Reaction spreading on percolating clusters

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(Received 7 March 2013; published 19 June 2013)

Reaction-diffusion processes in two-dimensional percolating structures are investigated. Two different problems are addressed: reaction spreading on a percolating cluster and front propagation through a percolating channel. For reaction spreading, numerical data and analytical estimates show a power-law behavior of the reaction product as $M(t) \sim t^{d_l}$, where d_l is the connectivity dimension. In a percolating channel, a statistically stationary traveling wave develops. The speed and the width of the traveling wave are numerically computed. While the front speed is a low-fluctuating quantity and its behavior can be understood using a simple theoretical argument, the front width is a high-fluctuating quantity showing a power-law behavior as a function of the size of the channel.

DOI: 10.1103/PhysRevE.87.062811 PACS number(s): 82.30.−b, 82.40.Qt

I. INTRODUCTION

Reaction-diffusion processes have been extensively studied in the past years as systems that are able to shed some light on various problems in different disciplines [1,2]. Recently, the importance of the nonhomogeneity of the medium over which the reaction and diffusion take place has been highlighted [3], since the qualitative and quantitative features of the spreading of the reaction process can depend on the presence of system irregularities. Many studies in recent years have been concerned with the reaction-diffusion process in heterogeneous media accomplishing different problems, ranging from epidemic evolution in heterogeneous networks [4] or intracellular calcium dynamics [5] to combustion in porous media [6]. In this context, studies of the reaction dynamics on percolating clusters appear very interesting for their physical relevance and their applications in many different scientific and technological fields [7–10]. For recent experimental and numerical results on reaction diffusion in heterogeneous media, see [11–16].

The study of reaction and diffusion dynamics on homogeneous substrates dates back to the Fisher-Kolmogorov-Petrovskii-Piskunov (FKPP) model [17],

$$\partial_t \theta = D\Delta\theta + \alpha g(\theta)\,, \tag{1}$$

where the scalar field θ represents the fractional concentration of the reaction products, D is the molecular diffusivity, $g(\theta)$ describes the reaction process, and α is the reaction rate, i.e., the inverse of the characteristic time, τ, of the reaction process. In the original model [17], $g(\theta)$ assumes a convex shape $g(\theta) = \theta(1-\theta)$. It is possible to show that under very general conditions [17], i.e., if $g(\theta)$ is a convex function and $g'(0) = 1$, a traveling wave develops with asymptotic speed and width given by

$$v_0 = 2\sqrt{\alpha D}, \quad \delta_0 = c\sqrt{D/\alpha},$$

where the constant c depends on the definition adopted for the computation of the front width.

Afterward, as previously mentioned, reaction-transport dynamics attracted a considerable amount of interest due to their relevance in an incredibly large number of chemical, biological, and physical systems [1,2]. In general, when dealing with a nontrivial environment for the reaction and diffusion process, it is possible to extend Eq. (1) in order to take into account the properties of the medium [18–20]:

$$\partial_t \theta = \hat{L}\theta + f(\theta), \tag{2}$$

where the linear operator $\hat{L}$ rules the transport process. An important class of processes of this type is the advection-reaction-diffusion processes, where $\hat{L} = -\mathbf{u}\cdot\nabla + D\Delta$ (e.g., see [18]). On the other hand, it is possible to extend the $\hat{L}$ operator in order to include cases of effective diffusion on fractal objects, $\hat{L} = \frac{1}{r^{d_f-1}}\frac{\partial}{\partial r}[k(r)r^{d_f-1}\frac{\partial}{\partial r}]$ [21], suitable to study reaction dynamics on fractals [22]. Moreover, in a recent paper [20], the reaction spreading on graphs has been considered; in such a case, the operator $\hat{L}$ is merely the Laplacian operator for graphs [23,24]. In the present paper, in the spirit of the cited works, we study the reaction and diffusion dynamics on percolation clusters, considering the spreading properties of such a process. In Sec. II, we present the model and some numerical details. Section III is devoted to the study of reaction spreading in a large percolating cluster, while front

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propagation in a percolating channel is discussed in Sec. IV. In Sec. V, the reader can find some conclusions.

