schemes. In the absence of management, the three coefficients may be chosen as $\beta(z) = 1$, $\chi(z) = 1$, and $\gamma(z) = 0$. One then finds $w(z) = w_0$, $X(z,x) = x/w_0$, and $T(z) = z/w_0^2$. These solutions correspond to the usual Ma breather, the first-order Peregrine, and the second-order Akhmediev rogue waves that were presented elsewhere [12–14].

We present here a few managed cases, in which the choice of the parameter functions leads to the controlled development of rogue waves. We study first the influence of the loss or gain in Eq. (1), in the absence of nonlinearity management, namely for $\chi(z) = 1$. From the condition (6), one obtains the diffraction coefficient $\beta(z) = \beta_0 \exp[-2\int_0^z \gamma(z)dz]$, where β_0 is the coefficient at $z = 0$. Hence, the width of the rogue wave becomes $w(z) = w_0 \exp[-2\int_0^z \gamma(z)dz]$, the effective distance $T = \int_0^z \exp\{[4\int_0^z \gamma(z)dz]\}dz/w_0^2$, and the similarity variable $X = x\exp[-2\int_0^z \gamma(z)dz]/w_0$. We present the corresponding rogue wave intensity distributions in Fig. 1. The figure shows the intensity profiles of the Ma breather, the Peregrine, and the Akhmediev rogue waves of Eq. (9), represented as functions of x and z, for a small loss. As a typical example, we choose $\gamma(z) = -0.01$. As can be seen in Fig. 1(a), the Ma breather has its peak gradually decreasing and its width expanding

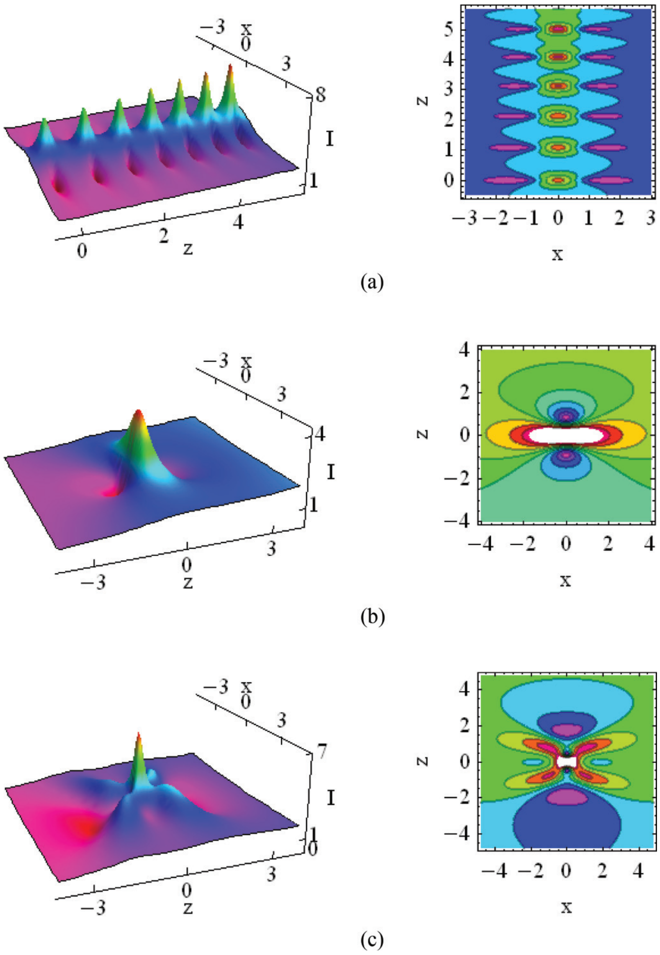

FIG. 3. (Color online) Rogue wave solutions with the exponential modulation of the diffraction and nonlinearity coefficients. Setup is the same as in Fig. 1. The parameters are given in the text.

along the propagation direction. For positive γ, these trends are reversed. It is noteworthy to observe that the Peregrine wave peak splits into two, owing to the diffraction effect in the lossy medium. Also, the usual symmetry of the Peregrine and Akhmediev solutions along the z direction is now gone.