II. MODEL

A natural model to study reaction and diffusion on a two-dimensional (2D) nonhomogeneous medium can be constructed starting from a generalization of Eq. (1) in which the transport operator, $\hat{L} = D(\mathbf{x})\Delta$, depends on the spatial variable:

$$\partial_t \theta(\mathbf{x},t) = D(\mathbf{x})\Delta\theta(\mathbf{x},t) + f(\theta(\mathbf{x},t)). \tag{3}$$

The shape and the spatial distribution of $D(\mathbf{x})$ enable us to take into account the properties of the medium and therefore to consider different physical and biological topics [25,26]. In this way, it is possible to study the reaction dynamics at the "microscopic" level without assuming any effective equation that can incorporate mainly qualitative features of the heterogeneous medium [21,22,27].

Since we are mainly interested in the scaling properties of the asymptotic behavior of the system, without weakening the results we consider the case in which the variable $D(x)$ can assume only two values, i.e., $D(x) = 0$ in forbidden spatial regions and $D(x) = D_0$ in permitted ones. The second step is to consider a spatial discretization of Eq. (3). The spatial region under examination has been discretized using a 2D Euclidean lattice, $\mathcal{L}$, where Δx is the lattice constant. Points $\mathbf{x}$ are replaced by sites of the lattice $s = (i, j)$.

The percolating clusters have been obtained as follows. Each site may be permitted (with probability p) or prohibited (with probability $1 - p$). If $p > p_c$, where $p_c \simeq 0.592\,746$ is the site percolation threshold for square lattices, there is a good chance that the reaction, starting from any of the permitted sites, can invade the system (percolation). We call $\mathcal{P}$ the set of permitted sites. In each permitted site, we have a value of the concentration field, $\theta_s(t) = \theta_{(i,j)}(t)$. Equation (3) can be discretized as follows:

$$\frac{d}{dt}\theta_s = \sum_{s'} C_{s,s'}\theta_{s'} + f(\theta_s), \tag{4}$$

where $\sum_{s'} C_{s,s'}\theta_{s'}$ is the discretization of the general transport operator $\hat{L} = D(x)\Delta\theta(\mathbf{x},t)$. Since we are working on a discrete structure, the value of the lattice spacing Δx is not particularly important (it can be "absorbed" in D_0 for D_0 and δ_0 large enough), therefore we assume $\Delta x = 1$.

To specify the quantity $C_{s,s'}$, we introduce the variable A_s that characterizes the permitted region of the lattice:

$$A_s = \begin{cases} 1 & \text{if } s \in \mathcal{P}, \\ 0 & \text{if } s \notin \mathcal{P}. \end{cases} \tag{5}$$

Given a site s, we can define $k_s = \sum_{|s'-s|=1} A_{s'}$ as the number of permitted nearest neighbors of s. Using these quantities and imposing the mass conservation of the diffusion operator, we can express $C_{s,s'}$ as

$$C_{s,s'} = D_0 \begin{cases} 0 & \text{if } |s - s'| > 1, \\ A_{s'} & \text{if } |s - s'| = 1, \\ k_s & \text{if } s = s'. \end{cases} \tag{6}$$

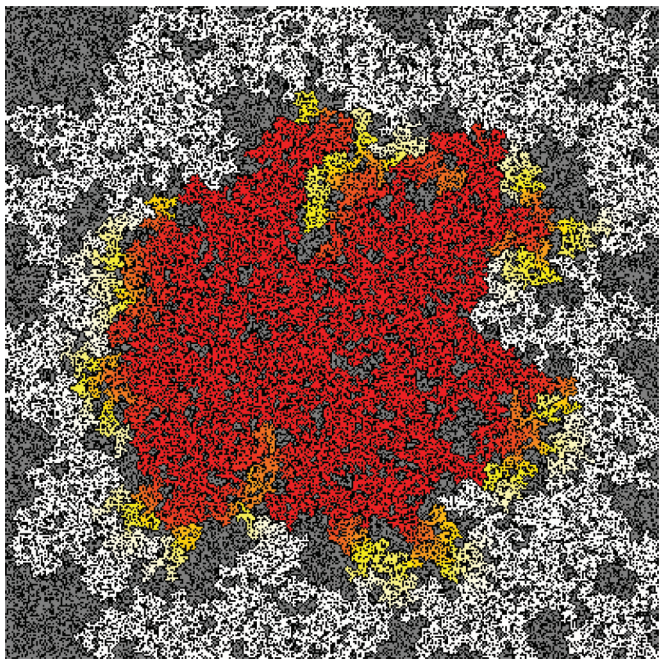

FIG. 1. (Color online) Reaction spreading on a square lattice. The red (dark grey) area contains reaction products, the yellow (light grey) area is the one where the reaction takes place, and the white area contains fresh material. The black dots indicate prohibited sites, whereas the various gray areas indicate regions of permitted sites that do not belong to the percolating cluster (we call them islands).

Therefore, the discretized transport term $\hat{L}\theta_s = \sum_{s'} C_{s,s'}\theta_{s'}$ becomes the discrete Laplacian of the lattice [23]:

$$\hat{L}\theta_s(t) = D_0\left(\sum_{|s'-s|=1} [A_{s'}\theta_{s'}(t)] - k_s\theta_s(t)\right). \tag{7}$$

Finally, the complete model of reaction diffusion on percolating clusters reads

$$\frac{d}{dt}\theta_s = D_0\left(\sum_{|s'-s|=1} [A_{s'}\theta_{s'}(t)] - k_s\theta_s(t)\right) + \alpha g(\theta_s), \tag{8}$$

where, following classical works [17], we choose $f(\theta) = \alpha g(\theta)$, where $g(\theta) = \theta_s(1 - \theta_s)$. From the numerical viewpoint, given the spatial discretization, the temporal derivative is computed via a fourth-order Runge-Kutta algorithm.

In the following, we study two different problems. The first concerns reaction spreading on a large 2D percolating cluster without a specific geometry (see Fig. 1), and starting from an initial condition $\theta_s(0) = 0$ except a single site, $\tilde{s}$, in which $\theta_{\tilde{s}}(0) = 1$. In the second problem, we study the front propagation features (speed and width of the traveling wave) in a 2D channel with dimensions $L_x \times L_y$ with $L_x \geqslant L_y$ (see Fig. 3). In the numerical computations, the lattice is dynamically modified in order to follow the reacting front, i.e., the domain considered in the computation moves rigidly downstream when in the upstream part of the reaction is extinguished. In all the simulations, without a lack of generality, we set $D_0 = 1$.

It is worth saying that, for $p < p_c$, the propagation is practically forbidden if the system is very large. For finite systems one has yet a possible propagation if p is not much

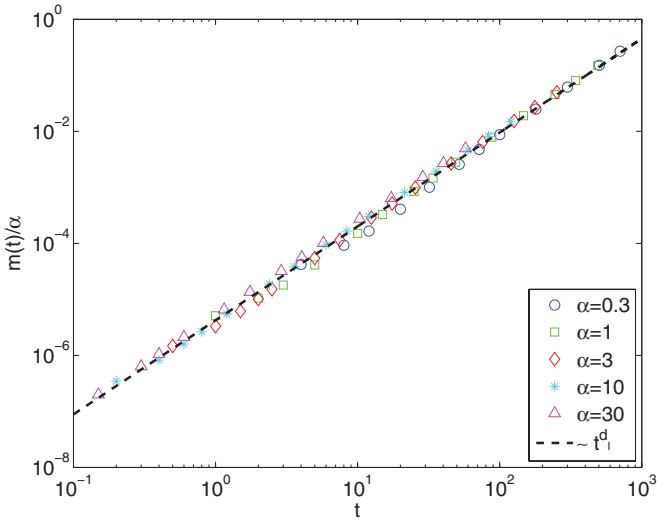

FIG. 2. (Color online) Time evolution at $p = 0.595 \approx p_c$ of the percentage of the quantity of products rescaled by α, $m(t)/\alpha$ vs t, together with the prediction $m(t)/\alpha \simeq t^{d_l}$ with $d_l \simeq 1.67$ (dashed line). To get smooth quantities, $m(t)$ is averaged over many realizations (≈5000) of lattices of size $L_x = 1000$ and $L_y = 1000$ which are portions of larger lattices.