It is evident that solution (9) is particularly useful for investigating the rogue waves through the periodic modulation of diffraction and nonlinearity coefficients. One can then obtain optically controlled systems, by choosing specific forms of the modulation functions, for specific problems. We consider an example illustrative of some fascinating features of our analytical solution (9), by considering a system in which the diffraction and the nonlinearity coefficients are chosen equal, $\beta(z) = \chi(z) = 1 + k\cos(\sigma z)$, where $-1 < k < 1$ and $\sigma(\neq 0)$ are arbitrary constants. The corresponding gain coefficient is given by Eqs. (5c) and (6), $\gamma(z) = k\sigma\sin(\sigma z)/[2(1 + k\cos\sigma z)]$, while the pulse width can be written as $w(z) = w_0/(1 + k\cos\sigma z)$. Also, $T = [6(3k^2 + 2)\sigma z + 9k(k^2 + 4)\sin\sigma z + 9k^2\sin(2\sigma z) + k^3 \sin(3\sigma z)]/12\sigma w_0^2$ and $X(z,x) = x(1 + k\cos\sigma z)/w_0$. We display intensity distributions in Fig. 2, where the parameters are chosen as $w_0 = 1$, $k = 0.2$, $\sigma = 1$. For the periodic modulation, the rogue waves propagate in a periodically modulated background. Propagation on a distorted background was also indicated in the first-ever experimental demonstration of a Peregrine soliton in an optical fiber generating femtosecond pulses [30].

To display some new solutions of Eq. (1), we consider $\beta(z)$ and $\chi(z)$ as exponential functions, without the loss term. The diffraction coefficient is chosen as $\beta(z) = be^{-cz} - ae^{cz}$, where b, a, and c are arbitrary constants. With this choice, from Eqs. (5c) and (6) we find the nonlinearity coefficient $\chi(z) = 2w_0(be^{-cz} - ae^{cz})/(be^{-cz} + ae^{cz})$ and the pulse width $w(z) = w_0(be^{-cz} + ae^{cz})/2$. It is noted that by selecting different values of a and b, one obtains asymmetric rogue waves. As an example, for the choice $a = b = 1$, one has $w(z) = w_0\cosh(cz)$. The intensity of the rogue wave is given by Eq. (9), where now $X(z,x) = x\,\text{sech}(cz)/w_0$ and $T = [\text{sech}(cz) - 1]/w_0^2 c$. In Fig. 3, we plot the corresponding rogue waves for the parameter $c = 0.04$. For the Ma breather, the peaks increase gradually.

In the end, we should stress again that our line of inquiry, involving management of rogue waves by diffraction, nonlinearity, and gain, opens venues into theoretical and experimental investigation of RS solutions to the generalized NLS equation. It also points to a way of achieving control of rogue waves, a goal certainly worth pursuing. While the whole field started almost accidentally, as an observation of freak waves in the ocean, it has moved, by now, to more firm theoretical and experimental grounds. On the experimental front, it appears that NL optics offers the best hope in advancing the field, most notably in using standard telecommunication fibers [31], in addition to multicomponent photoinduced plasmas [32] and, last but not least, the hydrodynamic water wave tanks [27].

IV. CONCLUSIONS

In conclusion, we have presented an analytical rogue wave solution of the generalized NLS equation with variable coefficients. By utilizing the similarity transformation, we have reduced the equation with variable coefficients into the standard NLS equation, whose rogue wave solutions are then transformed back into the solutions of the original equation. Our results show that the rogue waves can be well controlled through the choice of variable diffraction, nonlinearity, and gain or loss coefficients. Moreover, the method is applicable to the higher-order ($n \geqslant 3$) rogue waves of the generalized NLS equation. Since the understanding of rogue waves is very important in the $(2+1)$-dimensional models, which characterize the more realistic evolution in the transverse (x, y) plane, we will try to extend our study to the multidimensional NLS models.

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