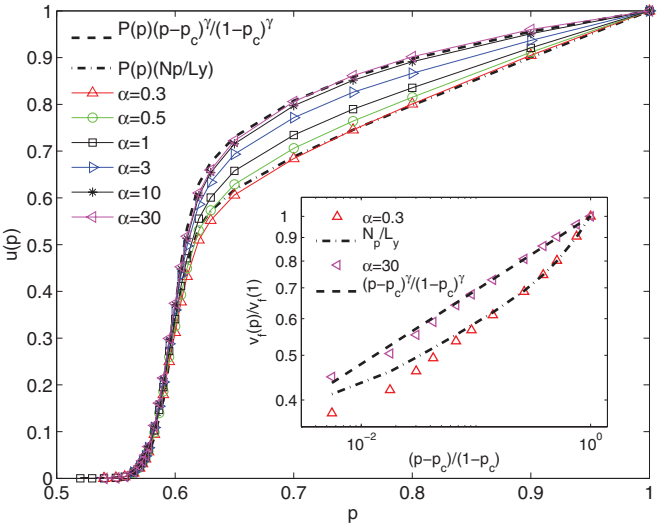

FIG. 4. (Color online) Average front speed $u(p)$ as a function of p for various α together with asymptotic behavior (15) and (16) (with $\gamma \simeq 0.16$). The channel length is taken to be $L_x = 100$ and $L_y = 100$. In the inset is shown in log-log scale the behavior of $v_f(p)/v_f(1)$ together with the theoretical prediction $v_f(p)/v_f(1) \sim N_p/L_y$ for a slow reaction rate and $v_f(p)/v_f(1) \sim [(p - p_c)/(1 - p_c)]^\gamma$ for a fast reaction rate.

lower than p_c, as can be seen in the following (in particular in Fig. 4).

III. REACTION SPREADING

An important quantity that characterizes the spreading of the reaction is the total mass of the reaction product, i.e., $M(t) = \sum_{s\in\mathcal{P}} \theta_s(t)$.

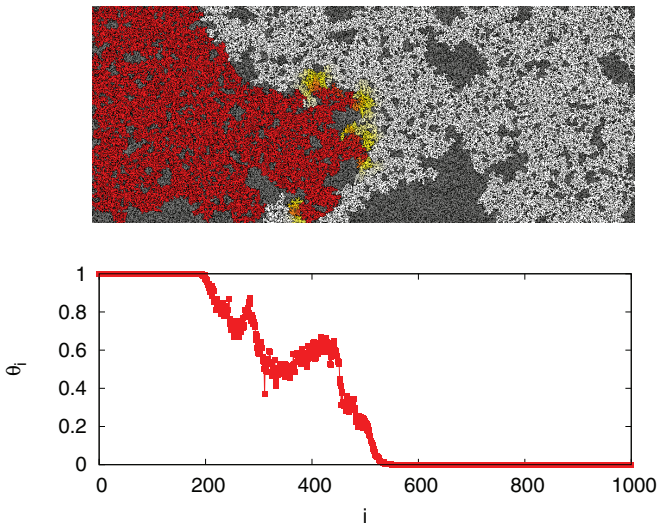

FIG. 3. (Color online) Snapshot of the reaction dynamics in a percolating cluster on a channel (see the caption of Fig. 1). The graph in the lower part of the figure shows the profile of the average of the front [see Eq. (11)] related to the snapshot.

Since this quantity depends on the number of permitted sites, we introduce the percentage of $m(t)$ over the lattice, i.e.,

$$m(t) = \frac{M(t)}{N} = \frac{\sum_{s\in\mathcal{P}} \theta_s(t)}{\sum_{s\in\mathcal{L}} A_s}, \tag{9}$$

where $N = \sum_{s\in\mathcal{L}} A_s = \sum_{s\in\mathcal{P}} 1$ is the number of permitted sites.

Let us briefly recall some relevant quantities in the statistical analysis of generic graphs: the fractal dimension, d_f, the connectivity dimension, d_l (also called chemical dimension), and the spectral dimension, d_s. The fractal dimension d_f [28] describes the scaling of the number of permitted sites in a sphere of radius r in the lattice, as $\mathcal{N}(r) \sim r^{d_f}$. The connectivity dimension, instead, measures the average number of sites connected to a given site in at most l steps, as $\mathcal{N}(l) \sim l^{d_l}$. The spectral dimension is related to diffusion processes on graphs and can be defined in terms of the return probability P_{ii} at site i for a random walker by $P_{ii}(t) \sim t^{-d_s/2}$, or equivalently in terms of the density of eigenvalues of the Laplacian operator [24]. The connectivity and fractal dimension can be obviously different and they are related via the mapping between the two distances r and l [29]. In particular, for site percolation in square lattices, the case of the present study, at percolation threshold $p \sim p_c$ one has $d_f \simeq 1.896$ but $d_l \simeq 1.67$ (and, for completeness, $d_s \simeq 1.36$).

Which is the right quantity that characterizes the reaction spreading? Numerical computations in agreement with analytical arguments [20] suggest that the chemical dimension is the right quantity. Starting from a single site with $\theta_i(0) = 1$, after t steps the number of sites reached by the field is $\mathcal{N}(t) \sim t^{d_l}$ [30]. Therefore, in the limit of very fast reaction, when each site reached by the field is immediately burnt (i.e., $\theta_s \simeq 1$), we can

expect

$$m(t) \sim t^{d_1}. \tag{10}$$

Figure 2 clearly shows the scaling of Eq. (10). Moreover, this figure reveals that the scaling (10) is valid not only in the fast reaction regime, and that the reaction rate is relevant only for the prefactor: $m(t) \simeq \alpha t^{d_1}$.

IV. FRONT PROPAGATION

The problem of the front propagation in reactive systems (classical reaction and diffusion processes, advection reaction and diffusion processes, reaction and diffusion in the presence of anomalous diffusion, etc.) has been extensively studied [1,2,19]. In some cases, under certain conditions, it is possible to show that the propagation is standard, i.e., there exists an asymptotic value for the speed v and the width δ of the propagating front. On the other hand, it is almost impossible (except in very special cases) to determine analytically the values of v and δ. Therefore, the numerical study of the speed and the thickness of the moving front is mandatory to obtain information about the spreading dynamics.

In the case of reaction processes on percolating clusters, if one considers an arbitrarily large (in any direction) lattice, the propagation generally is not standard since the total quantity of reaction products grows as a power law with a noninteger exponent, $m(t) \simeq \alpha t^{d_1}$. If the percolating cluster is embedded in a channel with a propagation direction, L_x, and a transversal direction, L_y, with $L_x \geqslant L_y$, a traveling wave takes place with a constant (on average) speed after a transient needed for the reaction product to invade the transversal direction of the channel. Therefore, we consider the model (8) with an initially empty 2D lattice, where $L_x \geqslant L_y$ and $\theta_{(i,j)}(0) = 0$. To reduce the transient, as boundary conditions we use $\theta_{(i=0,j)}(t) = 1$ and $\theta_{(i=L_x,j)}(t) = 0$ for the left and right edge, respectively. In the transversal direction, we have zero-flux (Neumann) boundary conditions, which are automatically guaranteed by the diffusion operator (7). Using the above boundary conditions, we expect the development of a front propagating with a fixed (on average) speed from the left to the right side of the lattice.

In Fig. 3, an example is shown of front propagation in a percolating cluster. The dynamic evolves through the horizontal direction with a fluctuating front depending on the position of the permitted sites. Because of such fluctuations, it is convenient to introduce the averaged field along the horizontal direction as the mean of the field $\theta_s(t)$ along the i direction,

$$\theta_i(t) = \frac{\sum_{j=1}^{L_y} A_{(i,j)}\theta_{(i,j)}(t)}{\sum_{j=1}^{L_y} A_{(i,j)}}. \tag{11}$$

Strictly speaking, given a percolating cluster, the moving front is not a traveling wave in the classical sense, since there does not exist a function $f(i)$ such as $\theta_i(t) = f(i - vt)$. This is due both to the random nature of the permitted sites on the lattice (i.e., the average stabilizes only at very large L_y) and the discrete nature of the lattice. But it is still possible to define averaged quantities such as the propagation speed or the front width as follows.

A. Front speed

In the case of traveling waves, we expect that the total mass of the reaction products increases, on average, linearly with time,

$$M(t) \simeq N_p v t, \tag{12}$$

where N_p is the averaged number of sites accessible by the reaction process in the vertical direction. The computation of N_p is a quite delicate point, thus the maximum amount of accessible sites in a single column of the lattice is neither L_y (since there are permitted and prohibited sites in the lattice) nor pL_y (since not the whole set of permitted sites belong to the percolating cluster). N_p can be estimated as follows. At the percolating threshold, $p \sim p_c$, the total number of points belonging to the percolating cluster in a square of size L_y is proportional to $L_y^{d_f}$. Since there are L_y rows in the square, one has $N_p \sim \frac{L_y^{d_f}}{L_y} = L_y^{d_f-1}$. Instead, if p is large enough to have one single big percolating cluster, without the presence of closed islands of permitted sites not connected to the principal percolating cluster, one has $N_p \simeq pL_y$. In the intermediate cases, it is possible to compute N_p numerically. Therefore, we can define the average front speed as

$$v_1 = \lim_{t\to\infty} \frac{M(t)}{N_p t}. \tag{13}$$

Another way to define v, which is much more sensitive to statistical fluctuations of the cluster structure, can be obtained starting from the dynamics of the model. Since the diffusion operator (7) is a mass-preserving term, the derivative of the total mass can be computed using Eq. (8),

$$v_2(t) = \frac{1}{N_p}\frac{d}{dt}M(t) = \frac{\alpha}{N_p}\sum_{s\in\mathcal{P}}\{\theta_s(t)[1-\theta_s(t)]\}. \tag{14}$$

Of course, $v_2(t)$ is a function of time and its fluctuations reflect the random nature of the percolating cluster. On the other hand, we expect, as confirmed from numerical simulation (not shown), that $\langle v_2(t)\rangle = v_1 = v_f$.

It is interesting to study the behavior of v_f as a function of p, the probability of having a permitted site. In fact, using different values of p, it is possible to model different degrees of nonhomogeneity, and we expect a different evolution of the reaction process. For $p = 1$, since the lattice is homogeneous, we expect to obtain the FKPP value $v_0 = 2\sqrt{\alpha D_0}$. This result is true for small α when $\delta_0 \sim \sqrt{D_0/\alpha}$ is larger than the lattice size (simulations not shown for the sake of brevity). On the contrary, for large α, because of the discrete nature of the lattice, the width of the FKPP front can be of the same order as, or even smaller than, the lattice step. In this case, there is a significant difference between the measured front speed and the FKPP value also for $p = 1$. Although this discrepancy does not invalidate our analysis, we choose to study only rescaled velocity $v_f(p)/v_f(1)$.

In the case of $p < 1$, especially for $p \sim p_c$, it is important to introduce the probability of having a percolating lattice, $P(p)$. We write $u(p) = P(p)v_f(p)/v_f(1)$ as the average velocity in a percolating cluster when the site probability is p. For value of p larger than p_c it is possible to give simple but valid arguments to explain the behavior of u. First of all, for small α values

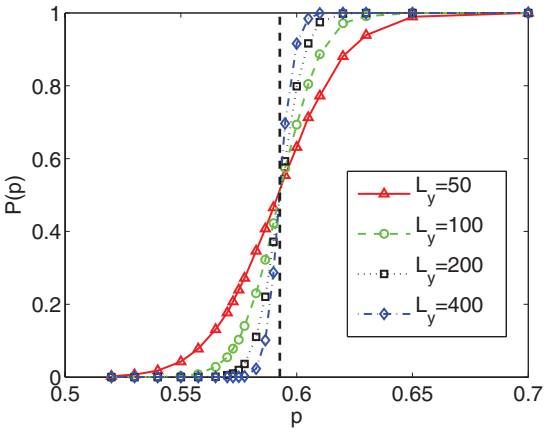

FIG. 5. (Color online) The probability $P(p)$ to percolate along a channel of size $L_x = L_y$ is shown for different L_y.

we expect a large front that regularizes the propagation. This is a kind of homogenization regime [18]. Practically, we can imagine the front proceeding almost as in a homogeneous medium excluding the region in which the propagation is prohibited. Therefore, we can write

$$u(p) = P(p)\frac{v_f(p)}{v_f(1)} \sim P(p)\frac{N_p}{L_y}. \tag{15}$$

Such a relation, when p is large, simplifies to $v_f(p) \sim p v_f(1)$.

In the other limit, for large α, we can use the following argument [32]. We know [from Eq. (10)] that $m(t) \sim t^{d_l}$. On the other hand, $m(t) \sim r(t)^{d_f}$. Therefore, $r(t) \sim t^{d_l/d_f}$ and $v = \frac{dr}{dt} \sim t^{d_l/d_f - 1} \sim r^{1-d_{\min}}$, where $d_{\min} = \frac{d_f}{d_l}$. Furthermore, if the linear size of the region is $r < \xi$, where ξ is the correlation length [9], the cluster is self-similar and then $v \sim \xi^{1-d_{\min}}$. Moreover, analysis of the percolation phase transition gives $\xi \sim |p - p_c|^{-\nu}$, with $\nu = 4/3$ for $d = 2$ [31], which gives the final scaling $v \sim (p - p_c)^{\gamma}$, where $\gamma = -\nu(1 - d_{\min})$. For the average velocity, the scaling is

$$u(p) = P(p)\frac{v_f(p)}{v_f(1)} \sim P(p)\left(\frac{p - p_c}{1 - p_c}\right)^{\gamma}. \tag{16}$$

Alternatively, a similar scaling had been derived through large deviation theory [22]. Both of the above behaviors are well observed in the numerical simulations, as shown in Fig. 4. It is worth noting that below the percolation threshold, the probability to have a percolating cluster tends to zero for a long enough channel; see Fig. 5. Nonetheless, in Fig. 4, it is possible to observe a very small velocity $u(p)$ for $p \lesssim p_c$. This result is basically due to the fact that, for finite size, $P(p)$ is not strictly zero for $p \lesssim p_c$; see Fig. 5.

Concerning the probability $P(p)$ of having a percolating lattice as a function of p, in the numerical simulations it is possible to compute $P(p)$ only for finite values of L_x and L_y. Moreover, in applications the cluster size is finite, and L_y can be small. Figure 5 shows $P(p)$ for different values of L_y in the case of $L_x = L_y$. Naturally for $L_y \to \infty$, $P(p)$ approaches the Heaviside step function $\Theta(p - p_c)$. Simulations (not reported here) show that for nonsquare lattices $L_x = nL_y$ with $n > 1$, while the front speed v_f does not change with n, the probability $P(p)$ is strongly influenced by n if n is large. Moreover, in the case of large n, also p_c changes, becoming dependent on both n and L_y.

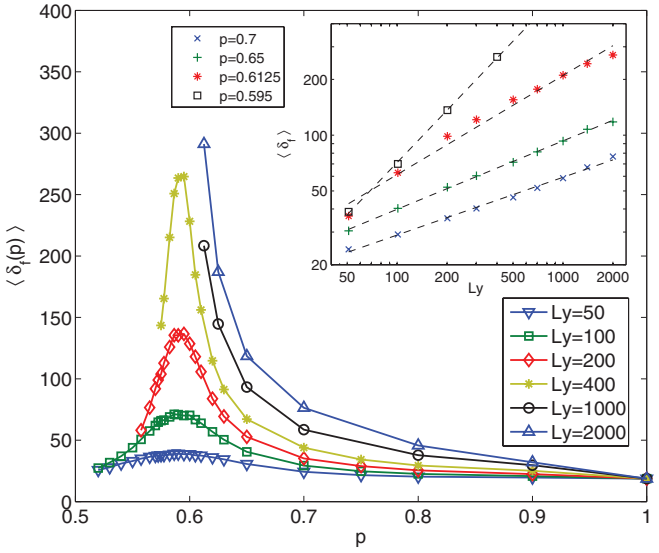

FIG. 6. (Color online) Averaged front width, $\langle \delta_f \rangle$, as a function of p at varying L_y, where the horizontal size of the lattice is $L_x = 5000L_y$. For all curves, $\alpha = 1$. In the inset it is shown that for fixed values of p, the scaling behavior of $\langle \delta_f \rangle \sim L_y^{\beta}$, where $\beta = 0.94$ for $p = 0.595$, $\beta = 0.54$ for $p = 0.6125$, $\beta = 0.37$ for $p = 0.65$, and $\beta = 0.31$ for $p = 0.7$.

B. Front width

For a two-dimensional propagating wave in random media, we can define various different measures of width. One important measure concerns the averaged width of the front along the propagation direction. It is analogous to the front width in the 1D FKPP traveling wave and measures the region along the x direction in which the reaction process is active

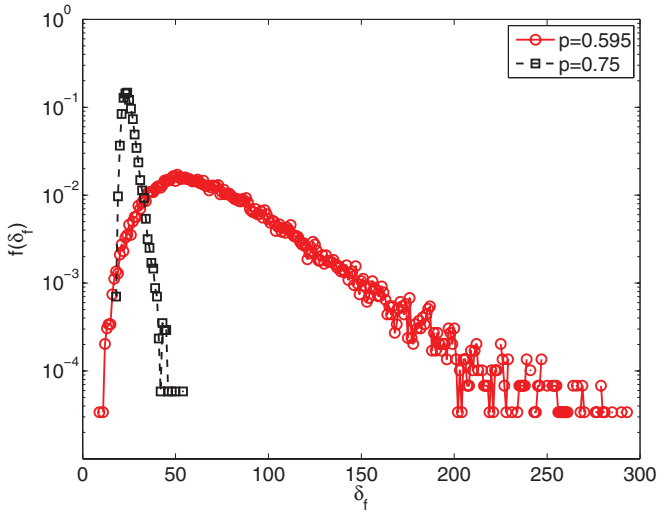

FIG. 7. (Color online) Two different probability density functions of δ_f for two values of p are shown: $p = 0.595 \approx p_c$ and $p = 0.75$, with $L_y = 100$, $L_x = 5000L_y$, and $\alpha = 1$. It is well evident that for $p \approx p_c$, fluctuations play a dominant role and large deviations are present.

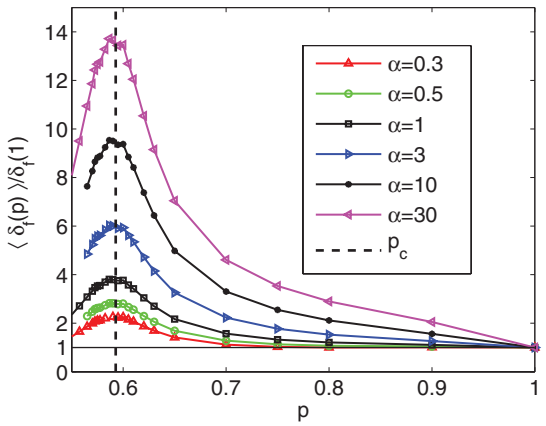

FIG. 8. (Color online) Normalized averaged front width for different values of the reaction rate with $L_y = 100$, $L_x = 5000L_y$. For small values of α fluctuations are smoothed, whereas they are maximal for large reaction rates.

(see Fig. 3). To define such a quantity, one can use $\theta_i(t)$, i.e., the average over the i direction of the field $\theta_s(t)$, defined in Eq. (11). Yet the averaged quantity $\theta_i(t)$ still suffers from large fluctuation, so we use a simplified observable that can give a good measure of the front width. First, we introduce an auxiliary quantity,

$$H_{(i,j)}(t) = \begin{cases} 1 & \text{if } 0.01 \leqslant \theta_{(i,j)}(t) \leqslant 0.99, \\ 0 & \text{elsewhere.} \end{cases} \quad (17)$$

Then we define $\delta_f(t)$, the front width, as the distance between the maximum and the minimum value of i such that $H_{(i,j)}(t) = 1$. In this way, we define a rectangle of size $L_y \times \delta_f(t)$ inside which there is the whole active front. Also, $\delta_f(t)$ is a strongly fluctuating quantity, therefore we study the statistical feature of $\delta_f(t)$, e.g., $\langle \delta_f \rangle$, as a function of p and L_y.

In principle, one can expect that given p and α, for L_y large enough, the front width reaches a constant value, as with the front speed. On the other hand, as shown in Fig. 6, the convergence depends on p: while for large p (near $p = 1$) there is an asymptotic value of $\langle \delta_f \rangle$, for values of p going to p_c there is no convergence at all. Notably, in the limit of very large clusters, the averaged front width diverges rapidly around $p \sim p_c$. As the inset of Fig. 6 shows, the scaling structure of the front width as a function of L_y at varying p is highly nontrivial and cannot be associated with a single scaling exponent [33].

Rather interesting is the presence of very large fluctuations of $\langle \delta_f \rangle$. Figure 7 shows how, for p near p_c, the typical value of the front width, ${\delta_f}^T$, given by the maximum of the probability density function, is of the same order of the fluctuation of $\langle \delta_f \rangle$ [measured as $\sqrt{\langle \delta_f^2(t) \rangle - \langle \delta_f(t) \rangle^2}$].

The above discussion is valid at fixed (and not too small) α. When α is small, the bare FKPP front width, δ_0, is large, and a large front width regularizes the reaction dynamics. If the bare front width is larger than the typical size of the prohibited islands (for a given p), we can expect that the random distribution of the islands does not affect too much the front propagation, with a net effect of diminishing the fluctuations and the dependence of the front on both p and L_y. On the other hand, for large α the bare front width is comparable with the lattice discretization. In this case, fluctuations become very strong due to the dependency of the front width both from p and L_y. Figure 8 explicates the above discussion.

V. CONCLUSION

Reaction and diffusion processes in heterogeneous media, because of their relevance in many real-world applications, play a central role in several different fields. In the present paper, starting from the basic equations, we have investigated the behavior of a simple reaction and diffusion process taking place in a heterogeneous medium, i.e., two-dimensional percolating structures. We show that for the reaction spreading on percolating clusters, the dynamics is ruled by the connectivity dimension, d_l [see Eq. (10)], and the reaction rate affects only the prefactor of the scaling. In the case of percolating clusters through a channel, the reaction and diffusion process develops a statistically stationary traveling wave. The speed and the width of the traveling wave are deeply influenced by the percolating transition together with finite-size effects that generate peculiar behaviors of both front speed and front width. Those effects are crucial since, in realistic problems, the channel over which the reaction takes place has necessarily a finite transversal length. Some recent numerical computations and experiments show the key role played by the flow heterogeneities on the chemical front dynamics [13–16].

ACKNOWLEDGMENTS

We thank R. Burioni for fruitful discussions.

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